

A CLT FOR DEPENDENT RANDOM VARIABLES WITH AN APPLICATION TO AN INFINITE SYSTEM OF INTERACTING DIFFUSION PROCESSES

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ABSTRACT. We present a central limit theorem for stationary random fields that are short-range dependent and asymptotically independent. As an application, we present a central limit theorem for an infinite family of interacting Itô-type diffusion processes.

1. INTRODUCTION

Let $\zeta = \{\zeta_k\}_{k \in \mathbb{Z}^d}$ be a stationary random field, centered and normalized so that $E(\zeta_0) = 0$ and $\text{Var}(\zeta_0) < \infty$. The principal aim of this article is to: (1) Prove the following central limit theorem for dependent random variables; and (2) Present an application of this theorem to interacting diffusion processes.

Theorem 1.1. *If the random field $\zeta = \{\zeta_k\}_{k \in \mathbb{Z}^d}$ is short-range dependent [see (2.1)], asymptotically independent [see Condition (AI)], and satisfies a minimal integrability condition [see (2.5)], then the distribution of $n^{-d/2} \sum_{k \in \{1, \dots, n\}^d} \zeta_k$ is asymptotically normal as $n \rightarrow \infty$.*

Short-range dependence and asymptotic independence will be recalled in due time. For now, it suffices to say that these are both natural conditions and both arise abundantly in the literature on time-series analysis (see for example Lahiri [16] and its references).

Our main application of Theorem 1.1 is a result about infinite systems of interacting diffusions. Before we describe that application, let us make two brief remarks that explain how Theorem 1.1 might relate to parts of a vast body of central limit theorems that already exist in the literature.

Remark 1.2. One can prove that if ζ is strongly mixing in the sense of Rosenblatt [20], then ζ is asymptotically independent; see Corollary 1.12 of Bradley [3, Vol. 1] for example when $d = 1$. It follows from this that Theorem 1.1 implies a well-known central limit theorem of Ibragimov [12] for strongly mixing sequences though; usually, the latter is cast when $d = 1$. A noteworthy difference between the proof of Theorem 1.1 and the proofs of CLTs for strongly mixing sequences is that the

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proof of Theorem 1.1 is non technical, and relies only on compactness arguments together with Paul Lévy’s classical characterization of standard Brownian motion as the unique mean-zero, continuous Lévy process with variance one at time one (this is in fact an immediate consequence of the Lévy-Khintchine formula; see, for example Bertoin [1]). The detailed bibliography of the three-volume book by Bradley [3] contains a very large number of pointers to the vast literature on CLT for stationary dependent sequences. See also the survey articles by Bradley [4] and Merlevède, Peligrad, and Utev [17].

Remark 1.3. It is easy to see that if ζ is an associated sequence of random variables in the sense of Esary, Proschan, and Walkup [10], then Theorem 1.1 reduces to the central limit theorem of Newman and Wright [18] for associated random variables (again, usually stated for $d = 1$).

The preceding remarks describe how Theorem 1.1 reduces to well-known theorems in specific settings. Next we describe a setting where we do not know whether there is association or strong mixing.

Consider the following infinite system of interacting Itô-type stochastic differential equations:

$$(1.1) \quad \begin{aligned} du_t(x) &= (Lu_t)(x) dt + \Phi(u_t(x)) dB_t(x) && \text{for all } t > 0 \text{ and } x \in \mathbb{Z}^d, \\ \text{subject to } u_0(x) &= 1 && \text{for all } x \in \mathbb{Z}^d, \end{aligned}$$

for a field $B = \{B(x)\}_{x \in \mathbb{Z}^d}$ of i.i.d. one-dimensional Brownian motions. Here, L denotes the generator of a continuous-time random walk on \mathbb{Z}^d , and the diffusion coefficient $\Phi : \mathbb{R} \mapsto \mathbb{R}$ is assumed to be Lipschitz continuous. Shiga and Shimizu [21] have shown that (1.1) has a unique adapted solution under these conditions.

Theorem 1.4. *For every Lipschitz-continuous function g and all $t \geq 0$,*

$$(1.2) \quad \left\{ n^{-d/2} \sum_{x \in \mathbb{Z}^d} (g(u_t(x)) - \mathbb{E}[g(u_t(0))]) \varphi(x/n); \varphi \in \mathcal{C} \right\} \xrightarrow{\text{fdd}} \{ \sigma_{g,t} W(\varphi); \varphi \in \mathcal{C} \},$$

where $\sigma_{g,t}^2 = \sum_{x \in \mathbb{Z}^d} \text{Cov}[g(u_t(0)), g(u_t(x))]$ is finite, in fact absolutely convergent.

In the above theorem, and throughout the paper, “ $\xrightarrow{\text{fdd}}$ ” refers to the weak convergence of all finite-dimensional distributions. Moreover \mathcal{C} denotes the class of all piecewise-continuous functions with compact support on \mathbb{R}^d .¹

Earlier, Deuschel [9] proved a version of Theorem 1.4 – where L is replaced by a more general nonlinear operator and Φ a less general function – using a martingale central limit theorem in place of Theorem 1.1. The present formulation of Theorem 1.4 includes the important special case of the “parabolic Anderson model.” That is when $\Phi(z) = \text{const} \times z$ for all $z \in \mathbb{R}$; see Carmona and Molchanov [5].

Theorem 1.4 can also be generalized to cover many other noise models; see Ref. [6–8] for analogous results in the continuous setting of SPDEs. Here, we will not study such more general results to maintain brevity.

The remainder of this paper is devoted to the proof of Theorems 1.1 and 1.4, which are proved in §2 and §3, respectively.

¹Piecewise continuous means functions which are continuous except on a finite number of hyperplanes.

We close this section with a brief description of the notation of this paper. Throughout we write $\|Z\|_k$ instead of $(E\|Z\|^k)^{1/k}$. We denote by Lip_f the Lipschitz constant of every function $f : \mathbb{R} \rightarrow \mathbb{C}$; that is,

$$\text{Lip}_f := \sup_{-\infty < a < b < \infty} \frac{|f(b) - f(a)|}{|b - a|}.$$

Rademacher’s theorem (see Federer [11, Theorem 3.1.6]) ensures that if $\text{Lip}_f < \infty$, then the weak derivative of f exists and is bounded almost everywhere in modulus by Lip_f .

2. A CLT FOR DEPENDENT VARIABLES

In this section we first recall the undefined terminology of Theorem 1.1, and then state and prove a more general theorem (Theorem 2.2) than Theorem 1.1 which turns out to be easier to prove directly.

2.1. Two definitions. We start with a general definition.

Definition 2.1. A sequence of N -dimensional random vectors $(X_{1,n}, \dots, X_{N,n})$, $N \geq 2$, is *asymptotically independent* as $n \rightarrow \infty$ if for all real numbers ξ_j , $1 \leq j \leq N$,

$$\lim_{n \rightarrow \infty} \left| E \left[\prod_{j=1}^n e^{i\xi_j X_{j,n}} \right] - \prod_{j=1}^N E \left[e^{i\xi_j X_{j,n}} \right] \right| = 0.$$

Recall that a stationary random field $\zeta = \{\zeta_k\}_{k \in \mathbb{Z}^d}$ is said to be *short-range dependent* when

$$(2.1) \quad \bar{\sigma}^2 := \sum_{k \in \mathbb{Z}^d} |\text{Cov}(\zeta_0, \zeta_k)| < \infty.$$

It will be helpful to first introduce some notation before we define the remaining undefined term in Theorem 1.1. We endow the collection \mathcal{C} of all piecewise-continuous functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ that have compact support with the $L^2(\mathbb{R}^d)$ norm. Let $\mathcal{U} \subset \mathcal{C}$ denote the collection of all linear combinations of indicator functions of upright boxes $Q \subset \mathbb{R}^d$ of the form $Q = (a_1, b_1] \times \dots \times (a_d, b_d]$, where $a_1 < b_1, \dots, a_d < b_d$ are real numbers. We always let $\text{supp}[\varphi]$ denote the support of the function $\varphi \in \mathcal{C}$. Moreover, the *separation* of two sets A and B is defined as

$$(2.2) \quad \text{sep}(A, B) = \inf \left\{ \max_{1 \leq i \leq d} |x_i - y_i| : x \in A, y \in B \right\} \quad \text{for } A, B \subset \mathbb{R}^d.$$

For every $\varphi \in \mathcal{C}$ we consider the sequence of random variables

$$(2.3) \quad S_n(\varphi) := n^{-d/2} \sum_{k \in \mathbb{Z}^d} \zeta_k \varphi(k/n) \quad [n \in \mathbb{N}].$$

Now we are ready to introduce the assumption of asymptotic independence.

