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NONLINEAR STOCHASTIC HEAT EQUATION DRIVEN BY SPATIALLY COLORED NOISE: MOMENTS AND INTERMITTENCY*

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Abstract In this article, we study the nonlinear stochastic heat equation in the spatial domain $\mathbb{R}^d$ subject to a Gaussian noise which is white in time and colored in space. The spatial correlation can be any symmetric, nonnegative and nonnegative-definite function that satisfies Dalang's condition. We establish the existence and uniqueness of a random field solution starting from measure-valued initial data. We find the upper and lower bounds for the second moment. With these moment bounds, we first establish some necessary and sufficient conditions for the phase transition of the moment Lyapunov exponents, which extends the classical results from the stochastic heat equation on $\mathbb{Z}^d$ to that on $\mathbb{R}^d$. Then, we prove a localization result for the intermittency fronts, which extends results by Conus and Khoshnevisan [9] from one space dimension to higher space dimension. The linear case has been recently proved by Huang et al [17] using different techniques.

Key words Stochastic heat equation; moment estimates; phase transition; intermittency; intermittency front; measure-valued initial data; moment Lyapunov exponents

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1 Introduction

In this article, we will study the following stochastic heat equation (SHE)

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\,\Delta\right) u(t,x) = \rho(u(t,x))\,\dot{M}(t,x), \quad x \in \mathbb{R}^d,\ t > 0\,, \\ u(0,\cdot) = \mu(\cdot), \end{cases} \tag{1.1}$$

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where $\dot{M}$ is a Gaussian noise (white in time and homogeneously colored in space), $\nu > 0$ is the diffusion parameter, and ρ is a globally Lipschitz continuous function satisfying linear growth condition:

$$L_\rho := \sup_{z\in\mathbb{R}} \frac{|\rho(z)|}{|z|} < \infty.$$

The initial data μ is a deterministic and locally finite (regular) Borel measure. Informally,

$$\mathbb{E}\left[\dot{M}(t,x)\dot{M}(s,y)\right] = \delta_0(t-s)f(x-y),$$

where δ_0 is the Dirac delta measure with unit mass at zero and f is a "correlation function" (that is, a nonnegative, nonnegative definite, and symmetric function that is not identically zero). The Fourier transform of f is denoted by $\hat{f}$

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} \exp\left(-\mathrm{i}\,\xi\cdot x\right) f(x)\mathrm{d}x.$$

In general, $\hat{f}$ is again a nonnegative and nonnegative-definite measure, which is usually called the spectral measure. When $\hat{f}$ is genuinely a measure, $\hat{f}(\xi)\mathrm{d}\xi$ is to be understood as $\hat{f}(\mathrm{d}\xi)$. For the existence of a random field solution (see Definition 2.3) to (1.1), a necessary condition for the correlation function f is Dalang's condition [10, 12]:

$$\Upsilon(\beta) := (2\pi)^{-d}\int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{\beta+|\xi|^2} < +\infty \quad \text{for some and hence for all } \beta > 0. \tag{1.2}$$

For the lower bound of the second moment, we will need the following assumption on ρ:

$$l_\rho := \inf_{x\in\mathbb{R}} \frac{\rho(x)}{|x|} > 0. \tag{1.3}$$

The case when $\rho(u) = \lambda u$, for some $\lambda\in\mathbb{R}$, is called the linear case or the Parabolic Anderson model.

The main contribution of this article is the finding of the point-wise moment formula for $\mathbb{E}\left[u(t,x)u(t,x')\right]$. In the linear case, that is, $\rho(u) = \lambda u$, we can write

$$\mathbb{E}\left[u(t,x)u(t,x')\right] = \lambda^{-2}\iint_{\mathbb{R}^{2d}} \mu(\mathrm{d}z)\mu(\mathrm{d}z')\,\mathcal{K}(t,x-z,x'-z';z'-z), \tag{1.4}$$

