Spatial ergodicity and central limit theorems for parabolic Anderson model with delta initial condition

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A R T I C L E I N F O

A B S T R A C T

Let $\{u(t,x)\}_{t>0,x\in\mathbb{R}}$ denote the solution to the parabolic Anderson model with initial condition δ_0 and driven by space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$, and let $\boldsymbol{p}_t(x) := (2\pi t)^{-1/2}\exp\{-x^2/(2t)\}$ denote the standard Gaussian heat kernel on the line. We use a non-trivial adaptation of the methods in our companion papers [6,7] in order to prove that the random field $x \mapsto u(t,x)/\boldsymbol{p}_t(x)$ is ergodic for every $t > 0$. And we establish an associated quantitative central limit theorem following the approach based on the Malliavin-Stein method introduced in Huang, Nualart, and Viitasaari [11].

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1. Introduction

Consider the *parabolic Anderson model*,

$$\partial_t u(t\,,x) = \tfrac{1}{2}\partial_x^2 u(t\,,x) + u(t\,,x)\eta(t\,,x), \qquad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

with delta initial condition $u(0) = \delta_0$, where η denotes space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. Following Walsh [18], we interpret the stochastic PDE (1.1) in the following mild form:

$$u(t\,,x) = \boldsymbol{p}_t(x) + \int\limits_{(0,t)\times\mathbb{R}} \boldsymbol{p}_{t-s}(x-y)u(s\,,y)\,\eta(\mathrm{d}s\,\mathrm{d}y), \tag{1.2}$$

where

$$\boldsymbol{p}_t(x) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-x^2/(2t)} \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R}.$$

Consider the following renormalization of the solution to (1.1):

$$U(t\,,x) := \frac{u(t\,,x)}{\boldsymbol{p}_t(x)} \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R}. \tag{1.3}$$

It is not too hard to prove that $\lim_{t\downarrow 0} U(t\,,x) = 1$ in $L^k(\Omega)$ for all $x \in \mathbb{R}$ and $k \geq 2$; see Lemma A.4 below. Therefore, we also define

$$U(0\,,x) := 1 \qquad \text{for all } x \in \mathbb{R},$$

throughout.

Amir, Corwin, and Quastel [1, Proposition 1.4] have shown that the process $U(t) := \{U(t\,,x)\}_{x\in\mathbb{R}}$ is stationary for every $t > 0$. The formulation (1.2) of the stochastic PDE (1.1) can be recast equivalently in terms of U as follows:

$$U(t\,,x) = 1 + \int\limits_{(0,t)\times\mathbb{R}} \frac{\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)}{\boldsymbol{p}_t(x)}\,U(s\,,y)\,\eta(\mathrm{d}s\,\mathrm{d}y).$$

Because[1]

$$\frac{\boldsymbol{p}_{t-s}(a)\boldsymbol{p}_s(b)}{\boldsymbol{p}_t(a+b)} = \boldsymbol{p}_{s(t-s)/t}\left(b - \frac{s}{t}(a+b)\right) \quad \text{for all } 0 < s < t \text{ and } a, b \in \mathbb{R}, \tag{1.4}$$

equation (1.2) can be recast as the following random evolution equation for U:

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[1] In fact, both sides of (1.4) represent the probability density of X_s at b where X denotes a Brownian bridge that emenates from zero and is conditioned to reach $a + b$ at time t.

$$U(t,x) = 1 + \int_{(0,t)\times\mathbb{R}} U(s,y)\boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right) \eta(\mathrm{d}s\,\mathrm{d}y). \tag{1.5}$$

The purpose of this paper is to study asymptotic properties of the stationary process $U(t)$, equivalently $u(t)/\boldsymbol{p}_t$. The main results are stated as the following three theorems.

Theorem 1.1. *The process $U(t)$ is weakly mixing, hence also ergodic, for every $t > 0$.*

It follows immediately from (1.5) that $\mathrm{E}[U(t,x)] = 1$. Therefore, Theorem 1.1 and the ergodic theorem together imply that for all $t \geq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N U(t,x)\,\mathrm{d}x = 1 \qquad \text{a.s. and in } L^1(\Omega). \tag{1.6}$$

In fact, Lemma 2.4 below implies that (1.6) holds in $L^k(\Omega)$ for every $k \geq 1$.

The next two theorems describe the rate of convergence in the ergodic theorem (1.6). In order to state those theorems, let us introduce

$$\mathcal{S}_{N,t} := \frac{1}{N} \int_0^N [U(t,x) - 1]\,\mathrm{d}x \qquad \text{for all } N > 0 \text{ and } t \geq 0. \tag{1.7}$$

Then we have the following quantitative central limit theorem.

Theorem 1.2. *For every $t > 0$ there exists a real number $c = c(t) > 0$ and $N_0 = N_0(t) > \mathrm{e}$ such that for all $N \geq N_0$,*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{N,t})}}, \mathrm{N}(0,1)\right) \leq c\sqrt{\frac{\log N}{N}}, \tag{1.8}$$

where d_{TV} denotes the total variation distance, and $\mathrm{N}(\mu, \sigma^2)$ denotes the normal law with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

Theorem 1.2 tacitly implies also that $\mathrm{Var}(\mathcal{S}_{N,t}) > 0$ for all N large. As part of the proof of Theorem 1.2, we in fact prove in Proposition 4.1 below that

$$\mathrm{Var}(\mathcal{S}_{N,t}) \sim \frac{2t\log N}{N} \qquad \text{as } N \to \infty. \tag{1.9}$$

Therefore, Theorem 1.2 implies that, for all $t > 0$,

$$\sqrt{\frac{N}{\log N}}\, \mathcal{S}_{N,t} \xrightarrow{\mathrm{d}} \mathrm{N}(0, 2t) \qquad \text{as } N \to \infty. \tag{1.10}$$

where "$\xrightarrow{\mathrm{d}}$" denotes convergence in distribution. Since the limiting variance $2t$ is a linear function of t, the above suggests the existence of a functional CLT with a Brownian limit. This is confirmed by the next result of this section.

Theorem 1.3. *Choose and fix a real number $T > 0$. Then, as $N \to \infty$,*

$$\sqrt{\frac{N}{\log N}}\, \mathcal{S}_{N,\bullet} \xrightarrow{C[0,T]} \sqrt{2}B, \tag{1.11}$$

where B denotes a standard one-dimensional Brownian motion, and "$\xrightarrow{C[0,T]}$" denotes weak convergence in the Banach space $C[0,T]$ of all continuous, real-valued functions on $[0,T]$, endowed with the uniform topology.

Theorem 1.2 indicates the convergence in total variation distance of the one-dimensional laws. It seems conceivable that one can obtain the convergence in total variation distance of the finite-dimensional distributions. Moreover, one might wonder if the weak convergence to Brownian motion in Theorem 1.3 can be replaced by convergence in total variation. We leave this question as an open problem for the interested readers.

Open problem: Does the process $\{\sqrt{N/\log N}\,\mathcal{S}_{N,t}\}_{t\in[0,T]}$ converge to $\{\sqrt{2}\,B_t\}_{t\in[0,T]}$ in total variation, as $N \to \infty$, for any $T > 0$?

Now let us compare our work with the existing ones to show the difficulties and hence the contributions of the current paper. First, regarding Theorem 1.1, in Chen et al. [6], we used Poincaré-type inequalities and Malliavin calculus in order to establish the spatial ergodicity for a large class of parabolic stochastic PDEs that include the parabolic Anderson model with *flat initial condition* $u(0) \equiv 1$. Broadly speaking, the method in [6] is also employed here in order to prove Theorem 1.1. However, because the initial profile of (1.1) is the singular measure δ_0, novel technical issues arise. Chief among them is the fact that the Malliavin derivative of the solution to (1.1) behaves radically differently from the case with constant initial data. This can be seen by comparing our Lemma 2.1 with Theorem 6.4 of [6]. As a result, the Poincaré-type inequality [see (2.1)] yields a $(\log N/N)$-decay rate, which is bigger than the $1/N$-rate obtained in the flat case [6], and the asymptotic variance (1.9) is likewise different from the case of flat initial data. The Poincaré-type inequality (2.1) is based on the Clark-Ocone formula, and the latter plays an import role not only in this context, but in fact throughout the paper.

Secondly, for Theorem 1.2, such total variation estimates for spatial averages of solutions to parabolic stochastic PDEs were introduced by Huang, Nualart, and Viitasaari [11] for the one-dimensional stochastic heat equation driven by a space-time white noise, and later extended in Huang, Nualart, Viitasaari, and Zheng [12] to the multidimensional stochastic heat equation driven by a noise whose spatially homogeneous covariance is a suitable Riesz kernel. The main ingredient in deriving such estimates is the Malliavin-Stein approach (see Nourdin and Peccati [13,14]) which provides a convergence rate, in

total variation distance, using a combination of Malliavin calculus and Stein's method for normal approximations. But unlike the case considered in Huang et al. [11] (see also Chen et al. [7]), where the initial condition was $u(0) \equiv 1$, in our setting the solution to (1.1) with delta initial condition is scaled by the heat kernel, and this produces asymptotic variance for spatial averages of order $\log(N)/N$; see (1.9). As a consequence, we need to normalize the average in (1.10) by the unconventional rate $\sqrt{N/\log N}$. Moreover, the $\sqrt{\log N/N}$-rate of convergence of the total variation distance in Theorem 1.2 is a natural one, which is of the same order as $\sqrt{\mathrm{Var}(\mathcal{S}_{N,t})}$ as $N \to \infty$ (see (1.9)). Such a relation also holds in the context of Malliavin-Stein approach to central limit theorems for other types of SPDEs; see [9,11,12,17]. Furthermore, the presence of these unexpected logarithmic factors is new in the literature, which shows the slow decorrelation of the random field $U(t)$, and can be attributed to the singularity of the delta initial condition.

Lastly, the functional central limit theorem stated in Theorem 1.3 is the counterpart in our framework of Theorem 1.2 in [11]. The convergence in law of finite-dimensional distributions is obtained using the Malliavin-Stein approach as in the proof of Theorem 1.2 in [11], but the proof of tightness, however, is more involved due to the singularity of the initial condition and requires computations which are different from those in [11] (see the proof of Proposition 6.1).

In the following, after introducing some preliminaries in §2, we first prove Theorem 1.1 in §3. Then we establish asymptotic results for the covariance of $\mathcal{S}_{N,t}$ in §4, which will be used in the proof of Theorems 1.2 and 1.3 in §5 and §6, respectively. Finally, some technical lemmas are proved in Appendix.

Let us close the Introduction with a brief description of the notation of this paper. For every $Z \in L^k(\Omega)$, we write $\|Z\|_k$ instead of $(\mathrm{E}[|Z|^k])^{1/k}$. Let Lip denote the class of all Lipschitz-continuous, real-valued functions on $\mathbb{R}$, and define for all $g : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{Lip}(g) := \sup_{-\infty < a < b < \infty} \frac{|g(b) - g(a)|}{|b - a|}.$$

Thus, $g \in \mathrm{Lip}$ if and only if $\mathrm{Lip}(g) < \infty$. Recall that if $g \in \mathrm{Lip}$, then Rademacher's theorem (see Federer [10, Theorem 3.1.6]) ensures that g has a weak derivative whose essential supremum is $\mathrm{Lip}(g)$. Let g' denote a given measurable version of that derivative. Throughout, we define

$$\log_+(x) := \log(\mathrm{e} + x) \qquad \text{for every } x \geq 0.$$

We also use "$\widehat{}$" to denote the Fourier transform, normalized so that

$$\hat{f}(x) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{ixy} f(y) \, \mathrm{d}y \qquad \text{for all } x \in \mathbb{R} \text{ and } f \in L^1(\mathbb{R}).$$

2. Preliminaries

2.1. Clark-Ocone formula

Let $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R})$. The Gaussian family $\{W(h)\}_{h \in \mathcal{H}}$ formed by the Wiener integrals

$$W(h) = \int\limits_{\mathbb{R}_+ \times \mathbb{R}} h(s,x)\, \eta(\mathrm{d}s\, \mathrm{d}x)$$

defines an *isonormal Gaussian process* on the Hilbert space $\mathcal{H}$. In this framework we can develop the Malliavin calculus (see Nualart [15]). We denote by D the derivative operator. Let $\{\mathcal{F}_s\}_{s \geq 0}$ denote the filtration generated by the space-time white noise η.

We recall the following Clark-Ocone formula (see Chen et al. [6, Proposition 6.3]):

$$F = \mathrm{E}[F] + \int\limits_{\mathbb{R}_+ \times \mathbb{R}} \mathrm{E}\left[D_{s,y}F \mid \mathcal{F}_s\right] \eta(\mathrm{d}s\, \mathrm{d}z) \qquad \text{a.s.,}$$

valid for every random variable F in the Gaussian Sobolev space $\mathbb{D}^{1,2}$. Thanks to Jensen's inequality for conditional expectations, the above Clark-Ocone formula readily yields the following Poincaré-type inequality, which plays an important role throughout the paper:

$$\left|\operatorname{Cov}(F,G)\right| \leq \int\limits_0^\infty \mathrm{d}s \int\limits_{-\infty}^\infty \mathrm{d}z \; \left\|D_{s,z}F\right\|_2 \left\|D_{s,z}G\right\|_2 \qquad \text{for all } F, G \in \mathbb{D}^{1,2}. \tag{2.1}$$

2.2. Malliavin derivative of $u(t,x)$

According to Chen, Hu, and Nualart [4, Proposition 5.1] (see Chen and Huang [5, Proposition 3.2] for the higher-dimensional case),

$$u(t,x) \in \bigcap_{k \geq 2} \mathbb{D}^{1,k} \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R},$$

and the corresponding Malliavin derivative $Du(t,x)$ satisfies the following stochastic integral equation: For $s \in (0,t)$,

$$D_{s,y}u(t,x) = \boldsymbol{p}_{t-s}(x-y)u(s,y) + \int\limits_{(s,t) \times \mathbb{R}} \boldsymbol{p}_{t-r}(x-z)D_{s,y}u(r,z)\, \eta(\mathrm{d}r\, \mathrm{d}z) \qquad \text{a.s.}$$

We offer the following estimate on the Malliavin derivative of $u(t,x)$.