(AI) For every $\delta > 0$ and all $\varphi_1, \varphi_2 \in \mathcal{U}$ such that $\text{sep}(\text{supp}[\varphi_1], \text{supp}[\varphi_2]) \geq \delta$, the random variables $S_n(\varphi_1)$ and $S_n(\varphi_2)$ are *asymptotically independent* as $n \rightarrow \infty$; that is,

$$(2.4) \quad \lim_{n \rightarrow \infty} \left| E \left[e^{iaS_n(\varphi_1) + ibS_n(\varphi_2)} \right] - E \left[e^{iaS_n(\varphi_1)} \right] E \left[e^{ibS_n(\varphi_2)} \right] \right| = 0 \quad \text{for all } a, b \in \mathbb{R}.$$

We conclude this section with a CLT for dependent variables. Let $W = \{W(\varphi); \varphi \in L^2(\mathbb{R}^d)\}$ denote the usual isonormal Gaussian process that is associated with white noise on \mathbb{R}^d . That is, W is a mean-zero Gaussian process with

$$\text{Cov}[W(\varphi_1), W(\varphi_2)] = \langle \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}^d)} \quad \text{for every } \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d).$$

Theorem 2.2. *If $\zeta = \{\zeta_k\}_{k \in \mathbb{Z}^d}$ is a stationary random field that satisfies (2.1) and (AI), and if*

$$(2.5) \quad n \mapsto S_n^2(\mathbf{1}_{(0,1]^d}) = n^{-d} \left(\sum_{k \in \{1, \dots, n\}^d} \zeta_k \right)^2 \text{ is uniformly integrable,}$$

then $\{S_n(\varphi); \varphi \in \mathcal{C}\} \xrightarrow{\text{fdd}} \{\sigma W(\varphi); \varphi \in \mathcal{C}\}$ as $n \rightarrow \infty$, where

$$(2.6) \quad \sigma^2 := \sum_{k \in \mathbb{Z}^d} \text{Cov}(\zeta_0, \zeta_k).$$

Let $\varphi = \mathbf{1}_{(0,1]^d}$ in order to deduce the following precise form of Theorem 1.1 from Theorem 2.2.

Corollary 2.3. *If $\zeta = \{\zeta_k\}_{k \in \mathbb{Z}^d}$ is a stationary random field that satisfies (2.1), (AI), and (2.5), then $n^{-d/2} \sum_{k \in \{1, \dots, n\}^d} \zeta_k \xrightarrow{d} N(0, \sigma^2)$ as $n \rightarrow \infty$, where σ^2 is defined in (2.6).*

Lemma 2.4. *Condition (2.5) implies that $n \mapsto S_n^2(\mathbf{1}_Q)$ is uniformly integrable for every $Q \in \mathcal{U}$.*

Proof. A finite sum of uniformly integrable (denoted by UI) random variables is UI. Therefore, by stationarity, it suffices to prove that $n \mapsto S_n^2(\mathbf{1}_Q)$ is UI for $Q = \prod_{k=1}^d (0, r_k]$, where $0 < r_k \leq 1$. We will prove this for $Q = (0, r] \times (0, 1]^{d-1}$ since this is all we will need later on; the same arguments can be applied to prove the uniform integrability for more general $Q \in \mathcal{U}$. To simplify the notation, we assume without loss of much generality that $d = 2$.

Choose and fix some $r \in (0, 1)$, and define $Q(\alpha) := (0, \alpha] \times (0, 1]$ for every $\alpha \in (0, 1]$. One can check from first principals that if $\{S_n^2(\mathbf{1}_{Q(\alpha)})\}_{n \in \mathbb{N}}$ is UI for some $\alpha \in (0, 1]$, then $\{S_n^2(\mathbf{1}_{\frac{1}{2}Q(\alpha)})\}_{n \in \mathbb{N}}$ is UI also. Hence, it follows from condition (2.5) and stationarity that $n \mapsto \{S_n^2(\mathbf{1}_{(0,1/2] + (0,1/2]^2})\}_{n \in \mathbb{N}}$ is UI. And because $Q(1/2)$ is the disjoint union of $(0, 1/2]^2$ and $(0, 1/2) + (0, 1/2]^2$, it follows that $n \mapsto S_n^2(\mathbf{1}_{Q(1/2)})$ is UI. By induction, we may deduce that $n \mapsto S_n^2(\mathbf{1}_{Q(1/2^k)})$ is UI for any $k \in \mathbb{N}$. Again, we use the fact that a finite sum of UI random variables is UI to see that $n \mapsto S_n^2(\mathbf{1}_{Q(q)})$ is UI for every real number $q \in (0, 1)$ of finite dyadic expansion, that is, q of the form $\sum_{i=1}^m x_i/2^i$ where $x_1, \dots, x_m \in \{0, 1\}$.

For every $\varepsilon > 0$ there exists a number $q \in (0, 1)$, with a finite dyadic expansion, such that $\|\mathbf{1}_{Q(|r-q|)}\|_{L^2(\mathbb{R}^2)} < \varepsilon$. Thus, the following is valid for all $K, L > 0$ and $n \in \mathbb{N}$:

$$\begin{aligned} \mathbb{E}[S_n^2(\mathbf{1}_{Q(r)}); |S_n(\mathbf{1}_{Q(r)})| > L] &= \mathbb{E} \left[\left| S_n(\mathbf{1}_{Q(q)}) + S_n(\mathbf{1}_{Q(r)} - \mathbf{1}_{Q(q)}) \right|^2; |S_n(\mathbf{1}_{Q(r)})| > L \right] \\ &\leq 2\mathbb{E} [S_n^2(\mathbf{1}_{Q(q)}); |S_n(\mathbf{1}_{Q(r)})| > L] + 2\mathbb{E} [S_n^2(\mathbf{1}_{Q(|r-q|)})] \\ &\leq 2\mathbb{E} [S_n^2(\mathbf{1}_{Q(q)}); |S_n(\mathbf{1}_{Q(q)})| > K] + \frac{2K^2}{L^2} \sup_{n \in \mathbb{N}} \mathbb{E} [S_n^2(\mathbf{1}_{Q(r)})] + 2\mathbb{E} [S_n^2(\mathbf{1}_{Q(|r-q|)})], \end{aligned}$$

thanks to Chebyshev’s inequality. We borrow in advance from Lemma 2.5 – see (2.7) – to see that²

$$\begin{aligned} & \lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} [S_n^2(\mathbf{1}_{Q(r)}) ; |S_n(\mathbf{1}_{Q(r)})| > L] \\ & \leq 2 \limsup_{n \rightarrow \infty} \mathbb{E} [S_n^2(\mathbf{1}_{Q(q)}) ; |S_n(\mathbf{1}_{Q(q)})| > K] + 2\sigma^2\varepsilon^2. \end{aligned}$$

Let $K \rightarrow \infty$, using the fact that $n \mapsto S_n^2(\mathbf{1}_{Q(q)})$ is UI, to conclude the uniform integrability of $n \mapsto S_n^2(\mathbf{1}_{Q(r)})$ from the fact that ε is arbitrary. \square

2.2. Tightness and weak convergence. Before we prove Theorem 2.2, we make some comments, by way of three lemmas, about tightness and weak convergence. The proof of Theorem 2.2 will be carried out in the next subsection.

Lemma 2.5. *For every $\varphi \in \mathcal{C}$ and all $n \in \mathbb{N}$,*
(2.7)

$$\mathbb{E} (|S_n(\varphi)|^2) \leq \bar{\sigma}^2 n^{-d} \sum_{k \in \mathbb{Z}^d} |\varphi(k/n)|^2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E} (|S_n(\varphi)|^2) = \sigma^2 \|\varphi\|_{L^2(\mathbb{R}^d)}^2,$$

where σ and $\bar{\sigma}$ are defined respectively in (2.6) and (2.1).

Proof. A change of variables shows that

$$\mathbb{E} (|S_n(\varphi)|^2) = n^{-d} \sum_{j \in \mathbb{Z}^d} \text{Cov}(\zeta_0, \zeta_j) \sum_{k \in \mathbb{Z}^d} \varphi(k/n)\varphi((j+k)/n).$$

This implies the first assertion of (2.7), since the Cauchy–Schwarz inequality for the counting measure implies that

$$n^{-d} \sum_{k \in \mathbb{Z}^d} |\varphi(k/n)\varphi((j+k)/n)| \leq n^{-d} \sum_{k \in \mathbb{Z}^d} |\varphi(k/n)|^2.$$

Moreover, since φ is piecewise continuous with compact support,

$$(2.8) \quad \lim_{n \rightarrow \infty} n^{-d} \sum_{k \in \mathbb{Z}^d} \varphi(k/n)\varphi((j+k)/n) = \|\varphi\|_{L^2(\mathbb{R}^d)}^2 \quad \text{boundedly, for every } j \in \mathbb{Z}^d.$$

Therefore, the remaining assertion of (2.7) follows from the dominated convergence theorem. \square

Lemma 2.6. *Let \mathcal{D} be a dense subset of \mathcal{C} in $L^2(\mathbb{R}^d)$, and suppose $S_n(\psi) \xrightarrow{d} \sigma W(\psi)$, as $n \rightarrow \infty$, for every $\psi \in \mathcal{D}$. Then, $\{S_n(\varphi); \varphi \in \mathcal{C}\} \xrightarrow{\text{fdd}} \{\sigma W(\varphi); \varphi \in \mathcal{C}\}$ as $n \rightarrow \infty$.*

Proof. Choose and fix an arbitrary function $\varphi \in \mathcal{C}$. We aim to prove that $S_n(\varphi) \xrightarrow{d} W(\varphi)$ as $n \rightarrow \infty$. The convergence of finite-dimensional distributions follows from this and the linearity of $\varphi \mapsto S_n(\varphi)$ and $\varphi \mapsto W(\varphi)$. For every $\varepsilon > 0$ there exists $\psi \in \mathcal{D}$ such that $\|\varphi - \psi\|_{L^2(\mathbb{R}^d)}^2 < \varepsilon$. Since $S_n(\psi) \xrightarrow{d} \sigma W(\psi)$ as $n \rightarrow \infty$, we can