for some kernel function $\mathcal{K}$. When ρ is nonlinear, the above moment formula turns into lower and upper bounds (see (2.9)–(2.11) below). In order to use this formula (1.4), one needs to find some easy-to-use, while nontrivial, upper and lower bounds for this kernel function $\mathcal{K}$. It turns out, as usual, that the upper bound is relatively easy to obtain, while to obtain some nontrivial lower bound is a task that is much more challenging. One of central aim of this article is to derive some nontrivial lower bounds for the nonlinear SHE. For this purpose, we need to introduce an asymmetric convolution operator "$\rhd$" as is defined in (2.4) below. In the end, we obtain a lower bound of the following form [1]: If the initial measure μ is nonnegative such that $\int_K \mu(\mathrm{d}z) > 0$ for some compact set $K\subseteq\mathbb{R}^d$, then for any $t>0$ and $x\in\mathbb{R}^d$, it holds that

$$\mathbb{E}\left[u(t,x)^2\right] \ge l_\rho^{-2}\left(\int_K G(t/2,z)\mu(\mathrm{d}z)\right)^2 \exp\left(-\frac{2|x|^2+C_1}{\nu t}\right) H_\nu\left(\frac{t}{2},x';C_2 l_\rho^2\right), \tag{1.5}$$

[1] See (3.2) below for the proof of (1.5).

where $C_1 > 0$, $C_2 > 0$, and $x' \in \mathbb{R}^d$ are some constants which depend on K, $G(t,x)$ denotes the heat kernel

$$G(t,x) = (2\pi\nu t)^{-d/2} \exp\left(-(2\nu t)^{-1}|x|^2\right), \quad t > 0,\ x \in \mathbb{R}^d,\ |x|^2 := x_1^2 + \cdots + x_d^2,$$

and H_ν is defined in (2.14) below. The exponential growth in t, if any, will be contributed by the function $H_\nu\left(\frac{t}{2}, x'; C_2 l_\rho^2\right)$. Unlike the linear case (that is, $\rho(u) = \lambda u$), where the p-th, $p \geq 2$, moments of the solution to (1.1) admit the Feynman-Kac representations, from which one can obtain sharp lower bounds as is done in [17], to the best of our knowledge, the lower bounds obtained in this article are the only nontrivial lower moment bounds for the nonlinear SHE (1.1); note that our lower bounds holds for any nonnegative and nonvanishing initial measure.

As a consequence of the upper bounds, we can allow the initial data to be any Borel measures subject to some mild integrability conditions; see (1.6). Similar upper bounds have also been recently obtained by Huang [16] for a more general class of second order operators than the Laplace operator studied in this article. The most interesting results of this article come from various applications of the lower bounds on the second moment. Firstly, these lower bounds allow us to establish several equivalent conditions for the phase transition of the second moment Lyapunov exponents. Secondly, we are able to use these lower bounds to establish the existence of intermittency fronts (for the nonlinear SHE (1.1)), which extend results by Conus and Khoshnevisan [9] from one space dimension to higher space dimension. For the linear case, similar results have been recently obtained by Huang [17] et al through the Feynman-Kac representations of the moments.

In the rest of this introduction section, we will explain in more details these consequences of our moment bounds.

1.1 Rough initial data

We first introduce some notation. By the Jordan decomposition, any Borel measure μ can be decomposed as $\mu = \mu_+ - \mu_-$, where $\mu_\pm$ are two nonnegative Borel measures with disjoint support. Denote $|\mu| := \mu_+ + \mu_-$. The requirement for the initial measure μ is that

$$\int_{\mathbb{R}^d} e^{-a|x|^2} |\mu|(\mathrm{d}x) < +\infty\ , \quad \text{for all } a > 0\ . \tag{1.6}$$

For convenience, we denote this set of measures by $\mathcal{M}_H\left(\mathbb{R}^d\right)$ and we call them the rough initial data. Hence, the phrase "rough initial data" conveys two properties, namely, the local regularity which is as regular as a Borel measure and the tail property which is controlled by (1.6).

The solution to the homogeneous equation is

$$J_0(t,x) := \int_{\mathbb{R}^d} G(t, x-y)\mu(\mathrm{d}y).$$

Then, condition (1.6) is equivalent to the condition that $J_0(t,x)$ with μ replaced by $|\mu|$ is finite for all $t > 0$ and $x \in \mathbb{R}^d$. This is an extension of the work [4] from $\mathbb{R}$ to $\mathbb{R}^d$. If the initial measure has a bounded density, then Dalang's condition (1.2) is a necessary and sufficient condition for (1.1) to have a random field solution; see [10, 12, 13, 19]. We will show that this statement is still true for all initial measures in $\mathcal{M}_H(\mathbb{R}^d)$, provided that either $\rho(u) = \lambda u$ is linear or the weak positivity (comparison) principle holds, namely,

$$u(t,x) \geq 0 \text{ a.s. for all } t > 0 \text{ and } x \in \mathbb{R}^d \text{ whenever } \mu \geq 0. \tag{1.7}$$