Lemma 2.1. *For every $T > 0$ and $k \geq 2$, there exists a real number $C_{T,k} > 0$ such that for $t \in (0,T)$ and $x \in \mathbb{R}$, and for almost every $(s,y) \in (0,t) \times \mathbb{R}$,*

$$\|D_{s,y}u(t,x)\|_k \leq C_{T,k}\, \boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y). \tag{2.2}$$

Proof. The proof is similar to the proof of Theorem 6.4 of Chen et al. [6]. Fix $t \in (0,T)$ and $x \in \mathbb{R}$. Let $u_0(t,x) = \boldsymbol{p}_t(x)$ for every $x \in \mathbb{R}$, and define iteratively, for every $n \in \mathbb{Z}_+$,

$$u_{n+1}(t,x) := \boldsymbol{p}_t(x) + \int\limits_{(0,t)\times\mathbb{R}} \boldsymbol{p}_{t-r}(x-z)u_n(r,z)\,\eta(\mathrm{d}r\,\mathrm{d}z). \tag{2.3}$$

Conus, Joseph, Khoshnevisan, and Shiu [8, Theorem 3.3] and Chen and Dalang [3, Theorem 2.4] found independently, and at the same time, that there exists a real number $c_{T,k} > 0$ such that for all $(s,y) \in (0,T] \times \mathbb{R}$,

$$\sup_{n\in\mathbb{Z}_+} \|u_n(s,y)\|_k \vee \|u(s,y)\|_k \leq c_{T,k}\,\boldsymbol{p}_s(y). \tag{2.4}$$

We apply the properties of the divergence operator [15, Prop. 1.3.8] in order to deduce from (2.3) that for almost every $(s,y) \in (0,t) \times \mathbb{R}$,

$$D_{s,y}u_{n+1}(t,x)$$
$$= \boldsymbol{p}_{t-s}(x-y)u_n(s,y) + \int\limits_{(s,t)\times\mathbb{R}} \boldsymbol{p}_{t-r}(x-z)D_{s,y}u_n(r,z)\,\eta(\mathrm{d}r\,\mathrm{d}z) \qquad \text{a.s.} \tag{2.5}$$

By (2.5), (2.4), and a suitable form of the Burkholder-Davis-Gundy inequality (BDG),

$$\|D_{s,y}u_{n+1}(t,x)\|_k^2$$
$$\leq 2c_{T,k}^2\,\boldsymbol{p}_{t-s}^2(x-y)\boldsymbol{p}_s^2(y) + 2c_k \int\limits_s^t \mathrm{d}r \int\limits_{-\infty}^{\infty} \mathrm{d}z\, \boldsymbol{p}_{t-r}^2(x-z)\|D_{s,y}u_n(r,z)\|_k^2, \tag{2.6}$$

where $c_k = 4k$; see [6, (5.6)]. Let $C_k := (2c_{T,k}^2) \vee (2c_k)$. We can iterate (2.6) to find that

$$\|D_{s,y}u_{n+1}(t,x)\|_k^2$$

$$\leq C_k\,\boldsymbol{p}_{t-s}^2(x-y)\boldsymbol{p}_s^2(y) + C_k^2\boldsymbol{p}_s^2(y) \int\limits_s^t \mathrm{d}r_1 \int\limits_{-\infty}^{\infty} \mathrm{d}z_1\, \boldsymbol{p}_{t-r_1}^2(x-z_1)\boldsymbol{p}_{r_1-s}^2(z_1-y)$$

$$+ \cdots + C_k^n\boldsymbol{p}_s^2(y) \int\limits_s^t \mathrm{d}r_1 \int\limits_{-\infty}^{\infty} \mathrm{d}z_1 \int\limits_s^{r_1} \mathrm{d}r_2 \int\limits_{-\infty}^{\infty} \mathrm{d}z_2 \cdots \int\limits_s^{r_{n-2}} \mathrm{d}r_{n-1} \int\limits_{-\infty}^{\infty} \mathrm{d}z_{n-1}\, \boldsymbol{p}_{t-r_1}^2(x-z_1)$$

$$\times \boldsymbol{p}_{r_1-r_2}^2(z_1-z_2) \times \cdots \times \boldsymbol{p}_{r_{n-1}-s}^2(z_{n-1}-y)$$

$$+ C_k^{n+1}\boldsymbol{p}_s^2(y) \int\limits_s^t \mathrm{d}r_1 \int\limits_{-\infty}^{\infty} \mathrm{d}z_1 \int\limits_s^{r_1} \mathrm{d}r_2 \int\limits_{-\infty}^{\infty} \mathrm{d}z_2 \cdots \int\limits_s^{r_{n-1}} \mathrm{d}r_n \int\limits_{-\infty}^{\infty} \mathrm{d}z_n\, \boldsymbol{p}_{t-r_1}^2(x-z_1)$$

$$\times\, \boldsymbol{p}_{r_1-r_2}^2(z_1-z_2)\times\cdots\times\boldsymbol{p}_{r_{n-1}-r_n}^2(z_{n-1}-z_n)\boldsymbol{p}_{r_n-s}^2(z_n-y). \tag{2.7}$$

In order to simplify the preceding expression, let us first use the elementary identity (1.4) in order to see that

$$\int_{-\infty}^{\infty}\boldsymbol{p}_{t-s}^2(x-y)\boldsymbol{p}_{s-r}^2(y-z)\,\mathrm{d}y=\sqrt{\frac{t-r}{4\pi(t-s)(s-r)}}\,\boldsymbol{p}_{t-r}^2(x-z).$$

Consequently,

$$\int_s^t\mathrm{d}r_1\int_{-\infty}^{\infty}\mathrm{d}z_1\int_s^{r_1}\mathrm{d}r_2\int_{-\infty}^{\infty}\mathrm{d}z_2\cdots\int_s^{r_{n-1}}\mathrm{d}r_n\int_{-\infty}^{\infty}\mathrm{d}z_n$$

$$\boldsymbol{p}_{t-r_1}^2(x-z_1)\boldsymbol{p}_{r_1-r_2}^2(z_1-z_2)\times\cdots\times\boldsymbol{p}_{r_{n-1}-r_n}^2(z_{n-1}-z_n)\boldsymbol{p}_{r_n-s}^2(z_n-y)$$

$$=(4\pi)^{-n/2}\,\boldsymbol{p}_{t-r}^2(x-y)$$

$$\times\int_s^t\mathrm{d}r_1\int_s^{r_1}\mathrm{d}r_2\cdots\int_s^{r_{n-1}}\mathrm{d}r_n\sqrt{\frac{t-s}{(t-r_1)(r_1-r_2)\cdots(r_{n-1}-r_n)(r_n-s)}}$$

$$=\left(\frac{t-s}{4\pi}\right)^{n/2}\boldsymbol{p}_{t-r}^2(x-y)\int_{0<r_n<\cdots<r_1<1}\frac{\mathrm{d}r_1\cdots\mathrm{d}r_n}{\sqrt{(1-r_1)(r_1-r_2)\cdots r_n}}$$

$$=\left(\frac{t-s}{4\pi}\right)^{n/2}\frac{\Gamma(1/2)^n}{\Gamma(n/2)}\,\boldsymbol{p}_{t-s}^2(x-y). \tag{2.8}$$

Together, (2.7) and (2.7)' yield

$$\|D_{s,y}u_{n+1}(t,x)\|_k^2\le\boldsymbol{p}_{t-s}^2(x-y)\boldsymbol{p}_s^2(y)\sum_{j=0}^{n}C_k^{j+1}\left(\frac{t-s}{4\pi}\right)^{j/2}\frac{\Gamma(1/2)^j}{\Gamma(j/2)}$$

$$\le\boldsymbol{p}_{t-s}^2(x-y)\boldsymbol{p}_s^2(y)\sum_{j=0}^{\infty}\frac{C_k^{j+1}T^{j/2}}{(4\pi)^{j/2}}\frac{\Gamma(1/2)^j}{\Gamma(j/2)}.$$

Since the above series is convergent, we can conclude that there exists $c'_{T,k}>0$ such that for almost every $(s,y)\in(0,t)\times\mathbb{R}$,

$$\sup_{n\ge0}\|D_{s,y}u_n(t,x)\|_k\le c'_{T,k}\,\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y). \tag{2.9}$$

Moreover, (1.4) and (2.9) together yield

$$\sup_{n\geq 0} \mathrm{E}\left(\|Du_n(t,x)\|^2_{\mathcal{H}}\right) \leq (c'_{T,2})^2 \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \; \boldsymbol{p}^2_{t-s}(x-y)\boldsymbol{p}^2_s(y)$$

$$= (c'_{T,k})^2\boldsymbol{p}^2_t(x) \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \; \boldsymbol{p}^2_{s(t-s)/t}\left(y-\frac{s}{t}x\right) \qquad (2.10)$$

$$= (c'_{T,k})^2\boldsymbol{p}^2_t(x) \int_0^t \sqrt{\frac{t}{4\pi s(t-s)}}\, \mathrm{d}s < \infty,$$

where we have used the semigroup property of the heat kernel in the final identity. It follows from (2.10) and the closability properties of the Malliavin derivative that there exists a subsequence $n(1) < n(2) < \cdots$ of positive integers such that $Du_{n(\ell)}(t,x)$ converges to $Du(t,x)$ in the weak topology of $L^2(\Omega;\mathcal{H})$. Then, we use a smooth approximation $\{\psi_\varepsilon\}_{\varepsilon>0}$ to the identity in $\mathbb{R}_+ \times \mathbb{R}$, and apply Fatou's lemma and duality for L^k-spaces, in order to find that for almost every $(s,y) \in (0,t) \times \mathbb{R}$ and for all $k \geq 2$,

$$\|D_{s,y}u(t,x)\|_k \leq \liminf_{\varepsilon\to 0} \left\| \int_0^{\infty} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y'\; D_{s',y'}u(t,x)\psi_\varepsilon(s-s',y-y') \right\|_k$$

$$\leq \liminf_{\varepsilon\to 0} \sup_{\|G\|_{k/(k-1)}\leq 1} \left| \int_0^{\infty} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y'\; \mathrm{E}\left[GD_{s',y'}u(t,x)\right]\psi_\varepsilon(s-s',y-y') \right|.$$

Choose and fix a random variable $G \in L^2(\Omega)$ such that $\|G\|_{k/(k-1)} \leq 1$. Because $Du_{n(\ell)}(t,x)$ converges weakly in $L^2(\Omega;\mathcal{H})$ to $Du(t,x)$ as $\ell \to \infty$, we can write

$$\left| \int_0^{\infty} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y'\; \mathrm{E}\left[GD_{s',y'}u(t,x)\right]\psi_\varepsilon(s-s',y-y') \right|$$

$$= \lim_{\ell\to\infty} \left| \int_0^{\infty} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y'\; \mathrm{E}\left[GD_{s',y'}u_{n(\ell)}(t,x)\right]\psi_\varepsilon(s-s',y-y') \right|$$

$$\leq \limsup_{\ell\to\infty} \int_0^{\infty} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y'\; \left\|D_{s',y'}u_{n(\ell)}(t,x)\right\|_k \psi_\varepsilon(s-s',y-y')$$

$$\leq c'_{T,k} \int_0^{\infty} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y'\; \mathbf{1}_{(0,t)}(s')\boldsymbol{p}_{t-s'}(x-y')\boldsymbol{p}_{s'}(y')\psi_\varepsilon(s-s',y-y').$$

Let $\varepsilon \to 0$ to conclude the proof of (2.2). $\quad\square$

2.3. The Malliavin-Stein method

Recall that if X and Y are random variables with respective probability distributions μ and ν on $\mathbb{R}$, then the total variation distance between X and Y is defined as

$$d_{\mathrm{TV}}(X,Y) = \sup_{B \in \mathcal{B}(\mathbb{R})} |\mu(B) - \nu(B)|,$$

where $\mathcal{B}(\mathbb{R})$ denotes the family of all Borel subsets of $\mathbb{R}$. The same sort of definition continues to hold when X and Y are abstract random variables on a topological space $\mathbb{X}$, except $\mathcal{B}(\mathbb{R})$ is replaced by $\mathcal{B}(\mathbb{X})$.

We abuse notation and let $d_{\mathrm{TV}}(F,\mathrm{N}(0,1))$ denote the total variation distance between the law of F and the $\mathrm{N}(0,1)$ law. The following bound on $d_{\mathrm{TV}}(F,\mathrm{N}(0,1))$ follows from a suitable combination of ideas from the Malliavin calculus and Stein's method for normal approximations; see Nualart and Nualart [16, Theorem 8.2.1].

Proposition 2.2. *Suppose that $F \in \mathbb{D}^{1,2}$ satisfies $\mathrm{E}(F^2) = 1$ and $F = \delta(v)$ for some v in the $L^2(\Omega)$-domain of the divergence operator δ. Then,*

$$d_{\mathrm{TV}}(F,\mathrm{N}(0,1)) \leq 2\sqrt{\mathrm{Var}\left(\langle DF,v\rangle_{\mathcal{H}}\right)}.$$

In the proof of Theorem 1.11 we will make use of the following generalization of a result of Nourdin and Peccati [14, Theorem 6.1.2].