²Lemma 2.5 and its proof do not refer to the present lemma. So this application of Lemma 2.5 is logically sound.

write, for all $\xi \in \mathbb{R}$,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \mathbb{E} e^{i\xi S_n(\varphi)} - \mathbb{E} e^{i\xi \sigma W(\varphi)} \right|^2 \\ & \leq 3 \limsup_{n \rightarrow \infty} \left| \mathbb{E} e^{i\xi S_n(\varphi)} - \mathbb{E} e^{i\xi S_n(\psi)} \right|^2 + 3 \left| \mathbb{E} e^{i\xi \sigma W(\varphi)} - \mathbb{E} e^{i\xi \sigma W(\psi)} \right|^2 \\ & \leq 3\xi^2 \limsup_{n \rightarrow \infty} \mathbb{E} (|S_n(\varphi) - S_n(\psi)|^2) + 3\xi^2 \sigma^2 \mathbb{E} (|W(\varphi) - W(\psi)|^2), \end{aligned}$$

since $|e^{iz} - e^{iy}| \leq |z - y|$ for all $z, y \in \mathbb{R}$. Property (2.7) now ensures that, for all $\xi \in \mathbb{R}$,

$$\limsup_{n \rightarrow \infty} \left| \mathbb{E} e^{i\xi S_n(\varphi)} - \mathbb{E} e^{i\xi \sigma W(\varphi)} \right|^2 \leq 6\xi^2 \sigma^2 \varepsilon \quad \text{for every } \xi \in \mathbb{R}.$$

This completes the proof since the left-hand side does not depend on ε . □

Lemma 2.7. *Let $\delta > 0$, and suppose that $(S_n(\psi_1), S_n(\psi_2))$ are asymptotically independent as $n \rightarrow \infty$ for all $\psi_1, \psi_2 \in \mathcal{U}$ such that $\text{sep}(\text{supp}[\psi_1], \text{supp}[\psi_2]) \geq \delta$ [see (2.2)]. Then, $(S_n(\varphi_1), \dots, S_n(\varphi_N))$ are asymptotically independent as $n \rightarrow \infty$ [see Definition 2.1] for all $\varphi_1, \dots, \varphi_N \in \mathcal{U}$ that satisfy*

$$\text{sep}(\text{supp}[\varphi_i], \text{supp}[\varphi_j]) \geq \delta \quad \text{for } 1 \leq i \neq j \leq N \text{ and integers } N \geq 2.$$

Proof. Set $\Phi_m := \varphi_1 + \dots + \varphi_m$ for all $m = 1, \dots, N$. For every $n \in \mathbb{N}$, let $D_{1,n} := 0$ and define

$$D_{m,n} := \left| \mathbb{E} \left[\prod_{j=1}^m e^{iS_n(\varphi_j)} \right] - \prod_{j=1}^m \mathbb{E} e^{iS_n(\varphi_j)} \right| = \left| \mathbb{E} e^{iS_n(\Phi_m)} - \prod_{j=1}^m \mathbb{E} e^{iS_n(\varphi_j)} \right|,$$

for $m = 2, \dots, N$. The linearity of S_n and the triangle inequality together imply that for every $n \in \mathbb{N}$ and $m = 2, \dots, N$,

$$\begin{aligned} D_{m,n} &= \left| \mathbb{E} e^{iS_n(\Phi_m)} - \mathbb{E} e^{iS_n(\Phi_{m-1})} \mathbb{E} e^{iS_n(\varphi_m)} \right| \\ &\quad + \left| \mathbb{E} e^{iS_n(\Phi_{m-1})} \mathbb{E} e^{iS_n(\varphi_m)} - \prod_{j=1}^m \mathbb{E} e^{iS_n(\varphi_j)} \right| \\ &\leq \left| \text{Cov} \left[e^{iS_n(\Phi_{m-1})}, e^{-iS_n(\varphi_m)} \right] \right| + D_{m-1,n}. \end{aligned}$$

Subtract $D_{m-1,n}$ from both sides and sum over m to find that

$$D_{N,n} \leq \sum_{m=2}^N \left| \text{Cov} \left[e^{iS_n(\Phi_{m-1})}, e^{iS_n(-\varphi_m)} \right] \right|.$$

The separation condition on the φ_j 's implies that $\text{sep}(\text{supp}[\Phi_{m-1}], \text{supp}[-\varphi_m]) \geq \delta$ for all $m = 2, \dots, N$, and hence $\lim_{n \rightarrow \infty} D_{N,n} = 0$ by asymptotic independence [see (2.4)]. Now relabel φ_j as $\xi_j \varphi_j$, where ξ_1, \dots, ξ_N are arbitrary nonzero constants, in order to deduce that $S_n(\varphi_1), \dots, S_n(\varphi_N)$ are asymptotically independent as $n \rightarrow \infty$. [This requires only the fact that $\xi_i \varphi_i$ has the same support as φ_i for $i = 1, \dots, N$.] □

2.3. Proof of Theorem 2.2. In this section, we assume that the conditions of Theorem 2.2 are met.

We first prove weak convergence in a special case, where the limit can be identified with a Brownian motion. Choose and fix real numbers a_1 and $a_2 < b_2, \dots, a_d < b_d$, and define

$$Q(r) = (a_1, a_1 + r] \times (a_2, b_2] \times \dots \times (a_d, b_d] \quad \text{for every } r \in (0, 1].$$

Observe that $Q(r) \in \mathcal{U}$ for every $r \in (0, 1]$. For every $n \in \mathbb{N}$, we define one-parameter processes Y_n and Y as follows:

$$Y_n(r) := S_n(\mathbf{1}_{Q(r)}) \quad \text{and} \quad Y(r) := \sigma W(\mathbf{1}_{Q(r)}) \quad \text{for every } r \in (0, 1].$$

It is immediate that Y is a one-dimensional Brownian motion with variance $\sigma^2 \prod_{i=2}^d (b_i - a_i)$. Our main objective is to prove the following specialized form of Theorem 2.2.

Proposition 2.8. $Y_n \xrightarrow{\text{fdd}} Y$ as $n \rightarrow \infty$.

Let us first deduce Theorem 2.2 from its specialized form Proposition 2.8. The proposition will be verified subsequently.

Proof of Theorem 2.2. Let $\mathcal{D} := \cup_{\delta > 0} \mathcal{D}_\delta$, where for every $\delta > 0$, \mathcal{D}_δ denotes the collection of all functions $\psi \in \mathcal{U}$ that have the form

$$(2.9) \quad \psi = \psi_1 + \dots + \psi_m, \quad \text{where } \psi_i = a_i \mathbf{1}_{Q_i},$$

where $m \in \mathbb{N}$, $a_1, \dots, a_m \in \mathbb{R} \setminus \{0\}$, and $Q_1, \dots, Q_m \in \mathcal{U}$ are upright boxes of the form $Q_i = (b_1^i, c_1^i] \times \dots \times (b_d^i, c_d^i]$ for real numbers $b_1^i < c_1^i, \dots, b_d^i < c_d^i$, $i = 1, \dots, m$, and satisfy

$$\text{sep}(Q_i, Q_j) \geq \delta \quad \text{whenever } 1 \leq i \neq j \leq m;$$

see (2.2). Because \mathcal{D} is dense in $L^2(\mathbb{R}^d)$, and hence also dense in \mathcal{C} , Lemma 2.6 will imply Theorem 2.2 once we prove that $S_n(\psi) \xrightarrow{d} \sigma W(\psi)$, as $n \rightarrow \infty$, for every $\psi \in \mathcal{D}$. With this aim in mind, let us choose and fix some $\delta > 0$ and $\psi \in \mathcal{D}_\delta$, and assume that ψ has the representation (2.9). By linearity, $S_n(\psi) = \sum_{i=1}^m S_n(\psi_i) =: \sum_{i=1}^m X_{i,n}$ a.s., where $X_{i,n} := S_n(\psi_i)$. The asymptotic independence condition in Theorem 2.2 and Lemma 2.7 ensure that $\{X_{i,n}\}_{i=1}^m$ describes an asymptotically independent sequence as $n \rightarrow \infty$; and Proposition 2.8 implies that $X_{i,n} \xrightarrow{d} \sigma W(\psi_i)$ as $n \rightarrow \infty$, for every $i = 1, \dots, m$. The asserted asymptotic independence then implies that $S_n(\psi) \xrightarrow{d} Y_1 + \dots + Y_m$ as $n \rightarrow \infty$, where Y_1, \dots, Y_m are independent, and the distribution of Y_i is the same as that of $\sigma W(\psi_i)$ for every $i = 1, \dots, m$. Because the supports of the ψ_i 's are disjoint, $W(\psi_1), \dots, W(\psi_m)$ are uncorrelated, hence independent, Gaussian random variables. In particular, $S_n(\psi) \xrightarrow{d} \sigma W(\psi_1) + \dots + \sigma W(\psi_m) = \sigma W(\psi)$ as $n \rightarrow \infty$; see (2.9) for the last identity. This concludes the proof of Theorem 2.2. \square

Proof of Proposition 2.8. We will prove this proposition in five steps.

Step 1. The laws of $\{Y_n(r)\}_{n \geq 1}$ are L^2 -bounded uniformly in $r \in (0, 1]$ and $n \in \mathbb{N}$, and hence also tight uniformly over all $r \in (0, 1]$.