One should not worry about this additional assumption (1.7). Historically, the comparison principle was proved with restrictions either on initial data or on the spatial correlation function f, or on both; see [7, 20, 24]. The most general form, that is, the one both under Dalang's condition (1.2) and for initial measure $\mu \in \mathcal{M}_H(\mathbb{R}^d)$, which is exactly what we need here, has recently been established by Chen and Huang [5]. In that article, the second moment upper bounds were extended to all p-th, $p \geq 2$, moment upper bounds of similar form.

1.2 Full intermittency and phase transitions

Using the moment formula (1.4), we will study the asymptotic behaviors of the solution. We first define the upper and lower (moment) Lyapunov exponents of order p ($p \geq 2$) by

$$\overline{m}_p(x) := \limsup_{t\to+\infty} \frac{1}{t} \log \mathbb{E}\left(|u(t,x)|^p\right), \quad \underline{m}_p(x) := \liminf_{t\to+\infty} \frac{1}{t} \log \mathbb{E}\left(|u(t,x)|^p\right). \tag{1.8}$$

If the initial data are homogeneous (that is, $\mu(\mathrm{d}x) = C\mathrm{d}x$ for some constant $C \in \mathbb{R}$), then neither $\overline{m}_p(x)$ nor $\underline{m}_p(x)$ depends on x. In this case, a solution is called fully intermittent if $\underline{m}_2 > 0$ and $m_1 = 0$ by Carmona and Molchanov [2, Definition III.1.1]. See [18] for a detailed discussion of the meaning of this intermittency property. By the same rationale, for non-homogeneous initial data, we call a solution fully intermittent if $\inf_{x\in\mathbb{R}^d} \underline{m}_2(x) > 0$ and $m_1(x) \equiv 0$ for all $x \in \mathbb{R}^d$.

Foondun and Khoshnevisan proved in [13, Theorem 1.8] and [14, Theorem 2] that if the correlation function f satisfies Dalang's condition (1.2) plus some other mild conditions, and if $\rho(x)$ satisfies (1.3), then for constant initial data, when l_ρ is sufficiently large, the second moment of the solution to (1.1) has at least exponential growth in time, that is,

$$\inf_{x\in\mathbb{R}^d} \overline{m}_2(x) > 0. \tag{1.9}$$

We will show that the condition that l_ρ should be sufficiently large is necessary in certain situations. Moreover, we will strengthen statement (1.9) into

$$\inf_{x\in\mathbb{R}^d} \underline{m}_2(x) > 0.$$

The change from the upper Lyapunov exponent $\overline{m}_2(x)$ to the lower one $\underline{m}_2(x)$ is highly nontrivial, and this is only doable if one has some nontrivial lower bounds on the second moment.

By the same reason, we are able to study the necessary and sufficient conditions for the phase transition of the second moment Lyapunov exponent. The phase transition for moment Lyapunov exponents are well known for the stochastic heat equation on $\mathbb{Z}^d$; see, for example, Carmona and Molchanov [2]. However, this is not clear especially for the nonlinear stochastic heat equation on $\mathbb{R}^d$. We say that the solution $u(t,x)$ to (1.1) admits a phase transition for the second moment if there exist two nonnegative constants $0 < \underline{\lambda}_c \leq \overline{\lambda}_c < \infty$, such that

$$\begin{cases} \sup_{x\in\mathbb{R}^d} \overline{m}_2(x) = 0 & \text{if } (l_\rho \leq)\ L_\rho < \underline{\lambda}_c, \\ \inf_{x\in\mathbb{R}^d} \underline{m}_2(x) > 0 & \text{if } \overline{\lambda}_c < l_\rho\ (\leq L_\rho). \end{cases} \tag{1.10}$$

Here, two parameters L_ρ and l_ρ play the same role as λ for the Anderson model $\rho(u) = \lambda u$. We will prove that under some mild conditions on the initial data, the phase transition happens if and only if

$$\Upsilon(0) := \lim_{\beta\to0} \Upsilon(\beta) < \infty; \tag{1.11}$$

see Theorem 1.3 for the precise statement. As a consequence, we have the following statements:

1. No phase transition occurs when $d = 1$ or 2.
2. Phase transition happens if and only if
$$d \geq 3 \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{f(z)}{|z|^{d-2}} \mathrm{d}z < \infty. \tag{1.12}$$
3. Let B_t be a Brownian motion on $\mathbb{R}^d$ starting from the origin with $\mathbb{E}(|B_t|^2) = \nu t$. Define, for $t > 0$,
$$\begin{aligned} k(t) &:= \mathbb{E}(f(B_t)) = \int_{\mathbb{R}^d} f(z) G(t, z) \mathrm{d}z, \\ h_1(t) &:= \mathbb{E}\left[\int_0^t f(B_s) \mathrm{d}s\right] = \int_0^t k(s) \mathrm{d}s. \end{aligned} \tag{1.13}$$
Then (1.11) holds if and only if
$$\lim_{t \to \infty} h_1(t) < \infty. \tag{1.14}$$

Remark 1.1 It is well-known that if $f \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz test function), then
$$h_1(\infty) = C_1 \int_{\mathbb{R}^d} \frac{f(x)}{|x|^{d-2}} \mathrm{d}x = C_2 \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{|\xi|^2} \mathrm{d}\xi = C_3 \Upsilon(0);$$
see, for example, [23, Lemma 2, Chapter 5] for the second equality. In this case, the equivalence among (1.11), (1.12), and (1.14) is clear.

Remark 1.2 Condition (1.12) sets restrictions on the behaviors of f both at the infinity and around zero. In particular, when $d \geq 3$, in order to have phase transition, the local integrability of f around zero is not enough, and the tails should be not too fat. Clearly, the integrability condition in (1.12) is stronger than Dalang's condition (1.2). If f is radial $f(x) = \tilde{f}(|x|)$, the integral condition in (1.12) reduces to $\int_0^\infty \tilde{f}(r) r \mathrm{d}r < \infty$.

We summarize these results in the following theorem. Recall that a measure $\mu > 0$ means that $\mu \geq 0$ (nonnegative) and $\mu \neq 0$ (non-vanishing, that is, $\int_{\mathbb{R}^d} |\mu|(\mathrm{d}x) \neq 0$).

Theorem 1.3 Suppose that the initial data $\mu \in \mathcal{M}_H(\mathbb{R})$ is such that $\mu > 0$ and
$$\int_{\mathbb{R}^d} e^{-\beta|x|} \mu(\mathrm{d}x) < +\infty \quad \text{for all } \beta > 0. \tag{1.15}$$
Then

(1) If $\Upsilon(0) < \infty$, then (1.10) holds for some nonnegative constants $0 < \underline{\lambda}_c \leq \overline{\lambda}_c < \infty$.

(2) If $\Upsilon(0) = \infty$, then $u(t, x)$ is fully intermittent, that is, $m_1(x) \equiv 0$ and $\inf_{x \in \mathbb{R}^d} \underline{m}_2(x) > 0$.

(3) The following three conditions are equivalent:
$$\text{Condition (1.11)} \Longleftrightarrow \text{Condition (1.12)} \Longleftrightarrow \text{Condition (1.14)}. \tag{1.16}$$

The proof of Theorem 1.3 is given at the end of Section 3.

Let us see some examples. Because the behaviors of $f(\cdot)$ matter both at the origin and at the infinity, we first take a look at the case when f is well behaved at zero, that is, $f(0) < \infty$. Examples of such kernel functions include the Ornstein-Uhlenbeck-type kernels $f(x) = \exp(-c|x|^\alpha)$ for $\alpha \in (0, 2]$ and $c > 0$, the Poisson kernel $f(x) = (1 + |x|^2)^{-(d+1)/2}$ and the

Cauchy kernel $f(x) = \prod_{j=1}^{d}(1+x_j^2)^{-1}$. All these examples satisfy condition (1.12) for $d \geq 3$, thus there is a phase transition for the second moment. When $\rho(u) = \lambda u$, the above results are proved using the Feynman-Kac representation of the solution by Nobel [21, Theorem 9], and its discrete counterpart ($\mathbb{Z}^d$ replaced by $\mathbb{R}^d$) has been well studied by Carmona and Molchanov [2].