Proposition 2.3. *Let $F = (F^{(1)}, \ldots, F^{(m)})$ be a random vector such that, for every $i = 1, \ldots, m$, $F^{(i)} = \delta(v^{(i)})$ for some $v^{(i)} \in \mathrm{Dom}\,[\delta]$. Assume additionally that $F^{(i)} \in \mathbb{D}^{1,2}$ for $i = 1, \ldots, m$. Let G be a centered m-dimensional Gaussian random vector with co-variance matrix $(C_{i,j})_{1 \leq i,j \leq m}$. Then, for every $h \in C^2(\mathbb{R}^m)$ that has bounded second partial derivatives,*

$$|\mathrm{E}(h(F)) - \mathrm{E}(h(G))| \leq \tfrac{1}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^{m} \mathrm{E}\left(\left|C_{i,j} - \langle DF^{(i)}, v^{(j)}\rangle_{\mathcal{H}}\right|^2\right)},$$

where

$$\|h''\|_\infty := \max_{1 \leq i,j \leq m} \sup_{x \in \mathbb{R}^m} \left|\frac{\partial^2 h(x)}{\partial x_i \partial x_j}\right|.$$

2.4. On the ergodic theorem (1.6)

Recall the definition (1.7) of $\mathcal{S}_{N,t}$ and observe that the ergodic theorem (1.6) can be recast in terms of the average integral $\mathcal{S}_{N,t}$ as follows:

$$\lim_{N \to \infty} \mathcal{S}_{N,t} = 0 \quad \text{a.s. and in } L^1(\Omega).$$

The following lemma proves that the ergodic theorem (1.6) holds in $L^k(\Omega)$ for every $k \geq 2$, hence also in $L^k(\Omega)$ for every $k \geq 1$. It also yields a quantitative upper bound of $O(\sqrt{\log(N)/N})$ on the rate of convergence in $L^k(\Omega)$ for every $k \geq 1$, with a constant that describes also the behavior of the limit uniformly in t when $t \ll 1$. Perhaps not surprisingly, the mentioned rate of convergence coincides with the rate of convergence to normality that was ensured by Theorem 1.2.

Lemma 2.4. *For all real numbers $k \geq 2$ and $T > 0$ there exists a number $A_{k,T} > 0$ such that*

$$\sup_{N \geq e} \left\| \sqrt{\frac{N}{\log N}} \, \mathcal{S}_{N,t} \right\|_k \leq A_{k,T} \sqrt{t \log_+(1/t)} \quad \text{uniformly for all } t \in (0,T),$$

where $\log_+(w) := \log(e + w)$ for all $w \geq 0$.

Proof. Choose and fix a real number $k \geq 2$. By the BDG inequality and (1.4),

$$\|\mathcal{S}_{N,t}\|_k^2 = \frac{1}{N^2} \left\| \int_{(0,t) \times \mathbb{R}} U(s,y) \left[\int_0^N \boldsymbol{P}_{s(t-s)/t} \left(y - \frac{s}{t} x \right) \mathrm{d}x \right] \eta(\mathrm{d}s \, \mathrm{d}y) \right\|_k^2$$

$$\leq \frac{c_k}{N^2} \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \, \|U(s,y)\|_k^2 \left[\int_0^N \boldsymbol{P}_{s(t-s)/t} \left(y - \frac{s}{t} x \right) \mathrm{d}x \right]^2,$$

uniformly for all $N, t > 0$. Apply (2.4) to see that

$$\|\mathcal{S}_{N,t}\|_k^2 \leq \frac{c_k \, c_{T,k}^2}{N^2} \int_0^t \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y \left[\int_0^N \boldsymbol{P}_{s(t-s)/t} \left(y - \frac{s}{t} x \right) \mathrm{d}x \right]^2,$$

uniformly for all $N > 0$ and $t \in (0,T)$. Now expand the square and appeal to the semigroup property of the heat kernel in order to find that, for every $N, t > 0$,

$$\int_{-\infty}^\infty \mathrm{d}y \left[\int_0^N \boldsymbol{P}_{s(t-s)/t} \left(y - \frac{s}{t} x \right) \mathrm{d}x \right]^2 = \int_0^N \mathrm{d}y \int_0^N \mathrm{d}z \, \boldsymbol{P}_{2s(t-s)/t} \left(\frac{s}{t} (y - z) \right)$$

$$= \left(\frac{t}{s} \right)^2 \int_0^{Ns/t} \mathrm{d}a \int_0^{Ns/t} \mathrm{d}b \, \boldsymbol{P}_{2s(t-s)/t}(a - b)$$

$$= \frac{Nt}{\pi s} \int_{-\infty}^\infty \left(\frac{1 - \cos z}{z^2} \right) \exp \left(-\frac{t(t-s)z^2}{N^2 s} \right) \mathrm{d}z;$$

see Lemma A.2 of the Appendix. Consequently, if $N > 0$ and $t \in (0, T)$, then

$$
\begin{aligned}
\|\mathcal{S}_{N,t}\|_k^2 &\leq \frac{t c_k c_{k,T}^2}{\pi N} \int\limits_{-\infty}^{\infty} \mathrm{d}x \left(\frac{1 - \cos x}{x^2} \right) \int\limits_0^t \frac{\mathrm{d}s}{s} \exp\left(-\frac{t(t-s)x^2}{N^2 s} \right) \\
&= \frac{c_k c_{k,T}^2 \log N}{\pi N} \int\limits_{-\infty}^{\infty} \left(\frac{1 - \cos x}{x^2} \right) G_{N,t}(x) \, \mathrm{d}x,
\end{aligned}
$$

where $G_{N,t}$ is defined in (A.1) below, in the Appendix. We may appeal to Lemma A.1 of the Appendix to conclude the result. $\square$

3. Proof of Theorem 1.1

Since weak mixing implies ergodicity, it suffices to prove that $U(t)$ is weakly mixing for every $t > 0$. We follow the proof of [6, Corollary 9.1] in order to reduce the proof of Theorem 1.1 to the verification of the following:

$$
\lim_{|x| \to \infty} \mathrm{Cov}\left[\mathcal{G}(x), \mathcal{G}(0) \right] = 0, \tag{3.1}
$$

where the functions $g_1, \ldots, g_k \in C_b^1(\mathbb{R})$ satisfy $g_j(0) = 0$ and $\mathrm{Lip}(g_j) = 1$ for every $j = 1, \ldots, k$,

$$
\mathcal{G}(x) := \prod_{j=1}^k g_j(U(t, x + \zeta^j)) \qquad \text{for all } x \in \mathbb{R},
$$

and $\zeta^1, \ldots, \zeta^k$ are fixed real numbers. Thus, it suffices to prove (3.1).

By the chain rule for the Malliavin derivative [15, Proposition 1.2.4],

$$
D_{s,z}\mathcal{G}(x) = \sum_{j_0=1}^k \left(\prod_{\substack{j=1 \\ j \neq j_0}}^k g_j\left(U(t, x + \zeta^j) \right) \right) g_{j_0}'\left(U(t, x + \zeta^{j_0}) \right) D_{s,z}U(t, x + \zeta^{j_0}).
$$

Therefore, the definition of the process U in (1.3), (2.4), and Lemma 2.1 together imply the existence of a real number $c = c(T, k)$ such that

$$
\begin{aligned}
\|D_{s,z}\mathcal{G}(x)\|_2 &\leq \sum_{j_0=1}^k \left(\prod_{j=1, j \neq j_0}^k \|g_j(U(t, x + \zeta^j))\|_{2k} \right) \|D_{s,z}U(t, x + \zeta^{j_0})\|_{2k} \\
&\leq c \sum_{j=1}^k \frac{\boldsymbol{p}_{t-s}(x + \zeta^j - z)\boldsymbol{p}_s(z)}{\boldsymbol{p}_t(x + \zeta^j)} = c \sum_{j=1}^k \boldsymbol{p}_{s(t-s)/t}\left(z - \frac{s}{t}(x + \zeta^j) \right),
\end{aligned}
$$

uniformly for all $0 < s < t \leq T$ and $x, z \in \mathbb{R}$; the equality holds due to (1.4). Now apply the Poincaré inequality (2.1) and the semigroup property of the heat kernel to see that

$$\left|\text{Cov}\left[\mathcal{G}(x), \mathcal{G}(0)\right]\right| \leq c^2 \sum_{j,\ell=1}^{k} \int_0^t \boldsymbol{p}_{2s(t-s)/t}\left(\frac{s}{t}(x + \zeta^j - \zeta^\ell)\right) \mathrm{d}s.$$

This implies (3.1), thanks to the dominated convergence theorem, and concludes the proof. □

4. Asymptotic behavior of the covariance

Recall from (1.7) that

$$\mathcal{S}_{N,t} = \frac{1}{N} \int_0^N [U(t, x) - 1] \, \mathrm{d}x,$$

where $U(t, x)$ was defined in (1.3). The following proposition provides the asymptotic behavior of the covariance function of the renormalized sequence of processes $\mathcal{S}_{N,t}$ as N tends to infinity.

Proposition 4.1. *For every $t_1, t_2 > 0$,*

$$\lim_{N \to \infty} \text{Cov}\left[\sqrt{\frac{N}{\log N}} \mathcal{S}_{N,t_1}, \sqrt{\frac{N}{\log N}} \mathcal{S}_{N,t_2}\right] = 2(t_1 \wedge t_2).$$

Proof. First, let us recall from Chen and Dalang [3, (2.31)] that, for all $s > 0$ and $z \in \mathbb{R}$,

$$\text{E}\left(|u(s, z)|^2\right) = \boldsymbol{p}_s^2(z)(1 + \theta(s)), \tag{4.1}$$

where

$$\theta(s) := \mathrm{e}^{s/4} \sqrt{s/2} \int_{-\infty}^{\sqrt{s/2}} \mathrm{e}^{-y^2/2} \, \mathrm{d}y \qquad \text{for all } s > 0. \tag{4.2}$$

By (1.2), the Itô-Walsh isometry, and (4.1),

$$\text{Cov}\left[U(t_1, x), U(t_2, y)\right]$$
$$= \frac{1}{\boldsymbol{p}_{t_1}(x)\boldsymbol{p}_{t_2}(y)} \int_0^{t_1 \wedge t_2} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}z \, \boldsymbol{p}_{t_1-s}(x-z)\boldsymbol{p}_{t_2-s}(y-z)\text{E}\left(|u(s, z)|^2\right)$$

$$= \frac{1}{\boldsymbol{p}_{t_1}(x)\boldsymbol{p}_{t_2}(y)} \int\limits_0^{t_1 \wedge t_2} \mathrm{d}s \int\limits_{-\infty}^{\infty} \mathrm{d}z \; \boldsymbol{p}_{t_1-s}(x-z)\boldsymbol{p}_{t_2-s}(y-z)\boldsymbol{p}_s^2(z)(1+\theta(s))$$

$$= \int\limits_0^{t_1 \wedge t_2} \mathrm{d}s \int\limits_{-\infty}^{\infty} \mathrm{d}z \; \boldsymbol{p}_{s(t_1-s)/t_1}\left(z - \frac{s}{t_1}x\right) \boldsymbol{p}_{s(t_2-s)/t_2}\left(z - \frac{s}{t_2}y\right)(1+\theta(s))$$

$$= \int\limits_0^{t_1 \wedge t_2} \boldsymbol{p}_{s[(t_1-s)/t_1+(t_2-s)/t_2]}\left(s\left[\frac{x}{t_1} - \frac{y}{t_2}\right]\right)(1+\theta(s))\,\mathrm{d}s$$

$$=: \int\limits_0^{t_1 \wedge t_2} \mathcal{P}_{s,t_1,t_2}(x,y)(1+\theta(s))\,\mathrm{d}s,$$

notation being clear from context. Let $\tau := 2t_1 t_2/(t_1 + t_2)$, so that we can write

$$\mathcal{P}_{s,t_1,t_2}(x,y) = \mathcal{P}_{s,\tau}\left(\frac{2(xt_2 - yt_1)}{t_1 + t_2}\right) \quad \text{for} \quad \mathcal{P}_{s,t}(w) = \boldsymbol{p}_{2s(t-s)/t}\left(\frac{sw}{t}\right).$$

If $t_1 < t_2$, then

$$\mathrm{Cov}\left[\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t_1}\,,\,\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t_2}\right] = \frac{1}{N\log N} \int\limits_0^N \mathrm{d}y \int\limits_0^N \mathrm{d}x \; \mathrm{Cov}\left[U(t_1,x),U(t_2,y)\right]$$

$$= \frac{1}{N\log N} \int\limits_0^{t_1} \mathrm{d}s\,(1+\theta(s)) \int\limits_0^N \mathrm{d}y \int\limits_0^N \mathrm{d}x \; \mathcal{P}_{s,\tau}\left(\frac{2(xt_2 - yt_1)}{t_1 + t_2}\right).$$

In order to simplify the exposition define

$$\tau_1 := \frac{2t_2}{t_1 + t_2} \quad \text{and} \quad \tau_2 = \frac{2t_1}{t_1 + t_2}.$$

We then change variables $[x \to x/\tau_1$ and $y \to y/\tau_2]$ to obtain

$$\mathrm{Cov}\left[\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t_1}\,,\,\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t_2}\right]$$

$$= \frac{1}{\tau_1 \tau_2 N\log N} \int\limits_0^{t_1} (1+\theta(s))\,\mathrm{d}s \int\limits_0^{N\tau_1} \mathrm{d}x \int\limits_0^{N\tau_2} \mathrm{d}y \; \mathcal{P}_{s,\tau}(x-y)$$

$$= \frac{\tau}{\tau_1 \tau_2 N\log N} \int\limits_0^{t_1} \left(\frac{1+\theta(s)}{s}\right)\,\mathrm{d}s \int\limits_0^{N\tau_1} \mathrm{d}x \int\limits_0^{N\tau_2} \mathrm{d}y \; \boldsymbol{p}_{2\tau(\tau-s)/s}(x-y),$$

where in the last equality we have used the scaling property,

$$\boldsymbol{p}_\sigma(\alpha w) = \alpha^{-1}\boldsymbol{p}_{\sigma/\alpha^2}(w), \qquad \text{valid for all } \sigma, \alpha > 0 \text{ and } w \in \mathbb{R}. \tag{4.3}$$

Since $\widehat{\mathbf{1}_{[0,a]}}(\xi) = a\widehat{\mathbf{1}_{[0,1]}}(a\xi)$ for all $a > 0$ and $\xi \in \mathbb{R}$, Parseval's identity ensures that

$$\text{Cov}\left[\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t_1}\,,\,\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t_2}\right]$$

$$= \frac{\tau}{2\pi\tau_1\tau_2 \log N}\int_0^{t_1}\left(\frac{1+\theta(s)}{s}\right)\mathrm{d}s\int_{-\infty}^{\infty}\widehat{\mathbf{1}_{[0,\tau_1]}}(w)\overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)}\exp\left(-\frac{(\tau-s)\tau}{s}\frac{w^2}{N^2}\right)\mathrm{d}w$$

$$= \frac{1}{2\pi\tau_1\tau_2}\int_{-\infty}^{\infty}\widehat{\mathbf{1}_{[0,\tau_1]}}(w)\overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)}G_{N,\tau}(w)\,\mathrm{d}w$$

$$\quad - \frac{\tau}{2\pi\tau_1\tau_2 \log N}\int_{t_1}^{\tau}\frac{\mathrm{d}s}{s}\int_{-\infty}^{\infty}\widehat{\mathbf{1}_{[0,\tau_1]}}(w)\overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)}\exp\left(-\frac{(\tau-s)\tau}{s}\frac{w^2}{N^2}\right)\mathrm{d}w$$

$$\quad + \frac{\tau}{2\pi\tau_1\tau_2 \log N}\int_0^{t_1}\frac{\theta(s)}{s}\,\mathrm{d}s\int_{-\infty}^{\infty}\widehat{\mathbf{1}_{[0,\tau_1]}}(w)\overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)}\exp\left(-\frac{(\tau-s)\tau}{s}\frac{w^2}{N^2}\right)\mathrm{d}w$$

$$=: A_N^{(1)} - A_N^{(2)} + A_N^{(3)},$$

where the function $G_{N,\tau}$ is defined in (A.1) below, in the Appendix. We plan to prove that

$$\lim_{N\to\infty}A_N^{(1)} = 2t_1 \quad \text{and} \quad \lim_{N\to\infty}A_N^{(2)} = \lim_{N\to\infty}A_N^{(3)} = 0. \tag{4.4}$$

These facts together conclude the proof of the proposition.