In order to see why, apply (2.7) in order to see that

$$(2.10) \quad \mathbb{E}(|Y_n(r)|^2) \leq \bar{\sigma}^2 n^{-d} \sum_{k \in \mathbb{Z}^d} \mathbf{1}_{Q(r)}(k/n) \leq \bar{\sigma}^2 n^{-d} \sum_{k \in \mathbb{Z}^d} \mathbf{1}_{Q(1)}(k/n)$$

for every $r \in (0, 1]$ and $n \in \mathbb{N}$. The final quantity in (2.10) is bounded uniformly in $n \in \mathbb{N}$ by $\bar{\sigma}^2$ times the upper Riemann sum of $\mathbf{1}_{Q(1)}$, and the latter is finite. This yields the desired L^2 -boundedness, and tightness follows from Chebyshev’s inequality.

Step 2. For every unbounded sequence $0 < n_1 < n_2 < \dots$ there exists a subsequence $n' = \{n'_k\}_{k=1}^\infty$ and random variables $\bar{Y} = \{\bar{Y}(r)\}_{r \in \mathbb{Q} \cap (0,1]}$ such that $Y_{n'_k} \xrightarrow{\text{fdd}} \bar{Y}$ as $k \rightarrow \infty$.

This follows from uniform tightness in Step 1 and Cantor’s diagonalization.

Step 3. \bar{Y} can be extended to a continuous process $\bar{Y} = \{\bar{Y}(r)\}_{r \in [0,1]}$.

Let n tend to infinity along the subsequence n' , and appeal to Step 2, Fatou’s lemma, and (2.7) in order to see that for every $R > r > 0$ with $r, R \in \mathbb{Q} \cap (0, 1]$,

$$\mathbb{E} (|\bar{Y}(R) - \bar{Y}(r)|^2) \leq \liminf_{k \rightarrow \infty} \mathbb{E} (|Y_{n'_k}(R) - Y_{n'_k}(r)|^2) = \sigma^2 |R - r| \prod_{i=2}^d (b_i - a_i).$$

The continuity of Y follows from this and Kolmogorov continuity theorem [14, Theorem 2.8].

Step 4. We can realize $\bar{Y} = \{\bar{Y}(r)\}_{r \in [0,1]}$ as an infinitely divisible process with stationary increments such that $\bar{Y}(0) = 0$ and $\mathbb{E}[\bar{Y}(r)] = 0$ for all $r \in [0, 1]$. Therefore, the process $\bar{Y} = \{\bar{Y}(r)\}_{r \in [0,1]}$ is a centered Brownian motion indexed and normalized such that $\text{Var}[\bar{Y}(1)] = \sigma^2$.

We first prove that $\bar{Y} = \{\bar{Y}(r)\}_{r \in [0,1]}$ is infinitely divisible. Let us choose and fix an integer $M \geq 1$ and $M + 1$ real numbers $0 =: r_0 < r_1 < \dots < r_M$. For every sufficiently small $\delta > 0$, there exist rational points $r_{1,\delta}^\pm, \dots, r_{M,\delta}^\pm \in \mathbb{Q}$ such that $0 = r_0 = r_{0,\delta}^+ < r_{1,\delta}^- < r_1 < r_{1,\delta}^+ < \dots < r_{j,\delta}^- < r_j < r_{j,\delta}^+ < \dots < r_{M,\delta}^- < r_M$, and $\delta \leq r_{j,\delta}^+ - r_{j,\delta}^- \leq 2\delta$ for all $j = 1, \dots, M - 1$ and $r_M - r_{M,\delta}^- \leq \delta$. We choose and fix such a δ in order to deduce from the asymptotic independence condition (AI) and Lemma 2.7 that $\{Y_n(r_{j,\delta}^-) - Y_n(r_{j-1,\delta}^+)\}_{j=1}^M$ are asymptotically independent as $n \rightarrow \infty$. Hence, the random variables $\bar{Y}(r_{1,\delta}^-), \bar{Y}(r_{2,\delta}^-) - \bar{Y}(r_{1,\delta}^+), \dots, \bar{Y}(r_{M,\delta}^-) - \bar{Y}(r_{M-1,\delta}^+)$ are independent. Moreover, we may appeal to the continuity of $\bar{Y} = \{\bar{Y}(r)\}_{r \in [0,1]}$ (see Step 3) in order to conclude that for all $\alpha_1, \dots, \alpha_M \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{E} \left[e^{i \sum_{j=1}^M \alpha_j (\bar{Y}(r_j) - \bar{Y}(r_{j-1}))} \right] \\ &= \lim_{\delta \rightarrow 0} \mathbb{E} \left[e^{i \sum_{j=1}^M \alpha_j (\bar{Y}(r_{j,\delta}^-) - \bar{Y}(r_{j-1,\delta}^+))} \right] \\ &= \lim_{\delta \rightarrow 0} \prod_{j=1}^M \mathbb{E} \left[e^{i \alpha_j (\bar{Y}(r_{j,\delta}^-) - \bar{Y}(r_{j-1,\delta}^+))} \right] = \prod_{j=1}^M \mathbb{E} \left[e^{i \alpha_j (\bar{Y}(r_j) - \bar{Y}(r_{j-1}))} \right]. \end{aligned}$$

It follows that the random variables $\bar{Y}(r_1), \bar{Y}(r_2) - \bar{Y}(r_1), \dots, \bar{Y}(r_M) - \bar{Y}(r_{M-1})$ are independent, whence $\bar{Y} = \{\bar{Y}(r)\}_{r \in [0,1]}$ is infinitely divisible.

Since $\zeta = \{\zeta_k\}_{k \in \mathbb{Z}^d}$ is stationary, the law of $Y_n(s) - Y_n(r)$ is the same as the distribution of $Y_n(s - r)$ whenever $0 < r < s$. Thus, we see that the distribution of $\bar{Y}(s) - \bar{Y}(r)$ is the same as that of $\bar{Y}(s - r)$ whenever $0 < r < s$ are rational. The continuity of \bar{Y} now ensures that the preceding holds in fact whenever $0 < r < s$. This proves that the process $\bar{Y} = \{\bar{Y}(r)\}_{r \in \mathbb{Q} \cap [0,1]}$ has stationary increments.

Because of Step 1, and since $E[Y_n(r)] = 0$ for all $r \in (0, 1]$ and $n \in \mathbb{N}$, it follows that $E[\bar{Y}(r)] = 0$ for all $r \in \mathbb{Q} \cap (0, 1]$. A second appeal to Step 1 and continuity (Step 3) shows that $E[\bar{Y}(r)] = 0$ for every $r \in [0, 1]$.

Finally, Lévy’s characteristic theorem of Brownian motion ensures that $\bar{Y} = \{\bar{Y}(r)\}_{r \in [0,1]}$ is a Brownian motion; see Bertoin [1]. Therefore, it remains to check that $\text{Var}[\bar{Y}(1)] = \sigma^2 \prod_{i=2}^d (b_i - a_i)$. Indeed, Step 2 ensures that $Y_{n'_k}(1) \xrightarrow{d} \bar{Y}(1)$ as $k \rightarrow \infty$. Therefore, the uniformly integrability condition (2.5) and Lemma 2.4 imply that $\text{Var}[\bar{Y}(1)] = \lim_{k \rightarrow \infty} E[Y_{n'_k}^2(1)] = \sigma^2 \prod_{i=2}^d (b_i - a_i)$, where the last equality is due to Lemma 2.5.

Step 5. We are ready to complete the proof of Proposition 2.8.

So far, we have proved that for every unbounded increasing sequence $\{n_k\}_{k=1}^\infty$ there exists a further unbounded increasing subsequence $\{n'_k\}_{k=1}^\infty$ such that the finite-dimensional distributions of $\{Y_{n'_k}(r)\}_{r \in \mathbb{Q} \cap (0,1]}$ converge to those of a Brownian motion \bar{Y} as $k \rightarrow \infty$, and the speed of that Brownian motion is always $\sigma^2 \prod_{i=2}^d (b_i - a_i)$. In particular, the law of \bar{Y} is the same as the law of Y regardless of the choice of the original subsequence $\{n_k\}_{k=1}^\infty$. This proves that the finite-dimensional distributions of $\{Y_n(r)\}_{r \in \mathbb{Q} \cap (0,1]}$ converge to those of $\{Y(r)\}_{r \in \mathbb{Q} \cap (0,1]}$.