As for the case when f blows up at zero, that is, $f(0) = \infty$, the typical examples are the Riesz kernels $f(x) = |x|^{-\alpha}$ with $\alpha \in (0, 2 \wedge d)$. They fail to satisfy the integrability condition in (1.12) because of their fat tails. Hence, there is no phase transition. Recently, this case has also been studied by Foondun, Liu, and Omaba [15]. They focus on the case with function-valued initial data and the Riesz kernel, and obtained a lower bound for the second moment which implies that $\inf_{x\in\mathbb{R}^d} \underline{m}_2(x) > 0$. Note that our results hold for more general initial data and general kernel functions. Nevertheless, results in [15] hold for the fractional Laplace operator.

1.3 Intermittency fronts

Another application of our moment formula (1.4) is the study of the intermittency fronts. Following [9], define the following growth indices:

$$\underline{\lambda}(p) := \sup\left\{\alpha > 0 : \liminf_{t\to\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log \mathbb{E}\left(|u(t,x)|^p\right) > 0\right\}, \tag{1.17}$$

$$\overline{\lambda}(p) := \inf\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log \mathbb{E}\left(|u(t,x)|^p\right) < 0\right\}. \tag{1.18}$$

These quantities characterize the propagation speed of "high peaks"; see [4, 9] for more details. The higher spatial dimension cases have more geometric properties than the one space dimensional case. Here we will give a rough characterization of the locations of the peaks using the space-time cones. Refined investigations in this direction can be an interesting, but separate, project.

Theorem 1.4 Suppose that ρ satisfies (1.3) and the initial data $\mu > 0$ satisfies that

$$\int_{\mathbb{R}^d} e^{\beta|x|}\mu(\mathrm{d}x) < +\infty, \quad \text{for some } \beta > 0. \tag{1.19}$$

Then, it holds that

$$0 \leq \sqrt{\nu\,\theta_*} \leq \underline{\lambda}(2) \leq \overline{\lambda}(2) \leq \frac{\sqrt{d}}{2}\left(\nu\beta + \frac{\theta}{\beta}\right) < +\infty, \tag{1.20}$$

where the two constants $\theta := \theta(\nu, L_\rho^2)$ and $\theta_* := \theta_*(\nu, l_\rho^2)$ are defined as follows:

$$\theta(\nu, L_\rho^2) := \inf\left\{\beta > 0 :\ \Upsilon\left(2\beta/\nu\right) < \frac{\nu}{2L_\rho^2}\right\}, \tag{1.21}$$

$$\theta_*(\nu, l_\rho^2) := \lim_{t\to\infty} \frac{1}{t} \log\left(\sum_{n=0}^{\infty}\left(\left[2\sqrt{3}\right]^{-d} l_\rho^2\right)^n h_1(t/n)^n\right). \tag{1.22}$$

Moreover, if $\Upsilon(0) = \infty$, then $\theta_*(\nu, l_\rho^2) > 0$ (strict inequality) for all $\nu > 0$ and $l_\rho > 0$. Otherwise, $\theta_*(\nu, l_\rho^2)$ is strictly positive only when either l_ρ is sufficiently large or ν is sufficiently small.

This theorem is proved in Section 4.

Note that two conditions (1.19) and (1.15) look similar. One may allow some tempered tails for the initial condition in Theorem 1.3. However, for Theorem 1.4, the initial conditions should be localized in the sense that they should have exponentially decaying tails.

Remark 1.5 When $d = 1$, $f = \delta_0$, $\rho(u) = \lambda u$, and the initial measure $\mu \geq 0$ satisfies (1.19), it is proved in [4] that

$$\frac{\lambda^2}{2} \leq \underline{\lambda}(2) \leq \overline{\lambda}(2) \leq \frac{\beta\nu}{2} + \frac{\lambda^4}{8\nu\beta},$$

in particular, when $\beta \geq \lambda^2/(2\nu)$, $\underline{\lambda}(2) = \overline{\lambda}(2) = \lambda^2/2$. On the other hand, as shown in Example 4, $\theta = \nu^{-1}\lambda^4$ and $\theta_* = (6\pi\nu e^2)^{-1}\lambda^4$. Hence, by (1.20), when $\beta \geq \lambda^2/\nu$, we have

$$0.0847335\, \lambda^2 \approx \frac{\lambda^2}{e\sqrt{6\pi}} \leq \underline{\lambda}(2) \leq \overline{\lambda}(2) \leq \lambda^2.$$

These estimates in (1.20) are not as sharp as those in [4] but they cover more general noises.