In order to understand the behavior of $A_N^{(1)}$ we first apply Lemma A.1 and the dominated convergence theorem, and then the Parseval identity, in order to verify the first of the three assertions in (4.4):

$$\lim_{N\to\infty}A_N^{(1)} = \frac{2\tau}{2\pi\tau_1\tau_2}\int_{-\infty}^{\infty}\widehat{\mathbf{1}_{[0,\tau_1]}}(w)\overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)}\,\mathrm{d}w = \frac{2\tau}{\tau_1\tau_2}\left\langle\mathbf{1}_{[0,\tau_1]}\,,\,\mathbf{1}_{[0,\tau_2]}\right\rangle_{L^2(\mathbb{R})} = 2t_1.$$

We study $A_N^{(2)}$ by making a change of variables $[s \to \tau/(s+1)]$ to find that

$$A_N^{(2)} = \frac{\tau}{2\pi\tau_1\tau_1 \log N}\int_0^{(t_2-t_1)/(t_2+t_1)}\frac{\mathrm{d}s}{1+s}\int_{-\infty}^{\infty}\widehat{\mathbf{1}_{[0,\tau_1]}}(w)\overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)}\exp\left(-\frac{\tau s w^2}{N^2}\right)\mathrm{d}w.$$

Since $\exp(-\tau s w^2/N^2) \leq 1$, this proves that $A_N^{(2)} = O(1/\log N) \to 0$ as $N \to \infty$. Therefore, it remains to prove the third assertion in (4.4) about $A_N^{(3)}$. For that, we change variables $[s \to \tau s]$ to obtain

$$
\left| A_N^{(3)} \right| \leq \frac{\tau}{2\pi\tau_1\tau_2 \log N} \int\limits_{-\infty}^{\infty} \left| \widehat{\mathbf{1}_{[0,\tau_1]}}(w) \overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)} \right| \mathrm{d}w \int\limits_0^1 \frac{\theta(\tau s)}{s} \exp\left(-\frac{(1-s)\tau}{s} \frac{w^2}{N^2} \right) \mathrm{d}s
$$

$$
= \frac{\tau}{2\pi\tau_1\tau_2 \log N} \int\limits_{-\infty}^{\infty} \left| \widehat{\mathbf{1}_{[0,\tau_1]}}(w) \overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)} \right| \mathrm{d}w \int\limits_0^\infty \frac{\theta(\tau/(r+1))}{r+1} \exp\left(-\frac{w^2\tau r}{N^2} \right) \mathrm{d}r.
$$

By the definition of the function θ in (4.2),

$$
\theta\left(\frac{\tau}{r+1} \right) \exp\left(-\frac{w^2\tau r}{N^2} \right) < \theta\left(\frac{\tau}{r+1} \right) \leq \mathrm{e}^{\tau/4} \sqrt{\frac{\tau\pi}{r+1}} \quad \text{for all } r > 0.
$$

Hence,

$$
\left| A_N^{(3)} \right| \leq \frac{\mathrm{e}^{\tau/4} t \sqrt{\tau\pi}}{2\pi\tau_1\tau_2 \log N} \int\limits_{-\infty}^{\infty} \left| \widehat{\mathbf{1}_{[0,\tau_1]}}(w) \overline{\widehat{\mathbf{1}_{[0,\tau_2]}}(w)} \right| \mathrm{d}w \times \int\limits_0^\infty \frac{\mathrm{d}r}{(r+1)^{3/2}} \to 0,
$$

as $N \to \infty$. This concludes the proof of (4.4) and hence the proof of the proposition. $\quad\square$

5. Proof of Theorem 1.2

For all $N, t, s > 0$ and $y \in \mathbb{R}$ define

$$
g_{N,t}(s,y) := \mathbf{1}_{(0,t)}(s) \frac{1}{N} \int\limits_0^N \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x \right) \mathrm{d}x \quad \text{and} \quad v_{N,t}(s,y) := g_{N,t}(s,y) U(s,y).
$$

(5.1)

Because of (1.7) and a stochastic Fubini argument,

$$
\mathcal{S}_{N,t} = \int\limits_{\mathbb{R}_+ \times \mathbb{R}} v_{N,t}(s,y)\, \eta(\mathrm{d}s\,\mathrm{d}y) = \delta(v_{N,t}) \qquad \text{a.s.,}
$$

(5.2)

owing to the fact that $v_{N,t}$ is an adapted random field and hence its stochastic integral agrees with its divergence (see Nualart [15, Chapter 1.3.3]). Our work so far shows that $\mathcal{S}_{N,t}$ is Malliavin differentiable, and that the following defines a version of the Malliavin derivative of $\mathcal{S}_{N,t}$:

$$
D_{r,z}\mathcal{S}_{N,t} = \mathbf{1}_{(0,t)}(r) v_{N,t}(r,z) + \mathbf{1}_{(0,t)}(r) \int\limits_{(r,t)\times\mathbb{R}} D_{r,z} v_{N,t}(s,y)\, \eta(\mathrm{d}s\,\mathrm{d}y).
$$

(5.3)

The key technical result of this section is the following proposition:

Proposition 5.1. *For every $T > 0$ there exists a real number $K_T > 0$ such that*

$$\sup_{t,\tau \in (0,T)} \mathrm{Var}\, \langle D\mathcal{S}_{N,t}\,, v_{N,\tau}\rangle_{\mathcal{H}} \le K_T \, \frac{(\log N)^3}{N^3} \qquad \text{for all } N \ge \mathrm{e}.$$

We plan to first prove Proposition 5.1. Then, we will use this proposition to prove Theorem 1.2. The key to the proof of Proposition 5.1 is the following simple decomposition, which is an immediate consequence of (5.3):

$$\langle D\mathcal{S}_{N,t}\,, v_{N,\tau}\rangle_{\mathcal{H}} = \mathcal{X}_{N,t,\tau} + \mathcal{Y}_{N,t,\tau}, \tag{5.4}$$

where

$$\mathcal{X}_{N,t,\tau} := \langle v_{N,t}\,, v_{N,\tau}\rangle_{\mathcal{H}}\,, \quad \text{and}$$

$$\mathcal{Y}_{N,t,\tau} := \int\limits_{0}^{\infty} \mathrm{d}r \int\limits_{-\infty}^{\infty} \mathrm{d}z \; v_{N,\tau}(r\,,z) \left(\int\limits_{(r,t)\times\mathbb{R}} D_{r,z} v_{N,t}(s\,,y)\,\eta(\mathrm{d}s\,\mathrm{d}y) \right). \tag{5.5}$$

The decomposition (5.4) ensures that

$$\mathrm{Var}\langle D\mathcal{S}_{N,t}\,, v_{N,\tau}\rangle_{\mathcal{H}} \le 2\mathrm{Var}(\mathcal{X}_{N,t,\tau}) + 2\mathrm{Var}(\mathcal{Y}_{N,t,\tau}). \tag{5.6}$$

Therefore, the bulk of the work is to establish bounds on the last two variances. Those require some effort and are carried out separately, using slightly different ideas, in Lemmas 5.3 and 5.4 respectively. In light of those lemmas and (5.6), the proof Proposition 5.1 is immediate, with no need for additional proof.

First let us observe that the mean of $\langle D\mathcal{S}_{N,t}\,, v_{N,\tau}\rangle_{\mathcal{H}}$ is carried by $\mathcal{X}_{N,t,\tau}$.

Lemma 5.2. *For every $T, N > 0$ and $t, \tau \in (0\,,T)$,*

$$\mathrm{E}\mathcal{Y}_{N,t,\tau} = 0 \quad \text{and} \quad \mathrm{E}\,\langle D\mathcal{S}_{N,t}\,, v_{N,\tau}\rangle_{\mathcal{H}} = \mathrm{E}\mathcal{X}_{N,t,\tau} = Cov(\mathcal{S}_{N,t}\,,\,\mathcal{S}_{N,\tau}).$$

Proof. Thanks to Gaussian integration by parts (see Nualart [15, (1.42)]), $\mathrm{E}(\langle DF\,, V\rangle_{\mathcal{H}}) = \mathrm{E}[F\delta(V)]$ for all $F \in \mathbb{D}^{1,2}$ and $V \in \mathrm{Dom}[\delta]$. Choose $F \equiv 1$ to observe the well-known fact that $\delta(V)$ has mean zero, and choose $F = \delta(U)$ to see that $\mathrm{E}(\langle D\delta(U)\,, V\rangle_{\mathcal{H}}) = Cov(\delta(U)\,, \delta(V))$ whenever $U, V \in \mathrm{Dom}[\delta]$. Thanks to (5.2) we can apply the preceding with $U = v_{N,t}$ and $V = v_{N,\tau}$ to see that $\mathcal{S}_{N,t} = \delta(U)$ and $\mathcal{S}_{N,\tau} = \delta(V)$ [from (5.2)], whence $\mathrm{E}\langle D\mathcal{S}_{N,t}\,, v_{N,\tau}\rangle_{\mathcal{H}} = Cov(\mathcal{S}_{N,t}\,, \mathcal{S}_{N,\tau})$. Since the Walsh integral has mean zero and U is adapted, $\mathrm{E}\mathcal{Y}_{N,t,\tau} = 0$; see (5.5). This and (5.4) together complete the proof. $\quad\square$

Lemma 5.3. *For every $T > 0$ there exists a real number $A_T > 0$ such that*

$$\sup_{t,\tau \in (0,T)} \mathrm{Var}(\mathcal{X}_{N,t,\tau}) \leq A_T \frac{(\log N)^3}{N^3} \quad \text{uniformly for every } N \geq \mathrm{e}.$$

Proof. Choose and fix $0 < t, \tau < T$ and $N \geq \mathrm{e}$. It follows readily from (5.5) and our efforts thus far that $\mathcal{X}_{N,t,\tau}$ is Malliavin differentiable, and the following is a version of the Malliavin derivative:

$$D_{r,z}\mathcal{X}_{N,t,\tau} = 2\mathbf{1}_{[0,t\wedge\tau]}(r) \int_r^{t\wedge\tau} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \, g_{N,t}(s,y)g_{N,\tau}(s,y)U(s,y)D_{r,z}U(s,y).$$

Moreover, it follows from this and the definition of the $\mathcal{H}$-norm that

$$\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2 = 4 \int_0^{t\wedge\tau} \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}z \left| \int_r^{t\wedge\tau} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \, g_{N,t}(s,y)g_{N,\tau}(s,y)U(s,y)D_{r,z}U(s,y) \right|^2.$$

According to (1.3), (2.4), and Lemma 2.1, whenever $0 < s, s' < T$ and $y, y' \in \mathbb{R}$, the following holds a.s. for a.e. every $(r, z) \in (s \wedge s', t) \times \mathbb{R}$:

$$\begin{aligned}
&|\mathrm{E}\left[U(s,y)D_{r,z}U(s,y)U(s',y')D_{r,z}U(s',y')\right]| \\
&\quad \leq c_{4,T}^2 \|D_{r,z}U(s,y)\|_4 \|D_{r,z}U(s',y')\|_4 \\
&\quad \leq c_{T,4}^2 C_{T,4}^2 \frac{\boldsymbol{p}_{s-r}(y-z)\boldsymbol{p}_r(z)}{\boldsymbol{p}_s(y)} \frac{\boldsymbol{p}_{s'-r}(y'-z)\boldsymbol{p}_r(z)}{\boldsymbol{p}_{s'}(y')} \\
&\quad =: \tfrac{1}{4} A_T \, \boldsymbol{p}_{r(s-r)/s}\left(z - \frac{r}{s}y\right) \boldsymbol{p}_{r(s'-r)/s'}\left(z - \frac{r}{s'}y'\right),
\end{aligned}$$

where we have appealed to (1.4) in the last line. Therefore,

$$\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right)$$

$$\begin{aligned}
&\leq A_T \int_0^{t\wedge\tau} \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}z \int_r^{t\wedge\tau} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \int_r^{t\wedge\tau} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y' \\
&\qquad \times g_{N,t}(s,y)g_{N,\tau}(s,y)g_{N,t}(s',y')g_{N,\tau}(s',y')\boldsymbol{p}_{r(s-r)/s}\left(z - \frac{r}{s}y\right) \boldsymbol{p}_{r(s'-r)/s'}\left(z - \frac{r}{s'}y'\right) \\
&= A_T \int_0^{t\wedge\tau} \mathrm{d}r \int_r^{t\wedge\tau} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \int_r^{t\wedge\tau} \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y' \\
&\qquad \times g_{N,t}(s,y)g_{N,\tau}(s,y)g_{N,t}(s',y')g_{N,\tau}(s',y')\boldsymbol{p}_{[r(s-r)/s]+[r(s'-r)/s']}\left(\frac{r}{s}y - \frac{r}{s'}y'\right),
\end{aligned}$$

thanks to the semigroup property of the heat kernel. Since $g_{N,\nu}(s,y) \le \frac{\nu}{s}N^{-1}$ for all $N > 0, \nu \ge s > 0$ and $y \in \mathbb{R}$, we may bound two of the g-terms from above, each by N^{-1}, in order to find that

$$
\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right) \le \frac{A_T}{N^2} \int_0^{t\wedge\tau} \mathrm{d}r \int_r^{t\wedge\tau} \frac{t\vee\tau}{s}\mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \int_r^{t\wedge\tau} \frac{t\vee\tau}{s'}\mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y'
$$

$$
\times g_{N,t\wedge\tau}(s,y)g_{N,t\wedge\tau}(s',y')\boldsymbol{p}_{[r(s-r)/s]+[r(s'-r)/s']}\left(\frac{r}{s}y - \frac{r}{s'}y'\right)
$$

$$
= \frac{A_T}{N^4} \int_0^{t\wedge\tau} \mathrm{d}r \int_r^{t\wedge\tau} \frac{t\vee\tau}{s}\mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \int_r^{t\wedge\tau} \frac{t\vee\tau}{s'}\mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}y' \int_0^N \mathrm{d}x \int_0^N \mathrm{d}x'
$$

$$
\times \boldsymbol{p}_{s(\{t\wedge\tau\}-s)/(t\wedge\tau)}\left(y - \frac{s}{t\wedge\tau}x\right) \boldsymbol{p}_{s'(\{t\wedge\tau\}-s')/(t\wedge\tau)}\left(y' - \frac{s'}{t\wedge\tau}x'\right)
$$

$$
\times \boldsymbol{p}_{[r(s-r)/s]+[r(s'-r)/s']}\left(\frac{r}{s}y - \frac{r}{s'}y'\right).
$$