In order to conclude Proposition 2.8, we need to show that for all integer $M \geq 1$ and for all $0 < r_1 < \dots < r_M \leq 1$, the characteristic function of $(Y_n(r_1), \dots, Y_n(r_M))$ converges to the characteristic function of $(Y(r_1), \dots, Y(r_M))$ as $n \rightarrow \infty$. For any $\varepsilon > 0$, we can choose $R_1, \dots, R_M \in \mathbb{Q}$ such that

$$(2.11) \quad r_k < R_k < r_k + \varepsilon \quad \text{and} \quad \|Y(r_k) - Y(R_k)\|_2 < \varepsilon \quad \text{for all } k = 1, \dots, M.$$

Define $\mathcal{E}_n(\boldsymbol{\alpha}, \boldsymbol{\beta}) := E[e^{i \sum_{k=1}^M \alpha_k Y_n(\beta_k)}]$ and $\mathcal{E}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := E[e^{i \sum_{k=1}^M \alpha_k Y(\beta_k)}]$ for all $\boldsymbol{\alpha} \in \mathbb{R}^M$ and $\boldsymbol{\beta} \in [0, 1]^M$. Our goal is to prove that $\lim_{n \rightarrow \infty} \mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{r}) = \mathcal{E}(\boldsymbol{\alpha}, \mathbf{r})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^M$. With this aim in mind, we can write

$$\begin{aligned} |\mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{r}) - \mathcal{E}(\boldsymbol{\alpha}, \mathbf{r})| &\leq |\mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{r}) - \mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{R})| \\ &\quad + |\mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{R}) - \mathcal{E}(\boldsymbol{\alpha}, \mathbf{R})| + |\mathcal{E}(\boldsymbol{\alpha}, \mathbf{R}) - \mathcal{E}(\boldsymbol{\alpha}, \mathbf{r})| \\ &\leq \sum_{k=1}^M |\alpha_k| \|Y_n(r_k) - Y_n(R_k)\|_2 + |\mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{R}) - \mathcal{E}(\boldsymbol{\alpha}, \mathbf{R})| + \sum_{k=1}^M |\alpha_k| \|Y(R_k) - Y(r_k)\|_2 \\ (2.12) \quad &\leq \sum_{k=1}^M |\alpha_k| \|Y_n(r_k) - Y_n(R_k)\|_2 + |\mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{R}) - \mathcal{E}(\boldsymbol{\alpha}, \mathbf{R})| + \varepsilon \sum_{k=1}^M |\alpha_k|, \end{aligned}$$

where the last inequality follows from (2.11). Since $\{Y_n(r)\}_{r \in \mathbb{Q} \cap (0,1]}$ $\xrightarrow{\text{fdd}}$ $\{Y(r)\}_{r \in \mathbb{Q} \cap (0,1]}$, the middle term in (2.12) vanishes as $n \rightarrow \infty$. Therefore, (2.7) implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} |\mathcal{E}_n(\boldsymbol{\alpha}, \mathbf{r}) - \mathcal{E}(\boldsymbol{\alpha}, \mathbf{r})| &\leq \sigma \prod_{i=2}^d (b_i - a_i)^{1/2} \sum_{k=1}^M |\alpha_k| (R_k - r_k)^{1/2} + \varepsilon \sum_{k=1}^M |\alpha_k| \\ &\leq (\sigma \sqrt{\varepsilon} \prod_{i=2}^d (b_i - a_i)^{1/2} + \varepsilon) \sum_{k=1}^M |\alpha_k|. \end{aligned}$$

This concludes the proof of Proposition 2.8 since $\varepsilon > 0$ is arbitrary. □

3. CLT FOR INFINITELY-MANY INTERACTING DIFFUSIONS

The central limit theorem in Theorem 2.2 can be applied to infinitely-many interacting diffusion processes, as indicated in the title of this paper.

3.1. Malliavin calculus. We introduce some elements of Malliavin calculus in order to establish the central limit theorem for infinitely-many interacting diffusion processes. Let $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{Z}^d)$, and recall that the Gaussian family $\{\eta(h)\}_{h \in \mathcal{H}}$ formed by the Wiener integrals $\mathcal{H} \ni h \mapsto \eta(h) = \int_0^\infty \sum_{x \in \mathbb{Z}^d} h(s, x) dB_s(x)$ defines an *isonormal Gaussian process*. In this framework, we can develop the Malliavin calculus as has been done, for instance, by Nualart [19].

We denote by D the derivative operator, and $\mathbb{D}^{1,2}$ the Gaussian Sobolev space generated by all $F \in L^2(\Omega)$ with derivative $DF \in L^2(\Omega; \mathcal{H})$. In accord with the Poincaré inequality,

$$(3.1) \quad |\text{Cov}(F, G)| \leq \int_0^\infty \sum_{z \in \mathbb{Z}^d} \|D_{s,z}F\|_2 \|D_{s,z}G\|_2 ds \quad \text{for all } F, G \in \mathbb{D}^{1,2}.$$

We will use this inequality extensively in the sequel.

3.2. Comments on the solution $u_t(x)$. The general theory of stochastic PDEs indexed by LCA groups (see Khoshnevisan and Kim [15]) implies that we may write the solution to (1.1) in the following *mild* form (variation of parameters): Almost surely for all $t > 0$ and $x \in \mathbb{Z}^d$,

$$(3.2) \quad u_t(x) = 1 + \int_0^t \sum_{y \in \mathbb{Z}^d} \mathbf{p}_{t-s}(y-x) \Phi(u_s(y)) dB_s(y),$$

where, for every $r \geq 0$ and $w \in \mathbb{Z}^d$, $\mathbf{p}_r(w) = P\{X_t = w\}$ for a continuous-time random walk $X = \{X_t\}_{t \geq 0}$ starting from the origin on \mathbb{Z}^d whose generator is L . It also follows from general theory [15] that for every real number $k \geq 2$ there exists a positive real number $L = L(k, \Phi)$ such that

$$(3.3) \quad \sup_{x \in \mathbb{Z}^d} E(|u_t(x)|^k) \leq Le^{Lt} \quad \text{and} \quad \sup_{n \geq 0} \sup_{x \in \mathbb{Z}^d} E(|u_n(t, x)|^k) \leq Le^{Lt} \quad \text{for all } t \geq 0,$$

where u_n denotes the n th stage of the Picard iteration approximation of u ; see (3.5).

Finally, let us record the following elementary fact.

Proposition 3.1. *The random field $\{u_t(x) : x \in \mathbb{Z}^d\}$ is stationary for every $t \geq 0$.*

We omit the proof as it follows along the same lines as in the proof of [6, Lemma 7.1], using the fact that the law of space-time white noise on $\mathbb{R}_+ \times \mathbb{Z}^d$ is translation invariant.

3.3. The Malliavin derivative of $u_t(x)$. The following is the main result of this section.

Proposition 3.2. *$u_t(x) \in \cap_{k \geq 2} \mathbb{D}^{1,k}$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{Z}^d$. Furthermore, for all real numbers $T > 0$ and $k \geq 2$ there exists a number $C = C_{k,T,\Phi} > 0$ such that*

$$(3.4) \quad \|D_{s,y}u_t(x)\|_k \leq C \mathbf{p}_{t-s}(x-y) \quad \text{for all } 0 < s < t < T \text{ and } x, y \in \mathbb{Z}^d.$$

The proof of Proposition 3.2 rests on the following sub-semigroup property of the squares of the transition functions of the underlying random walk.

Lemma 3.3. $\sum_{y \in \mathbb{Z}^d} [\mathbf{p}_t(x - y)\mathbf{p}_s(y - z)]^2 \leq [\mathbf{p}_{t+s}(x - z)]^2$ for all $t, s \geq 0$ and $x, z \in \mathbb{Z}^d$.

Proof. We may change variables in order to reduce the problem to the case that $z = 0$. Now suppose $z = 0$, and let X and X' denote two independent copies of the random walk whose generator is L . We may observe that

$$\begin{aligned} & \sum_{y \in \mathbb{Z}^d} [\mathbf{p}_t(x - y)\mathbf{p}_s(y)]^2 \\ &= \sum_{y \in \mathbb{Z}^d} \mathbb{P} \{X_{t+s} - X_s = x - y, X'_{t+s} - X'_s = x - y, X_s = y, X'_s = y\} \\ &= \mathbb{P} \{X_{t+s} = x, X'_{t+s} = x, X_s = X'_s\}. \end{aligned}$$

Drop the event $\{X_s = X'_s\}$ from the above to increase the latter probability to $[\mathbf{p}_{t+s}(x)]^2$; this yields the lemma. \square

Proof of Proposition 3.2. We will provide the details for the case that $k = 2$ and merely point out the key part that needs a small revision to extend the proof to $k > 2$.

The proof is carried out in a few relatively direct steps, similar to the proof of [8, Lemma 4.2]; see also [6, Theorem 6.4].

Step 1. Let u_n denote the n th stage of the Picard iteration approximation of u . That is, $u_0(t, x) := 1$ and

$$u_{n+1}(t, x) := 1 + \int_0^t \sum_{y \in \mathbb{Z}^d} \mathbf{p}_{t-s}(x - y)\Phi(u_n(s, y)) \, dB_s(y) \quad [n \in \mathbb{Z}_+, t > 0, x \in \mathbb{Z}^d]. \tag{3.5}$$

We claim that for all $n \in \mathbb{N}, t > 0, s \in (0, t)$, and $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} & \mathbb{E} (|D_{s,y}u_{n+1}(t, x)|^2) \\ & \leq [\mathbf{p}_{t-s}(x - y)]^2 \left\{ K e^{Ls} \sum_{\nu=0}^{n-1} 2^{\nu+1} \text{Lip}_{\Phi}^{2\nu} \frac{(t-s)^\nu}{\nu!} + 2^{n+1} \text{Lip}_{\Phi}^{2n} \frac{(t-s)^n}{n!} |\Phi(1)|^2 \right\}, \end{aligned}$$

where K is a positive real that depends only on Φ . Furthermore,

$$\mathbb{E} (|D_{s,y}u_1(t, x)|^2) = [\mathbf{p}_{t-s}(x - y)]^2 |\Phi(1)|^2, \tag{3.6}$$

for every $t > 0, s \in (0, t), x, y \in \mathbb{Z}^d$.