Theorem 1.4 holds for the nonlinear SHE. However, in the linear case, one may use the Feynman-Kac representations for the moments and a recent sharp time asymptotic result by X. Chen [8] to derive both sharp growth indices and the criteria for phase transitions for all p-th moments, $p \geq 2$. During preparation of this article, these arguments have been carried out by Huang, Lê, and Nualart [17]. Nevertheless, we would emphasize that our methods are very different and we allow the nonlinear dependence on the solution.

1.4 Some comments on the recent progress

Before the end of this introduction section, we would like to make some further comments on the recent progress since the first submission of this article in the Fall of 2015. With time passing, some results in this article are no longer new, such as the rough initial data part and the linear cases for Theorems 1.3 and 1.4. However, we believe that our results in the nonlinear setting are still new and valuable to the field, which in particular include both Theorem 1.3 and Theorem 1.4 (in case of nonlinear ρ). The workhorse behind these results is our convenient lower bound for the second moment as in (1.5). This lower bound works for both linear and nonlinear SHE. Even just for the linear SHE, compared to the Feynman-Kac representation as in [17], our lower bound (1.5) is much more explicit and therefore more convenient for applications.

Some techniques used in this article have been applied by Balan and the first author of this article in [1] to study a similar SHE with time-fractional noise (instead of noise "white in time" as in this article). In particular, one may find full details for the proof of Lemma 2.5 in [1, Lemma 3.8].

Under exactly the same settings as this article, Huang and the first author of this article have made some significant progress in [5]. They have obtained some convenient upper bounds for the general p-th moments, $p \geq 2$, through an application of the Burkholder-Davis-Gundy inequality. With these moment bounds, they have further established the Hölder regularity and the pathwise comparison principle. The results in [5] rely on the upper bound for the p-th moments and in particular they do not require the lower bound for the second moment obtained in this article.

Outline This article is organized as follows. We first study the existence and uniqueness of a random field solution to (1.1) under rough initial conditions in Section 2. The phase transition result (Theorem 1.3) is proved in Section 3. The growth indices result (Theorem 1.4) is proved in Section 4. Finally, some examples are listed in the Appendix.

Throughout this article, $||\cdot||_p$ denotes the $L^p(\Omega)$-norm.

2 Existence and Uniqueness

The upper and lower bounds of the second moment are derived in this section. We first state some prerequisites in Section 2.1. The main results (Theorem 2.4) concerning the existence/uniqueness and moment bounds are stated in Section 2.2. The proof of Theorem 2.4 is given in Section 2.3, where we prove one of the key results – Lemma 2.7 – of this article.

2.1 Some prerequisites

Throughout this subsection, let $R(x,y)$ be a nonnegative and nonnegative definite kernel in the sense that

$$\iint_{\mathbb{R}^{2d}} R(x,y)\psi(x)\psi(y)\mathrm{d}x\mathrm{d}y \geq 0, \quad \text{for all } \psi \in C_c^\infty\left(\mathbb{R}^d\right),$$

where $C_c^\infty\left(\mathbb{R}^d\right)$ denotes the set of smooth functions with compact support. Suppose that $R(x,y)$ satisfies the following condition:

$$\iint_{K\times K} R(x,y)\mathrm{d}x\mathrm{d}y < +\infty, \qquad \text{for all compact sets } K \in \mathbb{R}^d.$$

Associated with such R, there is a nonnegative and locally finite measure, denoted by μ_R, over $\mathbb{R}^d$, such that

$$\mu_R(K) := \iint_{K\times K} R(x,y)\mathrm{d}x\mathrm{d}y, \quad \text{for any Borel sets } K \subseteq \mathbb{R}^d.$$

Definition 2.1 A spatially R-correlated Gaussian noise that is white in time is an $L^2(\Omega)$-valued mean zero Gaussian process

$$\left\{F(\psi):\ \psi \in C_c^\infty\left(\mathbb{R}^{1+d}\right)\right\},$$

such that

$$\mathbb{E}\left[F(\psi)F(\phi)\right] = \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \psi(s,x)R(x,y)\phi\left(s,y\right)\mathrm{d}x\mathrm{d}y.$$

Note that if $R(x,y) = h(x-y)$ for some kernel h, then the above definition reduces to the spatially homogeneous Gaussian noise that is white in time [10]. In particular, if $h(x-y) = \delta_0(x-y)$, then this noise becomes the space-time white noise and the associated measure μ_R reduces to the Lebesgue measure on $\mathbb{R}^d$.