It follows from (4.3) that

$$
\boldsymbol{p}_{[r(s-r)/s]+[r(s'-r)/s']}\left(\frac{r}{s}y - \frac{r}{s'}y'\right) = \frac{s}{r}\boldsymbol{p}_{[s(s-r)/r]+[s^2(s'-r)/(s'r)]}\left(y - \frac{s}{s'}y'\right).
$$

Therefore, the semigroup property of the heat kernel implies the following:

$$
\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right) \le \frac{A_T(t\vee\tau)^2}{N^4} \int_0^{t\wedge\tau} \frac{\mathrm{d}r}{r} \int_r^{t\wedge\tau} \mathrm{d}s \int_r^{t\wedge\tau} \frac{\mathrm{d}s'}{s'} \int_{-\infty}^{\infty} \mathrm{d}y' \int_0^N \mathrm{d}x \int_0^N \mathrm{d}x'
$$

$$
\boldsymbol{p}_{s'(\{t\wedge\tau\}-s')/(t\wedge\tau)}\left(y' - \frac{s'}{t\wedge\tau}x'\right)
$$

$$
\times \boldsymbol{p}_{[s(s-r)/r]+[s^2(s'-r)/(s'r)]+[s(\{t\wedge\tau\}-s)/(t\wedge\tau)]}\left(\frac{s}{s'}y' - \frac{s}{t\wedge\tau}x\right).
$$

A repeated appeal to (4.3) yields

$$
\boldsymbol{p}_{[s(s-r)/r]+[s^2(s'-r)/(s'r)]+[s(\{t\wedge\tau\}-s)/(t\wedge\tau)]}\left(\frac{s}{s'}y' - \frac{s}{t\wedge\tau}x\right)
$$

$$
= \frac{s'}{s}\boldsymbol{p}_{[(s')^2(s-r)/(sr)]+[s'(s'-r)/r]+[(s')^2(\{t\wedge\tau\}-s)/\{s(t\wedge\tau)\}]}\left(y' - \frac{s'}{t\wedge\tau}x\right).
$$

And yet another appeal to the semigroup property reveals the following:

$$
\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right)
$$

$$
\le \frac{A_T(t\vee\tau)^2}{N^4} \int_0^{t\wedge\tau} \frac{\mathrm{d}r}{r} \int_r^{t\wedge\tau} \frac{\mathrm{d}s}{s} \int_r^{t\wedge\tau} \mathrm{d}s' \int_0^N \mathrm{d}x \int_0^N \mathrm{d}x'
$$

$$\times \boldsymbol{P}_{[(s')^2(s-r)/(sr)]+[s'(s'-r)/r]+[(s')^2(\{t\wedge\tau\}-s)/\{s(t\wedge\tau)\}]+[s'(\{t\wedge\tau\}-s')/(t\wedge\tau)]} \left(\frac{s'}{t\wedge\tau}(x-x') \right)$$

$$= \frac{A_T(t\vee\tau)^2(t\wedge\tau)}{N^4} \int\limits_0^{t\wedge\tau} \frac{\mathrm{d}s}{s} \int\limits_0^{t\wedge\tau} \frac{\mathrm{d}s'}{s'} \int\limits_0^{s\wedge s'} \frac{\mathrm{d}r}{r} \int\limits_0^N \mathrm{d}x \int\limits_0^N \mathrm{d}x'$$

$$\times \boldsymbol{P}_{[(t\wedge\tau)^2(s-r)/(sr)]+[(t\wedge\tau)^2(s'-r)/(s'r)]+[(t\wedge\tau)(\{t\wedge\tau\}-s)/s]+[(t\wedge\tau)(\{t\wedge\tau\}-s')/s']}(x-x'),$$

thanks also to scaling (4.3) and Fubini's theorem. Since

$$\frac{2(t\wedge\tau)(t\wedge\tau-r)}{r} = \frac{(t\wedge\tau)^2(s-r)}{sr} + \frac{(t\wedge\tau)^2(s'-r)}{s'r} + \frac{(t\wedge\tau)(t\wedge\tau-s)}{s}$$
$$+ \frac{(t\wedge\tau)(t\wedge\tau-s')}{s'},$$

we appeal to Lemma A.2 in order to find that

$$\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right)$$

$$\leq \frac{A_T(t\vee\tau)^2(t\wedge\tau)}{N^3\pi} \int\limits_0^{t\wedge\tau} \frac{\mathrm{d}s}{s} \int\limits_0^{t\wedge\tau} \frac{\mathrm{d}s'}{s'} \int\limits_0^{s\wedge s'} \frac{\mathrm{d}r}{r} \int\limits_{-\infty}^{\infty} \mathrm{d}z\, \varphi(z) \mathrm{e}^{-((t\wedge\tau)((t\wedge\tau)-r))z^2/(rN^2)}.$$

Integrating in the variables s and s' yields

$$\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right) \leq \frac{A_T(t\vee\tau)^2(t\wedge\tau)}{N^3\pi} \int\limits_0^{t\wedge\tau} \frac{\mathrm{d}r}{r} \left(\log\left(\frac{t\wedge\tau}{r}\right)\right)^2 \int\limits_{\mathbb{R}} \mathrm{e}^{-\frac{(t\wedge\tau)((t\wedge\tau)-r)}{r}\frac{z^2}{N^2}}\varphi(z)\mathrm{d}z,$$

Making the change of variables $\frac{(t\wedge\tau)-r}{r} = \theta$, allows us to write

$$\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right) \leq \frac{A_T(t\vee\tau)^2(t\wedge\tau)}{N^3\pi} \int\limits_{\mathbb{R}} \varphi(z)\mathrm{d}z \int\limits_0^{\infty} \mathrm{d}\theta \frac{1}{\theta+1} \left(\log(\theta+1)\right)^2 \mathrm{e}^{-\frac{(t\wedge\tau)\theta z^2}{N^2}}.$$

Integrating by parts and using the fact that

$$\left(\frac{1}{3}(\log(\theta+1))^3 \mathrm{e}^{-\frac{(t\wedge\tau)\theta z^2}{N^2}}\right)\Bigg|_{\theta=0}^{\theta=\infty} = 0,$$

we obtain

$$\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right) \leq \frac{A_T(t\vee\tau)^2(t\wedge\tau)}{3N^3\pi} \int\limits_{\mathbb{R}} \varphi(z)\mathrm{d}z \int\limits_0^{\infty} \mathrm{d}\theta \left(\log(\theta+1)\right)^3 \mathrm{e}^{-\frac{t\theta z^2}{N^2}} \frac{(t\wedge\tau)z^2}{N^2}$$

$$= \frac{A_T(t\vee\tau)^2(t\wedge\tau)}{3N^3\pi} \int\limits_{\mathbb{R}} \varphi(z)\mathrm{d}z \int\limits_0^{\infty} \mathrm{d}\theta \left(\log\left(\frac{N^2}{(t\wedge\tau)z^2}\theta+1\right)\right)^3 \mathrm{e}^{-\theta}.$$

Using the inequality

$$\log\left(\frac{N^2}{(t\wedge\tau)z^2}\theta+1\right) \leq 2\log N + \log(\theta+1) + \log\left(\frac{1}{t\wedge\tau}+1\right) + \log\left(\frac{1}{z^2}+1\right)$$

$$\leq \left(2\log N + \log\left(\frac{1}{t\wedge\tau}+1\right)\right)\left(1+\log(\theta+1)+\log\left(\frac{1}{z^2}+1\right)\right),$$

and taking into account that

$$C := \int_{\mathbb{R}}\varphi(z)\mathrm{d}z\int_0^\infty \mathrm{d}\theta\left(1+\log(\theta+1)+\log\left(\frac{1}{z^2}+1\right)\right)^3 e^{-\theta} < \infty,$$

we finally get

$$\mathrm{E}\left(\|D\mathcal{X}_{N,t,\tau}\|_{\mathcal{H}}^2\right) \leq \frac{CA_T(t\vee\tau)^2(t\wedge\tau)}{3N^3\pi}\left(2\log N + \log\left(\frac{1}{t\wedge\tau}+1\right)\right)^3,$$

which provides the desired estimate. □

Lemma 5.4. *For every $T > 0$ there exists a real number $A'_T > 0$ such that*

$$\sup_{t,\tau\in(0,T)} \mathrm{Var}\left(\mathcal{Y}_{N,t,\tau}\right) \leq A'_T\frac{(\log N)^3}{N^3} \quad \text{uniformly for every } N \geq \mathrm{e}.$$

Proof. Lemma 5.2 ensures that $\mathcal{Y}_{N,t,\tau}$ has mean zero, and hence

$$\mathrm{Var}\left(\mathcal{Y}_{N,t,\tau}\right) = \mathrm{E}\int_0^t \mathrm{d}r\int_{-\infty}^\infty \mathrm{d}z\int_0^\tau \mathrm{d}r'\int_{-\infty}^\infty \mathrm{d}z'\left(\int_{(r,t)\times\mathbb{R}} v_{N,t}(r,z)D_{r,z}v_{N,t}(s,y)\,\eta(\mathrm{d}s\,\mathrm{d}y)\right)$$

$$\times\left(\int_{(r',\tau)\times\mathbb{R}} v_{N,\tau}(r',z')D_{r',z'}v_{N,\tau}(s,y)\,\eta(\mathrm{d}s\,\mathrm{d}y)\right),$$

which, by Fubini's theorem, is

$$= \int_0^{t\wedge\tau} \mathrm{d}r\int_{-\infty}^\infty \mathrm{d}z\int_0^{t\wedge\tau} \mathrm{d}r'\int_{-\infty}^\infty \mathrm{d}z'\int_{r\vee r'}^{t\wedge\tau} \mathrm{d}s\int_{-\infty}^\infty \mathrm{d}y\, g_{N,t}(r,z)g_{N,\tau}(r',z')g_{N,t}(s,y)g_{N,\tau}(s,y)$$

$$\times\mathrm{E}\left[U(r,z)\cdot D_{r,z}U(s,y)\cdot U(r',z')\cdot D_{r',z'}U(s,y)\right].$$

Combine (1.3) and (2.4) with Lemma 2.1 in order to see that

$$|\mathrm{E}\left[U(r\,,z)\cdot D_{r,z}U(s\,,y)\cdot U(r'\,,z')\cdot D_{r',z'}U(s\,,y)\right]|$$
$$\leq c_{T,4}^2\|D_{r,z}U(s\,,y)\|_4\|D_{r',z'}U(s\,,y)\|_4$$
$$\leq c_{T,4}^2 C_{T,4}^2\,\frac{\boldsymbol{p}_{s-r}(y-z)\boldsymbol{p}_r(z)}{\boldsymbol{p}_s(y)}\frac{\boldsymbol{p}_{s-r'}(y-z')\boldsymbol{p}_{r'}(z')}{\boldsymbol{p}_s(y)}$$
$$=: L_T\,\boldsymbol{p}_{r(s-r)/s}\left(z-\frac{r}{s}y\right)\boldsymbol{p}_{r'(s-r')/s}\left(z'-\frac{r'}{s}y\right).$$

Plug this into the preceding identity for $\mathrm{Var}(\mathcal{Y}_{N,t,\tau})$ in order to see that

$$\mathrm{Var}(\mathcal{Y}_{N,t,\tau})\leq L_T\int\limits_0^{t\wedge\tau}\mathrm{d}r\int\limits_{-\infty}^{\infty}\mathrm{d}z\int\limits_0^{t\wedge\tau}\mathrm{d}r'\int\limits_{-\infty}^{\infty}\mathrm{d}z'\int\limits_{r\vee r'}^{t\wedge\tau}\mathrm{d}s\int\limits_{-\infty}^{\infty}\mathrm{d}y$$

$$\times g_{N,t}(r\,,z)g_{N,\tau}(r'\,,z')g_{N,t}(s\,,y)g_{N,\tau}(s\,,y)\boldsymbol{p}_{r(s-r)/s}\left(z-\frac{r}{s}y\right)\boldsymbol{p}_{r'(s-r')/s}\left(z'-\frac{r'}{s}y\right).$$