Proof of Step 1. We apply the properties of the divergence operator (see [19, Prop. 1.3.8]) in order to find from (3.5) that: (1) $D_{s,y}u_1(t, x) = \mathbf{p}_{t-s}(x - y)\Phi(1)$, which proves (3.6); and (2)

$$\begin{aligned} (3.7) \quad D_{s,y}u_{n+1}(t, x) &= \mathbf{p}_{t-s}(x - y)\Phi(u_n(s, y)) \\ & \quad + \int_s^t \sum_{z \in \mathbb{Z}^d} \mathbf{p}_{t-r}(x - z)D_{s,y}\Phi(u_n(r, z)) \, dB_r(z), \end{aligned}$$

for every $n \in \mathbb{N}$, $(t, x) \in (0, \infty) \times \mathbb{Z}^d$, and $(s, y) \in (0, t) \times \mathbb{Z}^d$. Because $|\Phi(z)| \leq |\Phi(0)| + \text{Lip}_\Phi |z|$ for all $z \in \mathbb{R}$, the preceding yields the following bounds:

$$\begin{aligned}
 (3.8) \quad & \|D_{s,y}u_{n+1}(t, x)\|_2 \\
 & \leq \mathbf{p}_{t-s}(x - y) (|\Phi(0)| + \text{Lip}_\Phi \|u_n(s, y)\|_2) \\
 & \quad + \left(\int_s^t \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{t-r}(x - z)]^2 \|D_{s,y}\Phi(u_n(r, z))\|_2^2 dr \right)^{1/2} \\
 & \leq \mathbf{p}_{t-s}(x - y) (|\Phi(0)| + \text{Lip}_\Phi \|u_n(s, y)\|_2) \\
 & \quad + \text{Lip}_\Phi \left(\int_s^t \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{t-r}(x - z)]^2 \|D_{s,y}u_n(r, z)\|_2^2 dr \right)^{1/2},
 \end{aligned}$$

thanks to the chain rule of Malliavin calculus for Lipschitz-continuous functions (see Nualart [19, Proposition 1.2.4]). In order to adapt the preceding the asserted bound for $\|D_{s,y}u_t(x)\|_k$ when $k > 2$, we apply the Burkholder-Davis-Gundy inequality instead, and obtain

$$\begin{aligned}
 \|D_{s,y}u_t(x)\|_k & \leq \mathbf{p}_{t-s}(x - y) (|\Phi(0)| + \text{Lip}_\Phi \|u_n(s, y)\|_2) \\
 & \quad + A \left(\int_s^t \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{t-r}(x - z)]^2 \|D_{s,y}u_n(r, z)\|_k^2 dr \right)^{1/2},
 \end{aligned}$$

where $A = A(k, \text{Lip}_\Phi)$. We continue with the case $k = 2$ from now on, but point out that we obtain the general form of the proposition by keeping track of the effect of using the above modification.

Recall (3.3) and let $K := (|\Phi(0)| + \sqrt{L} \text{Lip}_\Phi)^2$ in order to find that

$$\begin{aligned}
 (3.9) \quad & \|D_{s,y}u_{n+1}(t, x)\|_2^2 \leq 2K e^{Ls} [\mathbf{p}_{t-s}(x - y)]^2 \\
 & \quad + 2 \text{Lip}_\Phi^2 \int_s^t \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{t-q}(x - z)]^2 \|D_{s,y}u_n(q, z)\|_2^2 dq,
 \end{aligned}$$

where we have used the elementary inequality $(a + b)^2 \leq 2a^2 + 2b^2$, valid for every $a, b \in \mathbb{R}$. In particular, we may freeze the variables s and y in order to see that the following functions g_1, g_2, \dots , defined via

$$(3.10) \quad g_n(r, x) := \|D_{s,y}u_n(s + r, x + y)\|_2^2 \quad [n \in \mathbb{N}, r > 0, x \in \mathbb{Z}^d],$$

satisfy

$$(3.11) \quad g_1(r, x) \leq [\Phi(1)\mathbf{p}_r(x)]^2,$$

and

$$g_{n+1}(r, x) \leq 2K e^{Ls} [\mathbf{p}_r(x)]^2 + 2 \text{Lip}_\Phi^2 \int_0^r dq \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{r-q}(x - z)]^2 g_n(q, z),$$

for every $n \in \mathbb{N}$, $r > 0$, and $x \in \mathbb{Z}^d$. We may iterate this recursive inequality once in order to see that if $n \geq 2$ is an integer, $r > 0$, and $x \in \mathbb{Z}^d$, then

$$\begin{aligned} g_{n+1}(r, x) &\leq 2Ke^{Ls}[\mathbf{p}_r(x)]^2 + 4Ke^{Ls} \text{Lip}_{\Phi}^2 \int_0^r dq \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{r-q}(x-z)]^2 [\mathbf{p}_q(z)]^2 \\ &\quad + 4 \text{Lip}_{\Phi}^4 \int_0^r dq_1 \sum_{z_1 \in \mathbb{Z}^d} [\mathbf{p}_{r-q_1}(x-z_1)]^2 \int_0^{q_1} dq_2 \sum_{z_2 \in \mathbb{Z}^d} [\mathbf{p}_{q_1-q_2}(z_1-z_2)]^2 g_{n-1}(q_2, z_2) \\ &\leq 2Ke^{Ls}[\mathbf{p}_r(x)]^2 + 4Ke^{Ls} \text{Lip}_{\Phi}^2 r [\mathbf{p}_r(x)]^2 \\ &\quad + 4 \text{Lip}_{\Phi}^4 \int_0^r dq_1 \int_0^{q_1} dq_2 \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{r-q_2}(x-z)]^2 g_{n-1}(q_2, z), \end{aligned}$$

owing to sub-semigroup property of \mathbf{p}^2 [Lemma 3.3]. We may repeat once again to see that if $n \geq 3$ is an integer, $r > 0$, and $x \in \mathbb{Z}^d$, then

$$\begin{aligned} g_{n+1}(r, x) &\leq 2Ke^{Ls}[\mathbf{p}_r(x)]^2 + 4Ke^{Ls} \text{Lip}_{\Phi}^2 r [\mathbf{p}_r(x)]^2 \\ &\quad + 8Ke^{Ls} \text{Lip}_{\Phi}^4 \int_0^r dq_1 \int_0^{q_1} dq_2 \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{r-q_2}(x-z)]^2 [\mathbf{p}_{q_2}(z)]^2 \\ &\quad + 16 \text{Lip}_{\Phi}^6 \int_0^r dq_1 \int_0^{q_1} dq_2 \sum_{z \in \mathbb{Z}^d} [\mathbf{p}_{r-q_2}(x-z)]^2 \int_0^{q_2} dq_3 \sum_{z' \in \mathbb{Z}^d} [\mathbf{p}_{q_2-q_3}(z-z')]^2 g_{n-2}(q_3, z'), \end{aligned}$$

and so on. Continue this iteration process and deduce from (3.10) the asserted inequality for $E(|D_{s,y}u_{n+1}(t, x)|^2)$ after a change of variables [$s + t \leftrightarrow s + r$ and $x + y \leftrightarrow x$], using the simple fact that $\sum_{w \in \mathbb{Z}^d} [\mathbf{p}_{\tau}(w)]^2 \leq 1$ for all $\tau \geq 0$ in order to obtain the last term in the curly brackets. This and (3.11) together prove the validity of Step 1. □

Step 2. $\sup_{n \in \mathbb{Z}_+} E(\|Du_n(t, x)\|_{\mathcal{H}}^2) < \infty$ for all $t \geq 0$ and $x \in \mathbb{Z}^d$, where $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{Z}^d)$ was defined in §3.1.

Proof of Step 2. In accord with [19, Corollary 1.2.1], $D_{s,y}u_n(t, x) = 0$ when $s \geq t$. Therefore, Step 1 implies that $E(\|Du_n(t, x)\|_{\mathcal{H}}^2) \leq \int_0^t c_n(s, t) \|\mathbf{p}_s\|_{\ell^2(\mathbb{Z}^d)}^2 ds$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{Z}^d$ and $n \in \mathbb{N}$, where $c_1 \equiv |\Phi(1)|^2$ and

$$(3.12) \quad c_{n+1}(s, t) = Ke^{Ls} \sum_{\nu=0}^{n-1} 2^{\nu+1} \text{Lip}_{\Phi}^{2\nu} \frac{(t-s)^\nu}{\nu!} + 2^{n+1} \text{Lip}_{\Phi}^{2n} \frac{(t-s)^n}{n!} |\Phi(1)|^2.$$

Step 2 is a consequence of the above and the elementary fact that $\sup_{n \in \mathbb{N}} c_n(s, t) < \infty$. □

Finally, we complete the proof of Proposition 3.2 in a third, and final, step.

Step 3. $u_t(x) \in \mathbb{D}^{1,k}$ for every $k \geq 2$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{Z}^d$, and (3.4) holds for the parameter dependencies of Proposition 3.2.