We need some criteria to check whether a random field is predictable. As in [10], we extend F to a σ-finite L^2-valued measure $B \to F(B)$ defined for bounded Borel sets $B \in [0,\infty)\times\mathbb{R}^d$ and then define

$$M_t(A) := F([0,t]\times A), \quad A \in \mathcal{B}_b\left(\mathbb{R}^d\right).$$

Let $(\mathcal{F}_t, t \geq 0)$ be the filtration given by

$$\mathcal{F}_t := \sigma\left(M_s(A):\ 0 \leq s \leq t, A \in \mathcal{B}_b\left(\mathbb{R}^d\right)\right) \vee \mathcal{N}, \quad t \geq 0,$$

which is the natural filtration augmented by all P-null sets $\mathcal{N}$ in $\mathcal{F}$, where $\mathcal{B}_b(\mathbb{R}^d)$ is the collection of Borel measurable sets with finite Lebesgue measure. The family of subsets of

$[0,\infty)\times\mathbb{R}^d\times\Omega$, which contains all sets of the form $\{0\}\times A\times F_0$ and $(s,t]\times A\times F$, where $F_0\in\mathcal{F}_0$, $F\in\mathcal{F}_s$ for $0\le s<t$ and A is a rectangle in $\mathbb{R}^d$, is called the class of predictable rectangles. The σ-field generated by the predictable rectangles is called the predictable σ-field, which is denoted by $\mathcal{P}$. Sets in $\mathcal{P}$ are called predictable sets. A random field $X:\Omega\times[0,\infty)\times\mathbb{R}^d\mapsto\mathbb{R}$ is called predictable if X is $\mathcal{P}$-measurable.

For $p\ge 2$, denote $\mathcal{P}_p$ to be the set of all predictable and $L^2_R\left([0,\infty)\times\mathbb{R}^d;\ L^p(\Omega)\right)$ integrable random fields, namely, $f\in\mathcal{P}_p$ if and only if f is predictable and

$$||f||_{M,p}^2 := \iiint_{(0,\infty)\times\mathbb{R}^{2d}} R(x,y)\,||f(s,x)f(s,y)||_{\frac{p}{2}}\,\mathrm{d}s\mathrm{d}x\mathrm{d}y < +\infty\ , \tag{2.1}$$

where $||\cdot||_p$ denotes the $L^p(\Omega)$-norm. In particular, if $R(x,y)=\delta_0(x-y)$, then

$$||f||_{M,p}^2 = \iint_{(0,\infty)\times\mathbb{R}^d} ||f(s,x)||_p^2\,\mathrm{d}s\mathrm{d}x.$$

Clearly,

$$\mathcal{P}_2\supseteq\mathcal{P}_p\supseteq\mathcal{P}_q,\quad \text{for } 2\le p\le q<+\infty.$$

The following proposition is useful to check whether a random field belongs to $\mathcal{P}_p$ or not.

Proposition 2.2 Suppose that for some $t>0$ and $p\ge 2$, a random field

$$X=\left\{\ X(s,y):\ (s,y)\in\ (0,t)\ \times\mathbb{R}^d\ \right\}$$

has the following properties:
(i) X is adapted, that is, for all $(s,y)\in\ (0,t)\ \times\mathbb{R}^d$, $X(s,y)$ is $\mathcal{F}_s$-measurable;
(ii) X is jointly measurable with respect to $\mathcal{B}\left([0,\infty)\times\mathbb{R}^d\right)\times\mathcal{F}$;
(iii) $||X||_{M,p}<+\infty$.
Then, $X(\cdot,\circ)\,1_{(0,t)}(\cdot)$ belongs to $\mathcal{P}_p$.