We can apply first (5.1), and then the semigroup property of the heat kernel, in order to see that

$$\int\limits_{-\infty}^{\infty}g_{N,t}(r\,,z)\boldsymbol{p}_{r(s-r)/s}\left(z-\frac{r}{s}y\right)\mathrm{d}z$$

$$=\frac{1}{N}\int\limits_0^N\mathrm{d}x\int\limits_{-\infty}^{\infty}\mathrm{d}z\,\boldsymbol{p}_{r(s-r)/s}\left(z-\frac{r}{s}y\right)\boldsymbol{p}_{r(t-r)/t}\left(z-\frac{r}{t}x\right)$$

$$=\frac{1}{N}\int\limits_0^N\boldsymbol{p}_{[r(s-r)/s]+[r(t-r)/t]}\left(\frac{r}{s}y-\frac{r}{t}x\right)\mathrm{d}x.$$

Therefore,

$$\mathrm{Var}(\mathcal{Y}_{N,t,\tau})\leq\frac{L_T}{N^2}\int\limits_0^{t\wedge\tau}\mathrm{d}r\int\limits_0^{t\wedge\tau}\mathrm{d}r'\int\limits_{r\vee r'}^{t\wedge\tau}\mathrm{d}s\int\limits_{-\infty}^{\infty}\mathrm{d}y\int\limits_0^N\mathrm{d}x\int\limits_0^N\mathrm{d}x'\,g_{N,t}(s\,,y)g_{N,\tau}(s\,,y)$$

$$\times\boldsymbol{p}_{[r(s-r)/s]+[r(t-r)/t]}\left(\frac{r}{s}y-\frac{r}{t}x\right)\boldsymbol{p}_{[r'(s-r')/s]+[r'(\tau-r')/\tau]}\left(\frac{r'}{s}y-\frac{r'}{\tau}x'\right).$$

Since $g_{N,\nu}(s,y)\leq\frac{\nu}{s}N^{-1}$ for all $N>0,\nu\geq s>0$ and $y\in\mathbb{R}$,

$$\mathrm{Var}(\mathcal{Y}_{N,t,\tau})\leq\frac{t\tau L_T}{N^4}\int\limits_0^{t\wedge\tau}\mathrm{d}r\int\limits_0^{t\wedge\tau}\mathrm{d}r'\int\limits_{r\vee r'}^{t\wedge\tau}\frac{\mathrm{d}s}{s^2}\int\limits_{-\infty}^{\infty}\mathrm{d}y\int\limits_0^N\mathrm{d}x\int\limits_0^N\mathrm{d}x'$$

$$\times\boldsymbol{p}_{[r(s-r)/s]+[r(t-r)/t]}\left(\frac{r}{s}y-\frac{r}{t}x\right)\boldsymbol{p}_{[r'(s-r')/s]+[r'(\tau-r')/\tau]}\left(\frac{r'}{s}y-\frac{r'}{\tau}x'\right).$$

Now we use scaling [see (4.3)] to see that

$$\boldsymbol{P}_{[r(s-r)/s]+[r(t-r)/t]}\left(\frac{r}{s}y-\frac{r}{t}x\right) = \frac{s}{r}\,\boldsymbol{P}_{[s(s-r)/r]+[s^2(t-s)/(rt)]}\left(y-\frac{s}{t}x\right),$$

with an analogous expression holding for the version with the variables with the primes. This endeavor, and the semigroup property of the heat kernel, together yield

$$\mathrm{Var}(\mathcal{Y}_{N,t,\tau}) \leq \frac{t\tau L_T}{N^4}\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}r}{r}\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}r'}{r'}\int\limits_{r\vee r'}^{t\wedge\tau}\mathrm{d}s\int\limits_0^N\mathrm{d}x\int\limits_0^N\mathrm{d}x'\,\boldsymbol{P}_{\Gamma+\Gamma'}\left(\frac{s}{t}x-\frac{s}{\tau}x'\right),$$

with Γ and Γ' being the following functions whose variable-dependencies are excised for ease of exposition:

$$\Gamma := \frac{s(s-r)}{r}+\frac{s^2(t-s)}{rt},\qquad \Gamma' := \frac{s(s-r')}{r'}+\frac{s^2(\tau-s)}{r'\tau}.$$

A change of variables [$a=sx/t$, $a'=sx'/\tau$] yields

$$\mathrm{Var}(\mathcal{Y}_{N,t,\tau}) \leq \frac{(t\tau)^2 L_T}{N^4}\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}r}{r}\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}r'}{r'}\int\limits_{r\vee r'}^{t\wedge\tau}\mathrm{d}s\,s^{-2}\int\limits_0^{Ns/t}\mathrm{d}a\int\limits_0^{Ns/\tau}\mathrm{d}a'\,\boldsymbol{P}_{\Gamma+\Gamma'}(a-a')$$

$$\leq \frac{(t\tau)^2 L_T}{N^4}\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}r}{r}\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}r'}{r'}\int\limits_{r\vee r'}^{t\wedge\tau}\mathrm{d}s\,s^{-2}\int\limits_0^{Ns/(t\wedge\tau)}\mathrm{d}a\int\limits_0^{Ns/(t\wedge\tau)}\mathrm{d}a'\,\boldsymbol{P}_{\Gamma+\Gamma'}(a-a')$$

$$= \frac{(t\tau)^2 L_T}{\pi(t\wedge\tau)N^3}\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}s}{s}\int\limits_0^s\frac{\mathrm{d}r}{r}\int\limits_0^s\frac{\mathrm{d}r'}{r'}\int\limits_{-\infty}^\infty\mathrm{d}z\,\varphi(z)\mathrm{e}^{-(\Gamma+\Gamma')z^2(t\wedge\tau)^2/(2N^2s^2)},$$

where φ is defined in (6.5) below, and we have used Lemma A.2 in the equality. Since

$$(\Gamma+\Gamma')(t\wedge\tau)^2/s^2 \geq (t\wedge\tau)^2\left(\frac{s(s-r)}{sr}+\frac{s(s-r')}{sr'}\right)+2(t\wedge\tau)((t\wedge\tau)-s),$$

we obtain

$\mathrm{Var}(\mathcal{Y}_{N,t,\tau})$

$$\leq \frac{(t\tau)^2 L_T}{\pi(t\wedge\tau)N^3}\int\limits_{-\infty}^\infty\mathrm{d}z\,\varphi(z)\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}s}{s}\left(\int\limits_0^s\frac{\mathrm{d}r}{r}\mathrm{e}^{-((t\wedge\tau)((t\wedge\tau)-s)+(t\wedge\tau)^2 s(s-r)/(sr))z^2/(2N^2)}\right)^2$$

$$= \frac{(t\tau)^2 L_T}{\pi(t\wedge\tau)N^3}\int\limits_{-\infty}^\infty\mathrm{d}z\,\varphi(z)\int\limits_0^{t\wedge\tau}\frac{\mathrm{d}r}{r}\left(\int\limits_0^r\frac{\mathrm{d}s}{s}\mathrm{e}^{-((t\wedge\tau)((t\wedge\tau)-r)+(t\wedge\tau)^2 r(r-s)/(sr))z^2/(2N^2)}\right)^2,$$

where we simply switch s and r in the equality.

Making the change of variables $(r - s)/s = \theta$, yields

$$\int_0^r \frac{ds}{s} e^{-\frac{[(t\wedge\tau)((t\wedge\tau)-s)/s+(t\wedge\tau)^2(r-s)/(rs)]z^2}{2N^2}} = \int_0^\infty \frac{1}{1+\theta} e^{-\frac{(t\wedge\tau)z^2}{2N^2}(2\theta(t\wedge\tau+(t\wedge\tau)-r)/r)} d\theta.$$

As a consequence,

$\text{Var}(\mathcal{Y}_{N,t,\tau})$

$$\leq \frac{(t\tau)^2 L_T}{\pi(t\wedge\tau)N^3} \int_{-\infty}^\infty dz\, \varphi(z) \int_0^{t\wedge\tau} \frac{1}{r} e^{-\frac{z^2}{N^2}(t\wedge\tau)((t\wedge\tau)-r)/r} \left(\int_0^\infty \frac{1}{1+\theta} e^{-\frac{(t\wedge\tau)^2 z^2\theta}{rN^2}} d\theta\right)^2 dz dr$$

With the further change of variable $\frac{(t\wedge\tau)-r}{r} = \xi$, we obtain

$\text{Var}(\mathcal{Y}_{N,t,\tau})$

$$\leq \frac{(t\tau)^2 L_T}{\pi(t\wedge\tau)N^3} \int_{-\infty}^\infty dz\, \varphi(z) \int_0^\infty \frac{1}{1+\xi} e^{-\frac{(t\wedge\tau)z^2\xi}{N^2}} \left(\int_0^\infty \frac{1}{1+\theta} e^{-\frac{(t\wedge\tau)(\xi+1)z^2\theta}{N^2}} d\theta\right)^2 dz d\xi$$

$$\leq \frac{(t\tau)^2 L_T}{\pi(t\wedge\tau)N^3} \int_{-\infty}^\infty dz\, \varphi(z) \left(\int_0^\infty \frac{1}{1+\theta} e^{-\frac{(t\wedge\tau)z^2\theta}{N^2}} d\theta\right)^3 dz$$

$$= \frac{(t\tau)^2 L_T}{\pi(t\wedge\tau)N^3} \int_{-\infty}^\infty dz\, \varphi(z) \left(\int_0^\infty \frac{1}{\theta + \frac{(t\wedge\tau)z^2}{N^2}} e^{-\theta} d\theta\right)^3 dz.$$

We have

$$\int_0^\infty \frac{1}{\theta + \frac{(t\wedge\tau)z^2}{N^2}} e^{-\theta} d\theta \leq \int_1^\infty e^{-\theta} d\theta + \int_0^1 \frac{1}{\theta + \frac{(t\wedge\tau)z^2}{N^2}} d\theta = e^{-1} + \log\left(1 + \frac{N^2}{(t\wedge\tau)z^2}\right)$$

$$\leq e^{-1} + 2\log N + \log(1 + 1/(t\wedge\tau)) + \log(1 + z^{-2})$$

Taking into account that

$$\int_{\mathbb{R}} \varphi(z)(1 + \log(1 + z^{-2}))^3 dz < \infty,$$

we obtain the desired estimate and complete the proof. $\quad\square$

We now conclude this section with the following.

Proof of Theorem 1.2. From Proposition (5.1) [with $t = \tau$], we see that for all $T > 0$ there exists a number $K_T > 0$ such that

$$\operatorname{Var}\langle D\mathcal{S}_{N,t}, v_{N,t}\rangle_{\mathcal{H}} \leq K_T \frac{(\log N)^3}{N^3} \quad \text{for all } t \in (0, T) \text{ and } N \geq \mathrm{e}.$$

By (5.2) and Proposition 2.2,

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\operatorname{Var}(\mathcal{S}_{N,t})}}, Z\right)$$

$$\leq 2\sqrt{\operatorname{Var}\left\langle \frac{D\mathcal{S}_{N,t}}{\sqrt{\operatorname{Var}(\mathcal{S}_{N,t})}}, \frac{v_{N,t}}{\sqrt{\operatorname{Var}(\mathcal{S}_{N,t})}}\right\rangle_{\mathcal{H}}}$$

$$\leq 2\sqrt{K_T}\frac{(\log N)^{3/2}}{N^{3/2}\operatorname{Var}(\mathcal{S}_{N,t})} \quad \text{uniformly for all } t \in (0, T) \text{ and } N \geq \mathrm{e}.$$

Proposition 4.1 ensures that $\operatorname{Var}(\mathcal{S}_{N,t}) \sim 2t\log(N)/N$ as $N \to \infty$, which concludes the proof. □

6. Proof of Theorem 1.3

In order to prove Theorem 1.3 we need to establish the weak convergence of the finite-dimensional distributions, as well as tightness. The following addresses tightness.

Proposition 6.1 *(Tightness). For every $T > 0$, $k \geq 2$, and $\gamma \in (0, 1/6)$, there exists a number $L = L(T, k, \gamma) > 0$ such that for all $\varepsilon \in (0, 1]$,*

$$\sup_{0 < t \leq T} \mathrm{E}\left(|\mathcal{S}_{N,t+\varepsilon} - \mathcal{S}_{N,t}|^k\right) \leq L\varepsilon^{\gamma k}\left(\frac{\log N}{N}\right)^{k/2} \quad \text{uniformly for all } N \geq \mathrm{e}.$$

The proof of Proposition 6.1 hinges on the following lemma, which is a useful inequality when t stays away from zero.

Lemma 6.2. *For every $T > 0$, $k \geq 2$ and $\delta > 0$, there exists a number $K = K(T, k, \delta) > 0$ such that*

$$\mathrm{E}\left(|\mathcal{S}_{N,t+\varepsilon} - \mathcal{S}_{N,t}|^k\right) \leq \frac{K\varepsilon^{k/2}}{(t \wedge 1)^{k(1+\delta)/2}}\left(\frac{\log N}{N}\right)^{k/2},$$

uniformly for all $N \geq \mathrm{e}$, $t \in (0, T]$, and $\epsilon \in (0, 1)$.

Proof. Thanks to (1.5) and (1.7), we may write the following: For all $N, t > 0$,

$$
\mathcal{S}_{N,t+\varepsilon} - \mathcal{S}_{N,t} = \frac{1}{N} \int_0^N [U(t+\varepsilon, x) - U(t, x)]\, \mathrm{d}x
$$

$$
= \int_{(0,t)\times\mathbb{R}} U(s, y)\mathcal{A}(s, y)\, \eta(\mathrm{d}s\, \mathrm{d}y) + \int_{(t,t+\varepsilon)\times\mathbb{R}} U(s, y)\mathcal{B}(s, y)\, \eta(\mathrm{d}s\, \mathrm{d}y),
$$

almost surely, where

$$
\mathcal{A}(s, y) := \frac{1}{N} \int_0^N \left[\boldsymbol{p}_{s(t+\varepsilon-s)/(t+\varepsilon)}\left(y - \frac{sx}{t+\varepsilon}\right) - \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{sx}{t}\right) \right] \mathrm{d}x, \qquad \text{and}
$$

$$
\mathcal{B}(s, y) := \frac{1}{N} \int_0^N \boldsymbol{p}_{s(t+\varepsilon-s)/(t+\varepsilon)}\left(y - \frac{sx}{t+\varepsilon}\right) \mathrm{d}x,
$$

and the dependence on the parameters N and ε are subsumed for ease of notation. Thus,

$$
\|\mathcal{S}_{N,t+\varepsilon} - \mathcal{S}_{N,t}\|_k \le T_{\mathcal{A}} + T_{\mathcal{B}}, \tag{6.1}
$$

where

$$
T_{\mathcal{A}} := \left\| \int_{(0,t)\times\mathbb{R}} U(s, y)\mathcal{A}(s, y)\, \eta(\mathrm{d}s\, \mathrm{d}y) \right\|_k \qquad \text{and}
$$

$$
T_{\mathcal{B}} := \left\| \int_{(t,t+\varepsilon)\times\mathbb{R}} U(s, y)\mathcal{B}(s, y)\, \eta(\mathrm{d}s\, \mathrm{d}y) \right\|_k.
$$

We will estimate $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ separately and in reverse order.