Proof of Step 3. Again we consider only the case $k = 2$; the general case is proved similarly. Since $u_0 \equiv 1$, Step 3 has content only when $t > 0$. With this comment in mind, let us choose and fix $t > 0$ and $x \in \mathbb{Z}^d$. General theory [15] ensures that $\lim_{n \rightarrow \infty} u_n(t, x) = u_t(x)$ in $L^2(\Omega)$ for every $t \geq 0$ and $x \in \mathbb{Z}^d$. Therefore, the closeability properties of the Malliavin derivative (see Nualart [19, Lemma 1.2.3])

and Step 2 together imply that $Du_n(t, x) \rightarrow Du(t, x)$, as $n \rightarrow \infty$, in the weak topology of $L^2(\Omega; L^2(\mathbb{R}_+ \times \mathbb{Z}^d))$; and moreover, that $u_t(x) \in \mathbb{D}^{1,2}$. Now, we apply Cantor’s diagonalization in order to see that there exists an unbounded sequence $\{n(\ell)\}_{\ell=1}^\infty$ of positive integers such that for every $y \in \mathbb{Z}^d$, $D_{\bullet,y}u_{n(\ell)}(t, x) \rightarrow D_{\bullet,y}u(t, x)$, as $\ell \rightarrow \infty$, in the weak topology of $L^2(\Omega; L^2(\mathbb{R}_+))$. We next use a bounded and smooth approximation $\{\psi_\varepsilon\}_{\varepsilon>0}$ to the identity in \mathbb{R}_+ , and apply Fatou’s lemma and the self-duality of L^2 spaces in order to find that

$$(3.13) \quad \begin{aligned} \|D_{s,y}u_t(x)\|_2 &\leq \liminf_{\varepsilon \downarrow 0} \left\| \int_0^s D_{s',y}u_t(x)\psi_\varepsilon(s-s') \, ds' \right\|_2 \\ &= \liminf_{\varepsilon \downarrow 0} \sup_{\|G\|_2 \leq 1} \left| \int_0^s \mathbb{E}[GD_{s',y}u_t(x)] \psi_\varepsilon(s-s') \, ds' \right|, \end{aligned}$$

for all $y \in \mathbb{Z}^d$ and almost every $s \in (0, t)$. Choose and fix a random variable $G \in L^2(\Omega)$ such that $\mathbb{E}(|G|^2) \leq 1$. For all $y \in \mathbb{Z}^d$, $D_{\bullet,y}u_{n(\ell)}(t, x) \rightarrow D_{\bullet,y}u_t(x)$, as $\ell \rightarrow \infty$, in the weak topology of $L^2(\Omega; L^2(\mathbb{R}_+))$. Thus, we find that for all $y \in \mathbb{Z}^d$ and almost all $s \in (0, t)$,

$$\begin{aligned} \left| \int_0^s \mathbb{E}[GD_{s',y}u_t(x)] \psi_\varepsilon(s-s') \, ds' \right| &= \lim_{\ell \rightarrow \infty} \left| \int_0^s \mathbb{E}[GD_{s',y}u_{n(\ell)}(t, x)] \psi_\varepsilon(s-s') \, ds' \right| \\ &\leq \limsup_{\ell \rightarrow \infty} \int_0^s \|D_{s',y}u_{n(\ell)}(t, x)\|_2 \psi_\varepsilon(s-s') \, ds' \\ &\leq \lim_{\ell \rightarrow \infty} \sqrt{c_n(\ell)(s, t)} \int_0^s \mathbf{p}_{t-s'}(x-y)\psi_\varepsilon(s-s') \, ds', \end{aligned}$$

owing to Step 1, where c_n was defined in (3.12). Letting $\varepsilon \rightarrow 0$ to deduce the result from (3.13), as well as the boundedness and the continuity of $s \mapsto \mathbf{p}_{t-s}(x-y)$ for every $t > 0$ and $x, y \in \mathbb{Z}^d$. □

3.4. Proof of Theorem 1.4. Let us make a small observation before we begin the proof of Theorem 1.4: Thanks to the Poincaré inequality (3.1) and the chain rule of Malliavin calculus [19, Proposition 1.2.4],

$$|\text{Cov}[g(u_t(0)), g(u_t(x))]| \leq \text{Lip}_g^2 \int_0^t \sum_{z \in \mathbb{Z}^d} \|D_{s,z}u_t(0)\|_2 \|D_{s,z}u_t(x)\|_2 \, ds.$$

Therefore, Proposition 3.2 yields

$$(3.14) \quad \sum_{x \in \mathbb{Z}^d} |\text{Cov}[g(u_t(0)), g(u_t(x))]| \leq A \int_0^t \sum_{x,z \in \mathbb{Z}^d} \mathbf{p}_{t-s}(-z)\mathbf{p}_{t-s}(x-z) \, ds < \infty,$$

for a real number $A = A(K, \text{Lip}_g, t, \text{Lip}_\Phi, L)$. We now proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4. Choose and fix some $t > 0$ throughout, and define

$$\zeta_k := g(u_t(k)) - \mathbb{E}[g(u_t(0))] \quad \text{for every } k \in \mathbb{Z}^d.$$

By Proposition 3.1 and (3.3), $\{\zeta_k\}_{k \in \mathbb{Z}^d}$ is stationary, $\mathbb{E}[\zeta_0] = 0$, and $\text{Var}(\zeta_0) < \infty$. Furthermore, (3.14) assures us that

$$\sigma_{g,t}^2 = \sum_{k \in \mathbb{Z}^d} \text{Cov}(\zeta_0, \zeta_k) = \sum_{k \in \mathbb{Z}^d} \text{Cov}(g(u_t(0)), g(u_t(k)))$$

is an absolutely convergent sum. We verify uniform integrability (2.5) next.

For every $\varphi \in \mathcal{C}$ define, following (2.3),

$$(3.15) \quad S_n(\varphi) = n^{-d/2} \sum_{k \in \mathbb{Z}^d} \{g(u_t(k)) - \mathbb{E}[g(u_t(0))]\} \varphi(k/n)$$

$$(3.16) \quad = n^{-d/2} \sum_{k \in \mathbb{Z}^d} \left(\sum_{y \in \mathbb{Z}^d} \int_0^t \mathbb{E} [D_{s,y}[g(u_t(k))] \mid \mathcal{F}_s] dB_s(y) \right) \varphi(k/n),$$

where $\mathcal{F}_s :=$ the σ -algebra generated by $\{B_r(y); y \in \mathbb{Z}^d, r \in [0, s]\}$, and we have used the Clark–Ocone formula in the last line. We apply Minkowski’s inequality and the Burkholder-Davis-Gundy inequality in order to see from the above that for all $p \geq 2$ there exists $c_p > 0$ such that for every $n \in \mathbb{N}$,

$$\begin{aligned} & \|S_n(\varphi)\|_p^2 \\ & \leq \frac{c_p}{n^d} \sum_{y \in \mathbb{Z}^d} \sum_{k, k' \in \mathbb{Z}^d} |\varphi(k/n)\varphi(k'/n)| \int_0^t \|\mathbb{E} [D_{s,y}[g(u_t(k))] \mid \mathcal{F}_s] \mathbb{E} [D_{s,y}[g(u_t(k'))] \mid \mathcal{F}_s]\|_{p/2} ds \\ & \leq \frac{c_p}{n^d} \sum_{y \in \mathbb{Z}^d} \sum_{k, k' \in \mathbb{Z}^d} |\varphi(k/n)\varphi(k'/n)| \int_0^t \|D_{s,y}[g(u_t(k))]\|_p \|D_{s,y}[g(u_t(k'))]\|_p ds, \end{aligned}$$

the last line valid thanks to the Cauchy-Schwarz inequality and Jensen’s inequality for conditional expectations. Thus, the chain rule of Malliavin derivative (see [19, Proposition 1.2.4]) yields

$$\|S_n(\varphi)\|_p^2 \leq \frac{c_p \text{Lip}_g^2}{n^d} \sum_{y \in \mathbb{Z}^d} \sum_{k, k' \in \mathbb{Z}^d} |\varphi(k/n)\varphi(k'/n)| \int_0^t \|D_{s,y}u_t(k)\|_p \|D_{s,y}u_t(k')\|_p ds.$$

Proposition 3.2 can now be used to deduce that, uniformly for all $n \in \mathbb{N}$,

$$\begin{aligned} \|S_n(\varphi)\|_p^2 & \leq \frac{\text{const}}{n^d} \cdot \sum_{y \in \mathbb{Z}^d} \sum_{k, k' \in \mathbb{Z}^d} |\varphi(k/n)\varphi(k'/n)| \int_0^t \mathbf{p}_{t-s}(k-y)\mathbf{p}_{t-s}(k'-y) ds \\ & = \frac{\text{const}}{n^d} \cdot \sum_{k, k' \in \mathbb{Z}^d} |\varphi(k/n)\varphi(k'/n)| \int_0^t \mathbf{p}_{2(t-s)}(k-k') ds [\text{semigroup property}] \\ & \leq \frac{\text{const}}{n^d} \cdot \sum_{k \in \mathbb{Z}^d} |\varphi(k/n)|^2 \int_0^t \sum_{j \in \mathbb{Z}^d} \mathbf{p}_{2(t-s)}(j) ds [\text{Cauchy-Schwarz inequality}]. \end{aligned}$$

Since $\mathbf{p}_{2(t-s)}(j)$ sums up to 1, the integral is equal to t , whence $\sup_{n \in \mathbb{N}} \|S_n(\varphi)\|_p < \infty$ for every $\varphi \in \mathcal{C}$. Because $p > 2$, the uniform integrability condition (2.5) follows.