To estimate $T_{\mathcal{B}}$ we appeal to the BDG inequality (with BDG constant c_k) as follows:

$$
T_{\mathcal{B}}^2 \le c_k \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, \|U(s, y)\|_k^2 |\mathcal{B}(s, y)|^2 \le c_k c_{k,T}^2 \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, |\mathcal{B}(s, y)|^2
$$

$$
= \frac{c_k c_{k,T}^2}{N^2} \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \int_0^N \mathrm{d}x_1 \int_0^N \mathrm{d}x_2\, \boldsymbol{p}_{s(t+\varepsilon-s)/(t+\varepsilon)}\left(y - \frac{sx_1}{t+\varepsilon}\right)
$$

$$
\times \boldsymbol{p}_{s(t+\varepsilon-s)/(t+\varepsilon)}\left(y - \frac{sx_2}{t+\varepsilon}\right),
$$

where we used (2.4) to deduce the second inequality. Rearrange the integrals and compute the dy-integral first to see from the semigroup property of the heat kernel that

$$T_{\mathcal{B}}^2 \le \frac{c_k c_{k,T}^2}{N^2} \int\limits_t^{t+\varepsilon} \mathrm{d}s \int\limits_0^N \mathrm{d}x_1 \int\limits_0^N \mathrm{d}x_2 \, \boldsymbol{p}_{2s(t+\varepsilon-s)/(t+\varepsilon)} \left(\frac{s(x_1-x_2)}{t+\varepsilon} \right)$$

$$= \frac{c_k c_{k,T}^2 (t+\varepsilon)^2}{N^2} \int\limits_t^{t+\varepsilon} \frac{\mathrm{d}s}{s^2} \int\limits_0^{sN/(t+\varepsilon)} \mathrm{d}x_1 \int\limits_0^{sN/(t+\varepsilon)} \mathrm{d}x_2$$

$$\times \boldsymbol{p}_{2s(t+\varepsilon-s)/(t+\varepsilon)}(x_1-x_2),$$

after a change of variables. Since the dx_2-integral is bounded above by one, it follows that

$$T_{\mathcal{B}}^2 \le \frac{c_k c_{k,T}^2 (t+\varepsilon)}{N} \int\limits_t^{t+\varepsilon} \frac{\mathrm{d}s}{s} < \frac{c_k c_{k,T}^2 (t+\varepsilon)}{Nt} \varepsilon. \tag{6.2}$$

The estimation of $T_{\mathcal{A}}$ is more involved, though it starts in the same way as did the process of bounding $T_{\mathcal{B}}$. Namely, we write, using the BDG inequality,

$$T_{\mathcal{A}}^2 \le c_k \int\limits_0^t \mathrm{d}s \int\limits_{-\infty}^{\infty} \mathrm{d}y \, \|U(s,y)\|_k^2 |\mathcal{A}(s,y)|^2$$

$$\le c_k c_{k,T}^2 \int\limits_0^t \mathrm{d}s \int\limits_{-\infty}^{\infty} \mathrm{d}y \, |\mathcal{A}(s,y)|^2 [\text{by } (2.4)] \tag{6.3}$$

$$= \frac{c_k c_{k,T}^2}{2\pi} \int\limits_0^t \mathrm{d}s \int\limits_{-\infty}^{\infty} \mathrm{d}\xi \, \left| \widehat{\mathcal{A}(s)}(\xi) \right|^2 = \frac{t c_k c_{k,T}^2}{2\pi N} \int\limits_0^t \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^{\infty} \mathrm{d}\xi \, \left| \widehat{\mathcal{A}(s)}(t\xi/(Ns)) \right|^2,$$

owing to Plancherel's theorem and a change of variables. The correct change of variables is slightly tricky to find. But once we have it set up, as we have done above, we note that

$$\widehat{\mathcal{A}(s)}(t\xi/(Ns)) = \frac{1}{N} \int\limits_0^N \left[\exp\left(i\frac{tx\xi}{N(t+\varepsilon)} - \frac{t^2(t+\varepsilon-s)\xi^2}{2s(t+\varepsilon)N^2} \right) - \exp\left(i\frac{x\xi}{N} - \frac{t(t-s)\xi^2}{2sN^2} \right) \right] \mathrm{d}x$$

$$= \int\limits_0^1 \left[\exp\left(i\frac{ty\xi}{t+\varepsilon} - \frac{t^2(t+\varepsilon-s)\xi^2}{2s(t+\varepsilon)N^2} \right) - \exp\left(iy\xi - \frac{t(t-s)\xi^2}{2sN^2} \right) \right] \mathrm{d}y$$

$$= J_1 + J_2,$$

where

$$
J_1 := \int\limits_0^1 e^{ity\xi/(t+\varepsilon)}\, \mathrm{d}y \times \left[\exp\left(-\frac{t^2(t+\varepsilon-s)\xi^2}{2s(t+\varepsilon)N^2} \right) - \exp\left(-\frac{t(t-s)\xi^2}{2sN^2} \right) \right], \quad \text{and}
$$

$$
J_2 := \int\limits_0^1 \left[\exp\left(i\frac{ty\xi}{t+\varepsilon} \right) - \exp(iy\xi) \right] \mathrm{d}y \times \exp\left(-\frac{t(t-s)\xi^2}{2sN^2} \right).
$$

Since $(a+b)^2 \le 2a^2 + 2b^2$ for all $a,b \in \mathbb{R}$, we see from (6.3) that

$$
T_{\mathcal{A}}^2 \le \frac{2tc_k c_{k,T}^2}{2\pi N} \int\limits_0^t \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^\infty \mathrm{d}\xi\, |J_1|^2 + \frac{2tc_k c_{k,T}^2}{2\pi N} \int\limits_0^t \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^\infty \mathrm{d}\xi\, |J_2|^2. \tag{6.4}
$$

Define,

$$
\varphi(z) := \frac{1-\cos z}{z^2} \qquad \text{for all } z \in \mathbb{R} \setminus \{0\}, \tag{6.5}
$$

and $\varphi(0) = 1/2$ to preserve continuity. It is then easy to see that

$$
\begin{aligned}
|J_1| &= \sqrt{2\varphi\left(\frac{t\xi}{t+\varepsilon} \right)}\, \left| \exp\left(-\frac{t^2(t+\varepsilon-s)\xi^2}{2s(t+\varepsilon)N^2} \right) - \exp\left(-\frac{t(t-s)\xi^2}{2sN^2} \right) \right| \\
&= \sqrt{2\varphi\left(\frac{t\xi}{t+\varepsilon} \right)}\, \exp\left(-\frac{t(t-s)\xi^2}{2sN^2} \right) \left| 1 - \exp\left(-\frac{\varepsilon t\xi^2}{2(t+\varepsilon)N^2} \right) \right|
\end{aligned}
$$

Therefore,

$$
\int\limits_0^t \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^\infty \mathrm{d}\xi\, |J_1|^2
$$

$$
\le 2 \int\limits_0^t \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^\infty \mathrm{d}\xi\, \varphi\left(\frac{t\xi}{t+\varepsilon} \right) \exp\left(-\frac{t(t-s)\xi^2}{sN^2} \right) \left| 1 - \exp\left(-\frac{\varepsilon t\xi^2}{2(t+\varepsilon)N^2} \right) \right|^2
$$

$$
\le C \int\limits_0^t \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^\infty \mathrm{d}\xi\, \frac{1}{\xi^2} \exp\left(-\frac{t(t-s)\xi^2}{sN^2} \right) \left| 1 - \exp\left(-\frac{\varepsilon t\xi^2}{2(t+\varepsilon)N^2} \right) \right|^2
$$

$$
\le \frac{C}{N} \int\limits_1^\infty \frac{\mathrm{d}r}{r} \int\limits_{-\infty}^\infty \mathrm{d}z\, \frac{1}{z^2} \exp\left(-t(r-1)z^2 \right) \left| 1 - \exp\left(-\frac{\varepsilon t z^2}{2(t+\varepsilon)} \right) \right|^2
$$

$$
\le \frac{C}{N} \int\limits_1^\infty \frac{\mathrm{d}r}{r} \int\limits_{-\infty}^\infty \mathrm{d}z\, \frac{1}{z^2} \exp\left(-t(r-1)z^2 \right) \frac{\varepsilon t z^2}{2(t+\varepsilon)}
$$

$$\leq \frac{C\varepsilon}{2N} \int\limits_{1}^{\infty} \frac{\mathrm{d}r}{r} \int\limits_{-\infty}^{\infty} \mathrm{d}z \ \exp\left(-t(r-1)z^2\right)$$

$$= \frac{C\varepsilon}{N} \int\limits_{1}^{\infty} \frac{1}{r\sqrt{t(r-1)}} \mathrm{d}r$$

$$= \frac{C\varepsilon}{N\sqrt{t}}, \tag{6.6}$$

where in the third step we have changed the variables $z = \xi/N$ and $r = t/s$, in the fourth step we have applied the inequality $(1 - e^{-x^2})^2 \leq 1 - e^{-x^2} \leq x^2$, and the constant C is a generic constant that may change values at each appearance.

Next, we estimate the same quantity but where J_1 is replaced by J_2. A few lines of computation show that

$$\int\limits_{0}^{1} \left[\exp\left(i\frac{ty\xi}{t+\varepsilon}\right) - \exp(iy\xi)\right] \mathrm{d}y = \frac{e^{i\xi}}{i\xi}\left[\exp\left(\frac{-i\varepsilon\xi}{t+\varepsilon}\right) - 1\right] + \frac{\varepsilon}{it\xi}\left[\exp\left(\frac{it\xi}{t+\varepsilon}\right) - 1\right],$$

provided that $\xi \neq 0$. Because $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$,

$$\left|\int\limits_{0}^{1} \left[\exp\left(i\frac{ty\xi}{t+\varepsilon}\right) - \exp(iy\xi)\right] \mathrm{d}y\right|^2 \leq \frac{4}{\xi^2}\left[1 - \cos\left(\frac{\varepsilon\xi}{t+\varepsilon}\right)\right] + \frac{4\varepsilon^2}{t^2\xi^2}\left[1 - \cos\left(\frac{t\xi}{t+\varepsilon}\right)\right]$$

$$\leq \frac{2}{\xi^2}\left(\frac{\varepsilon\xi}{t+\varepsilon}\right)^2 + \frac{2\varepsilon^2}{t^2\xi^2}\left(\frac{t\xi}{t+\varepsilon}\right)^2 < \frac{4\varepsilon^2}{t^2+\varepsilon^2},$$

since $1 - \cos\theta \leq \frac{1}{2}\theta^2$ for all $\theta \in \mathbb{R}$. Alternatively, we could have used the tautological bound, $1 - \cos\theta \leq 2$ in order to deduce

$$\left|\int\limits_{0}^{1} \left[\exp\left(i\frac{ty\xi}{t+\varepsilon}\right) - \exp(iy\xi)\right] \mathrm{d}y\right|^2 \leq \frac{8}{\xi^2} + \frac{8\varepsilon^2}{t^2\xi^2} \leq \frac{8}{\xi^2}\left(\frac{t^2+\varepsilon^2}{t^2}\right).$$

Combine the preceding two bounds in order to see that

$$\left|\int\limits_{0}^{1} \left[\exp\left(i\frac{ty\xi}{t+\varepsilon}\right) - \exp(iy\xi)\right] \mathrm{d}y\right|^2 \leq 8\left\{\left(\frac{\varepsilon^2}{t^2+\varepsilon^2}\right) \wedge \left(\frac{t^2+\varepsilon^2}{t^2\xi^2}\right)\right\}.$$

Consequently,

$$\int\limits_{0}^{t} \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^{\infty} \mathrm{d}\xi \ |J_2|^2 \leq 8 \int\limits_{0}^{t} \frac{\mathrm{d}s}{s} \int\limits_{-\infty}^{\infty} \mathrm{d}\xi \ \exp\left(-\frac{t(t-s)\xi^2}{sN^2}\right)\left[\left(\frac{\varepsilon^2}{t^2+\varepsilon^2}\right) \wedge \left(\frac{t^2+\varepsilon^2}{t^2\xi^2}\right)\right]$$

$$= \frac{8 \log N}{t} \int_{-\infty}^{\infty} G_{N,t}(\xi) \left[\left(\frac{\varepsilon^2}{t^2 + \varepsilon^2} \right) \wedge \left(\frac{t^2 + \varepsilon^2}{t^2 \xi^2} \right) \right] d\xi,$$

where $G_{N,t}$ is defined in (A.1) in the Appendix. Lemma A.1 of the Appendix now tells us that

$$
\begin{aligned}
\int_0^t \frac{ds}{s} \int_{-\infty}^{\infty} d\xi \, |J_2|^2 & \\
&\leq 56 \log(N) \log_+(1/t) \int_{-\infty}^{\infty} \left[\left(\frac{\varepsilon^2}{t^2 + \varepsilon^2} \right) \wedge \left(\frac{t^2 + \varepsilon^2}{t^2 \xi^2} \right) \right] \log_+(1/|\xi|) \, d\xi \\
&= 56 \log(N) \log_+(1/t) \left(\frac{t^2 + \varepsilon^2}{t^2} \right) \int_{-\infty}^{\infty} \left[\left(\frac{\varepsilon^2 t^2}{(t^2 + \varepsilon^2)^2} \right) \wedge \frac{1}{\xi^2} \right] \log_+(1/|\xi|) \, d\xi \\
&< \frac{560 \log(N) \log_+(1/t) \varepsilon}{t};
\end{aligned}
\tag{6.7}
$$

see Lemma A.3 in the Appendix. Combine (6.4) with (6.6) and (6.7) in order to find that

$$T_{\mathcal{A}}^2 \leq a_{T,k,\delta} \frac{\log N}{N} \frac{\varepsilon}{t^{1+\delta}},$$

where $a_{T,k,\delta}$ is a real number depends only on (T, k, δ). We combine this bound with (6.2) and then (6.1) to conclude the proof. $\quad\square$

We are now ready for the following.