In light of (3.14) and Theorem 2.2 it remains to prove that $\{S_n(\varphi); \varphi \in \mathcal{C}\}$ satisfies the asymptotic independence condition (AI). First, note that for all $z \in \mathbb{Z}^d$ and almost every $s \in (0, t)$,

$$\begin{aligned} & D_{s,z}S_n(\varphi) \\ & = n^{-d/2} \sum_{k \in \mathbb{Z}^d} D_{s,z}g(u_t(k))\varphi(k/n) = n^{-d/2} \sum_{k \in \mathbb{Z}^d} g'(u_t(k))D_{s,z}u_t(k)\varphi(k/n) \quad \text{a.s.} \end{aligned}$$

Therefore, we once again invoke the chain rule of Malliavin derivative (see Nualart [19, Proposition 1.2.4]) in order to see that for all $p \geq 2$, $n \in \mathbb{N}$, and $z \in \mathbb{Z}^d$, and for almost every $s \in (0, t)$,

$$(3.17) \quad \|D_{s,z}S_n(\varphi)\|_p \leq \frac{A_{p,t}}{n^{d/2}} \sum_{k \in \mathbb{Z}^d} \mathbf{p}_{t-s}(k-z)|\varphi(k/n)|,$$

for a number $A_{p,t} > 0$, where the last inequality follows from Proposition 3.2.

Choose and fix $a, b \in \mathbb{R}$, $\delta > 0$, and $\varphi_1, \varphi_2 \in \mathcal{U}$ such that

$$(3.18) \quad \text{sep}(\text{supp}[\varphi_1], \text{supp}[\varphi_2]) \geq \delta.$$

Our remaining goal is to prove that

$$\mathcal{C}_n := \left| \mathbb{E} \left[e^{iaS_n(\varphi_1) + ibS_n(\varphi_2)} \right] - \mathbb{E} \left[e^{iaS_n(\varphi_1)} \right] \mathbb{E} \left[e^{ibS_n(\varphi_2)} \right] \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\mathcal{C}_n = |\text{Cov}(e^{iaS_n(\varphi_1)}, e^{-ibS_n(\varphi_2)})|$, (3.1) and (3.17) together imply that

$$\begin{aligned} \mathcal{C}_n &\leq |ab| \sum_{z \in \mathbb{Z}^d} \int_0^t \|D_{s,z} S_n(\varphi_1)\|_2 \|D_{s,z} S_n(\varphi_2)\|_2 \, ds \\ &\leq \frac{\text{const}}{n^d} \sum_{k,m,z \in \mathbb{Z}^d} |\varphi_1(k/n) \varphi_2(m/n)| \int_0^t \mathbf{p}_s(k-z) \mathbf{p}_s(m-z) \, ds, \end{aligned}$$

uniformly in $n \in \mathbb{N}$. Observe that $\sum_{z \in \mathbb{Z}^d} \mathbf{p}_s(k-z) \mathbf{p}_s(m-z) = \mathbb{P}\{X_s - X'_s = k - m\}$ where X and X' are i.i.d. copies of a random walk with generator L . Thus, we can re-index the sums to find that

$$(3.19) \quad \mathcal{C}_n \leq \text{const} \cdot \sum_{\ell \in \mathbb{Z}^d} \int_0^t \mathbb{P}\{X_s - X'_s = \ell\} \, ds \cdot \frac{1}{n^d} \sum_{m \in \mathbb{Z}^d} \left| \varphi_1 \left(\frac{m+\ell}{n} \right) \varphi_2 \left(\frac{m}{n} \right) \right|.$$

The Cauchy-Schwarz inequality and (2.8) [with $j = 0$] together imply that the final quantity $n^{-d} \sum_{m \in \mathbb{Z}^d} |\varphi_1((m+\ell)/n) \varphi_2(m/n)|$ in (3.19) is bounded uniformly over all $n \in \mathbb{N}$ and $m \in \mathbb{Z}^d$. Therefore, an appeal to the dominated convergence theorem assures us that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{\ell \in \mathbb{Z}^d} \int_0^t \mathbb{P}\{X_s - X'_s = \ell\} \, ds \cdot \frac{1}{n^d} \sum_{m \in \mathbb{Z}^d} \left| \varphi_1 \left(\frac{m+\ell}{n} \right) \varphi_2 \left(\frac{m}{n} \right) \right| \\ &= t \int_{\mathbb{R}^d} |\varphi_1(y) \varphi_2(y)| \, dy = 0, \end{aligned}$$

owing to (2.8) and (3.18). This and (3.19) together imply that $\lim_{n \rightarrow \infty} \mathcal{C}_n = 0$, and complete the proof of the theorem. \square

Remark 3.4. An anonymous referee informs us a functional version of our Theorem 1.4. Consider $\varphi = \mathbf{1}_{Q(r)}$ (see the definition of $\mathbf{1}_{Q(r)}$ in §2.3). Then, as a process in $r \in [0, 1]$, the left-hand side of (1.2) converges in distribution to Brownian motion in the space $C[0, 1]$. This is guaranteed by [2, Theorem 19.2] since the moment estimate and asymptotic independence established in the proof of Theorem 1.4 verify the conditions of [2, Theorem 19.2]. Meanwhile, the same anonymous referee also informs us that when $d = 1$, our Theorem 2.2 can be deduced from Theorem 4.5 of Jakubowski [13].

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REFERENCES

- [1] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR1406564
- [2] Patrick Billingsley, *Convergence of probability measures*, John Wiley & Sons, Inc., New York-London-Sydney, 1968. MR0233396
- [3] Richard C. Bradley, *Introduction to strong mixing conditions. Vol. 2*, Kendrick Press, Heber City, UT, 2007. MR2325295
- [4] Richard C. Bradley, *Basic properties of strong mixing conditions. A survey and some open questions*, Probab. Surv. **2** (2005), 107–144, DOI 10.1214/154957805100000104. Update of, and a supplement to, the 1986 original. MR2178042
- [5] René A. Carmona and S. A. Molchanov, *Parabolic Anderson problem and intermittency*, Mem. Amer. Math. Soc. **108** (1994), no. 518, viii+125, DOI 10.1090/memo/0518. MR1185878
- [6] L. Chen, D. Khoshnevisan, D. Nualart, and F. Pu, *Spatial ergodicity for SPDEs via Poincaré-type inequalities*, Preprint, [arXiv:1907.11553](https://arxiv.org/abs/1907.11553), 2019.
- [7] L. Chen, D. Khoshnevisan, D. Nualart, and F. Pu, *Poincaré inequality, and central limit theorems for parabolic stochastic partial differential equations*, Preprint, [arXiv:1912.01482](https://arxiv.org/abs/1912.01482), 2019.
- [8] Davar Khoshnevisan, David Nualart, and Fei Pu, *Spatial Stationarity, Ergodicity, and CLT for Parabolic Anderson Model with Delta Initial Condition in Dimension $d \geq 1$* , SIAM J. Math. Anal. **53** (2021), no. 2, 2084–2133, DOI 10.1137/20M1350418. MR4242879
- [9] Jean-Dominique Deuschel, *Central limit theorem for an infinite lattice system of interacting diffusion processes*, Ann. Probab. **16** (1988), no. 2, 700–716. MR929072
- [10] J. D. Esary, F. Proschan, and D. W. Walkup, *Association of random variables, with applications*, Ann. Math. Statist. **38** (1967), 1466–1474, DOI 10.1214/aoms/1177698701. MR217826
- [11] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR0257325
- [12] I. A. Ibragimov, *Some limit theorems for stationary processes* (Russian, with English summary), Teor. Veroyatnost. i Primenen. **7** (1962), 361–392. MR0148125
- [13] A. Jakubowski, *Asymptotic independent representations for sums and order statistics of stationary sequences*, Available for download at <http://www-users.mat.uni.torun.pl/~adjakubo/hab.pdf>, 1991.
- [14] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991, DOI 10.1007/978-1-4612-0949-2. MR1121940
- [15] Davar Khoshnevisan and Kunwoo Kim, *Nonlinear noise excitation of intermittent stochastic PDEs and the topology of LCA groups*, Ann. Probab. **43** (2015), no. 4, 1944–1991, DOI 10.1214/14-AOP925. MR3353819
- [16] S. N. Lahiri, *A necessary and sufficient condition for asymptotic independence of discrete Fourier transforms under short- and long-range dependence*, Ann. Statist. **31** (2003), no. 2, 613–641, DOI 10.1214/aos/1051027883. Dedicated to the memory of Herbert E. Robbins. MR1983544
- [17] Florence Merlevède, Magda Peligrad, and Sergey Utev, *Recent advances in invariance principles for stationary sequences*, Probab. Surv. **3** (2006), 1–36, DOI 10.1214/154957806100000202. MR2206313
- [18] C. M. Newman and A. L. Wright, *An invariance principle for certain dependent sequences*, Ann. Probab. **9** (1981), no. 4, 671–675. MR624694
- [19] David Nualart, *The Malliavin calculus and related topics*, 2nd ed., Probability and its Applications (New York), Springer-Verlag, Berlin, 2006. MR2200233
- [20] M. Rosenblatt, *Central limit theorem for stationary processes*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Univ. California Press, Berkeley, Calif., 1972, pp. 551–561. MR0402869
- [21] Tokuzo Shiga and Akinobu Shimizu, *Infinite-dimensional stochastic differential equations and their applications*, J. Math. Kyoto Univ. **20** (1980), no. 3, 395–416, DOI 10.1215/kjm/1250522207. MR591802

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