Proof of Proposition 6.1. We assume without incurring loss in generality that $T > 1/e$. Choose and fix two arbitrary numbers $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. On one hand, Lemma 6.2 implies that, uniformly for all $\varepsilon \in (0, 1/e)$, $N \geq e$, and $t \in (\varepsilon^\beta, T]$,

$$\|\mathcal{S}_{N,t+\varepsilon} - \mathcal{S}_{N,t}\|_k \leq M \varepsilon^{(1-2\beta(1+\delta))/2} \sqrt{\frac{\log N}{N}}, \tag{6.8}$$

with $M := K^{1/k}$. [The condition $T > 1/e$ is there merely to ensure that $(\varepsilon^\beta, T] \neq \varnothing$]. On the other hand, Lemma 2.4 implies the existence of a real number $M' = M'(T, k, \alpha)$ such that, uniformly for all $N \geq e$ and $t \in (0, \varepsilon^\beta]$,

$$\|\mathcal{S}_{N,t+\varepsilon} - \mathcal{S}_{N,t}\|_k \leq \|\mathcal{S}_{N,t+\varepsilon}\|_k + \|\mathcal{S}_{N,t}\|_k \leq M' \varepsilon^{\beta\alpha/2} \sqrt{\frac{\log N}{N}}. \tag{6.9}$$

Choose $\beta = (2 + \alpha + 2\delta)^{-1}$ to match the exponents of ε in (6.8) and (6.9) and hence conclude the asserted inequality of the proposition with $L := M \vee M'$ and $\gamma := \alpha/\{2(2+$

$\alpha + 2\delta)\}$. To finish the proof we note that γ can be any number in $(0, 1/6)$ since $\alpha \in (0, 1)$ and $\delta > 0$ are arbitrary. $\quad \square$

Armed with Proposition 6.1, we conclude the section with the following.

Proof of Theorem 1.3. Choose and fix some $T > 0$. By Lemma 2.4 and Proposition 6.1, a standard application of Kolmogorov's continuity theorem and the Arzelà-Ascoli theorem ensures that $\{\sqrt{N/\log(N)}\,\mathcal{S}_{N,\bullet}\}_{N \geq e}$ is a tight net of processes on $C[0, T]$. Therefore, it remains to prove that the finite-dimensional distributions of the process $t \mapsto \sqrt{N/\log N}\,\mathcal{S}_{N,t}$ converge to those of $\sqrt{2}B$; see for example Billingsley [2].

Let us choose and fix some $T > 0$ and $m \geq 1$ points $t_1, \ldots, t_m \in (0, T)$. Proposition 4.1 ensures that, for every $i, j = 1, \ldots, m$,

$$\text{Cov}\left(\mathcal{S}_{N,t_i}, \mathcal{S}_{N,t_j}\right) \sim 2(t_i \wedge t_j)\frac{\log N}{N} \qquad \text{as } N \to \infty. \tag{6.10}$$

Therefore, there exists $N_0 > 0$ such that

$$\text{Var}(\mathcal{S}_{N,t_i}) \geq t_i \frac{\log N}{N} \qquad \text{for every } i = 1, \ldots, m \text{ and } N > N_0. \tag{6.11}$$

Choose and fix an arbitrary $N > N_0$, and consider the following random variables:

$$F_i := \frac{\mathcal{S}_{N,t_i}}{\sqrt{\text{Var}(\mathcal{S}_{N,t_i})}} \qquad \text{for } i = 1, \ldots, m,$$

and define $C_{i,j} := \text{Cov}(F_i, F_j)$ for every $i, j = 1, \ldots, m$. We will write $F := (F_1, \ldots, F_m)$, and let $G = (G_1, \ldots, G_m)$ denote a centered Gaussian random vector with covariance matrix $C = (C_{i,j})_{1 \leq i,j \leq m}$.

Recall from (5.1) the random fields $v_{N,t_1}, \ldots, v_{N,t_m}$, and define rescaled random fields $V_1, \ldots, V_m$ as follows:

$$V_i := \frac{v_{N,t_i}}{\sqrt{\text{Var}(\mathcal{S}_{N,t_i})}} \qquad \text{for } i = 1, \ldots, m.$$

According to (5.2), $F_i = \delta(V_i)$ for all $i = 1, \ldots, m$. Lemma 5.2 ensures that $\text{E}\langle DF_i, V_j\rangle_{\mathcal{H}} = C_{i,j}$ for all $i, j = 1, \ldots, m$. Therefore, Lemma 2.3 ensures that

$$|\text{E}h(F) - \text{E}h(G)| \leq \tfrac{1}{2}\|h''\|_{\infty} \sqrt{\sum_{i,j=1}^{m} \text{Var}\langle DF_i, V_j\rangle_{\mathcal{H}}},$$

for all $h \in C_b^2(\mathbb{R}^m)$. Proposition 5.1 and (6.11) together assure us that

$$\text{Var}\langle DF_i, V_j\rangle_{\mathcal{H}} = \frac{\text{Var}\langle D\mathcal{S}_{N,t_i}, v_{N,t_j}\rangle_{\mathcal{H}}}{\text{Var}(\mathcal{S}_{N,t_i})\text{Var}(\mathcal{S}_{N,t_j})} \leq \frac{K_T \log N}{N \min_{1 \leq k \leq m} t_k}.$$

whence

$$|\mathrm{E}h(F) - \mathrm{E}h(G)| \le c\|h''\|_\infty \sqrt{\log N}/\sqrt{N}, \tag{6.12}$$

for $c = \frac{1}{2}\sqrt{K_T/\min_{1\le k\le m} t_k}$.

Now we let $N \to \infty$: Thanks to (6.10), $C_{i,j} \to (t_i \wedge t_j)/\sqrt{t_i t_j}$ whence G converges weakly to $(B_{t_i}/\sqrt{t_i})_{1\le i\le m}$ as $N \to \infty$. Therefore, it follows from (6.12) that F converges weakly to $(B_{t_i}/\sqrt{t_i})_{1\le i\le m}$ as $N \to \infty$. One more appeal to (6.10) shows that

$$\sqrt{\frac{N}{\log N}}\left(\frac{\mathcal{S}_{N,t_1}}{\sqrt{2t_1}}, \ldots, \frac{\mathcal{S}_{N,t_m}}{\sqrt{2t_m}}\right) \xrightarrow{\mathrm{d}} \left(\frac{B_{t_1}}{\sqrt{t_1}}, \ldots, \frac{B_{t_m}}{\sqrt{t_m}}\right) \qquad \text{as } N \to \infty.$$

It follows from this fact that the finite-dimensional distributions of $t \mapsto \sqrt{N/\log N}\,\mathcal{S}_{N,t}$ converge to those of $\sqrt{2}\,B$ as $N \to \infty$. This verifies the remaining goal of this proof. $\quad\square$

Acknowledgment

Appendix A

We include in this section a few technical results that have been used along the paper. In order to describe the first result, define

$$G_{N,t}(x) := \frac{t}{\log N}\int_0^t \exp\left(-\frac{(t-s)t}{s}\cdot\frac{x^2}{N^2}\right)\frac{\mathrm{d}s}{s}$$

$$\text{for all } N, t > 0 \text{ and } x \in \mathbb{R}\setminus\{0\}. \tag{A.1}$$

Lemma A.1. *For every $t > 0$ and $x \in \mathbb{R}\setminus\{0\}$,*

$$\sup_{N\ge \mathrm{e}} G_{N,t}(x) \le 7t\log_+(1/t)\log_+(1/|x|),$$

where we recall that $\log_+(w) := \log(\mathrm{e} + w)$ for all $w \ge 0$. Moreover,

$$\lim_{N\to\infty} G_{N,t}(x) = 2t \qquad \text{for every } t > 0 \text{ and } x \in \mathbb{R}. \tag{A.2}$$

Proof. We change variables in order to see that

$$
G_{N,t}(x) = \frac{t}{\log N} \int_0^\infty \frac{\mathrm{e}^{-s}}{s + \frac{tx^2}{N^2}} \, \mathrm{d}s = \frac{t}{\log N} (A_N - B_N + C_N),
$$

where

$$
A_N := \int_0^1 \frac{\mathrm{d}s}{s + \frac{tx^2}{N^2}} = \log\left(\frac{N^2}{tx^2} + 1\right), \quad B_N := \int_0^1 \frac{1 - \mathrm{e}^{-s}}{s + \frac{tx^2}{N^2}} \, \mathrm{d}s, \quad C_N := \int_1^\infty \frac{\mathrm{e}^{-s}}{s + \frac{tx^2}{N^2}} \, \mathrm{d}s.
$$

This proves (A.2) because $B_N, C_N \in (0,1)$. Next, we observe that

$$
\frac{N^2}{tx^2} + 1 \le N^2 \left(\mathrm{e} + t^{-1}\right) \left(\mathrm{e} + |x|^{-2}\right),
$$

whence

$$
A_N \le 2 \log N + \log_+(1/t) + 2 \log_+(1/|x|) \le 5 \log(N) \log_+(1/t) \log_+(1/|x|),
$$

for all $N \ge \mathrm{e}$, $t > 0$, and all non-zero x. This does the job since $B_N + C_N \le 2$, which is manifestly less than or equal to $2 \log_+(1/t) \log_+(1/|x|)$. $\quad\square$

The following lemma provides a useful heat-kernel formula.

Lemma A.2. *For all $N, t > 0$, we have*

$$
\int_0^N \mathrm{d}x_1 \int_0^N \mathrm{d}x_2 \; \boldsymbol{p}_t(x_1 - x_2) = \frac{N}{\pi} \int_{-\infty}^\infty \varphi(z) \mathrm{e}^{-tz^2/(2N^2)} \, \mathrm{d}z,
$$

where $\varphi(z)$ was defined in (6.5).

Proof. Plancherel's theorem implies that

$$
\int_0^N \mathrm{d}x_1 \int_0^N \mathrm{d}x_2 \; \boldsymbol{p}_t(x_1 - x_2) = \frac{1}{2\pi} \int_{-\infty}^\infty |\widehat{\boldsymbol{1}_{[0,N]}}(y)|^2 \mathrm{e}^{-ty^2/2} \, \mathrm{d}y
$$

$$
= \frac{N^2}{2\pi} \int_{-\infty}^\infty |\widehat{\boldsymbol{1}_{[0,1]}}(Ny)|^2 \mathrm{e}^{-ty^2/2} \, \mathrm{d}y.
$$

A change of variables $[z = Ny]$ implies the lemma, since $|\widehat{\boldsymbol{1}_{[0,1]}}(z)|^2 = 2\varphi(z)$ for all $z \in \mathbb{R}$. $\quad\square$

Finally, we mention the following simple inequality.

Lemma A.3. *For every $\varepsilon \in (0,1)$,*

$$\int_{-\infty}^{\infty} \left(\varepsilon \wedge \frac{1}{z^2} \right) \log_+(1/|z|) \, \mathrm{d}z < 10\sqrt{\varepsilon}.$$

Proof. Let $J(\varepsilon)$ denote the integral in question. Because $\varepsilon < 1$ and $\log(2\mathrm{e}) \le 2$,

$$J(\varepsilon) = 4 \int_{1/\mathrm{e}}^{\infty} \left(\varepsilon \wedge \frac{1}{z^2} \right) \mathrm{d}z + 2\varepsilon \int_0^{1/\mathrm{e}} \log(1/z) \, \mathrm{d}z < 4\varepsilon \int_{1/\mathrm{e}}^{\infty} \left(1 \wedge \frac{1}{\varepsilon z^2} \right) \mathrm{d}z + 2\varepsilon,$$

since $z \mapsto \log(1/z)$ defines a probability density function on $(0,1)$ and $0 < \varepsilon < 1$. Change variables to see that

$$J(\varepsilon) < 4\sqrt{\varepsilon} \int_{\sqrt{\varepsilon}/\mathrm{e}}^{\infty} \left(1 \wedge \frac{1}{r^2} \right) \mathrm{d}r + 2\varepsilon = 8\sqrt{\varepsilon} + 2\left(1 - \frac{2}{\mathrm{e}} \right) \varepsilon,$$

which readily implies the result since $\varepsilon < \sqrt{\varepsilon}$. □

Lemma A.4. *Let $c_{T,k}$ be the constant defined in* (2.4) *and set $C_T := \pi^{1/4} 2^{-1/2} c_{T,2}$. Then,*

$$\sup_{x \in \mathbb{R}} \|U(t,x) - 1\|_2 \le C_T t^{1/4} \qquad \textit{for all } t \in (0,T].$$

Proof. Owing to (1.5), $\mathrm{E}[U(t,x)] = 1$ for all $t \in (0,T]$ and $x \in \mathbb{R}$, and

$$\mathrm{Var}[U(t,x)] = \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \, \left| \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x \right) \right|^2 \mathrm{E}\left(|U(s,y)|^2 \right)$$

$$\le c_{T,2}^2 \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \, \left| \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x \right) \right|^2 \qquad \text{[see (1.3) and (2.4)]}$$

$$= c_{T,2}^2 \int_0^t \boldsymbol{p}_{2s(t-s)/t}(0)\mathrm{d}s = c_{T,2}^2 \sqrt{\pi t/4},$$

thanks to the semigroup property of the heat kernel and a few computations. This completes the proof. □

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