This proposition is an extension of Dalang & Frangos's result in [11, Proposition 2] in the two senses: (1) the second moment of X can blow up at $s=0$ or $s=t$, which is the case, for example, when the initial data is the Dirac delta measure; (2) the condition that X is $L^2(\Omega)$-continuous has been removed. The proof of this proposition follows essentially the same arguments as the proof of the case where $d=1$ and the noise is white in both space and time variables; see [4, Proposition 3.1]. Proposition 2.2 will be used in the Picard iterations in the proof of Theorem 2.4.

2.2 Statement of the result

We formally write the SPDE (1.1) in the integral form

$$u(t,x)=J_0(t,x)+I(t,x), \tag{2.2}$$

where

$$I(t,x):=\iint_{[0,t]\times\mathbb{R}^d} G(t-s,x-y)\rho(u(s,y))M(\mathrm{d}s,\mathrm{d}y).$$

The above stochastic integral is understood in the sense of Walsh [10, 25].

Definition 2.3 A process $u=\left(u(t,x),\ (t,x)\in(0,\infty)\times\mathbb{R}^d\right)$ is called a random field solution to (1.1) if
(1) u is adapted, that is, for all $(t,x)\in(0,\infty)\times\mathbb{R}^d$, $u(t,x)$ is $\mathcal{F}_t$-measurable;

(2) u is jointly measurable with respect to $\mathcal{B}\left((0,\infty)\times\mathbb{R}^d\right)\times\mathcal{F}$;
(3) $||I(t,x)||_2<+\infty$ for all $(t,x)\in(0,\infty)\times\mathbb{R}^d$;
(4) The function $(t,x)\mapsto I(t,x)$ mapping $(0,\infty)\times\mathbb{R}^d$ into $L^2(\Omega)$ is continuous;
(5) u satisfies (2.2) a.s., for all $(t,x)\in(0,\infty)\times\mathbb{R}^d$.

Denote

$$J_1(t,x,x'):=J_0(t,x)J_0(t,x')$$

and $g(t,x,x'):=\mathbb{E}\left[u(t,x)u(t,x')\right]$. Then by Itô's isometry, g satisfies the following integral equation (for $\rho(u)=\lambda u$)

$$\begin{aligned} g(t,x,x') = J_1(t,x,x') + \lambda^2\int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}z\mathrm{d}z'\; g(s,z,z') \\ \times G(t-s,x-z)G(t-s,x'-z')f(z-z'). \end{aligned} \tag{2.3}$$

Replacing the function g on the r.h.s. of (2.3) by (2.3) itself repeatedly suggests the following definitions. For $h,w:[0,\infty)\times\mathbb{R}^{3d}\mapsto\mathbb{R}$, define the operation "$\rhd$", which depends on f, as follows:

$$\begin{aligned} (h\rhd w)(t,x,x';y) := \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}z\mathrm{d}z'\; h(t-s,x-z,x'-z';y-(z-z')) \\ \times w(s,z,z';y)\, f(y-(z-z')). \end{aligned} \tag{2.4}$$

By change of variables,

$$\begin{aligned} (h\rhd w)(t,x,x';y) := \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}z\mathrm{d}z'\; h(s,z,z';y-[(x-z)-(x'-z')]) \\ \times w(t-s,x-z,x'-z';y)\, f(y-[(x-z)-(x'-z')]). \end{aligned} \tag{2.5}$$

Note that for general f, this convolution-type operator is not symmetric, $h\rhd w\neq w\rhd h$, except for some special cases, such as, $f\equiv 1$ or $f=\delta_0$. Operators of this type have been studied in Chen's thesis [3, Chapter 3] [2]. Some calculations show that by introducing the additional variable y, this operator becomes associative, that is, for $h,w,v:[0,\infty)\times\mathbb{R}^{3d}\mapsto\mathbb{R}$,

$$((h\rhd w)\rhd v)(t,x,x';y) = (h\rhd(w\rhd v))(t,x,x';y);$$

see Lemma 4 below. We will use the following convention: If h is a function from $[0,\infty)\times\mathbb{R}^{2d}$ to $\mathbb{R}$, when applying the operation $\rhd$ to h, we write $\hat{h}(t,x,x';y):=h(t,x,x')$.

For $t>0$ and $x,x',y\in\mathbb{R}^d$, define

$$\mathcal{L}_n(t,x,x';y) := \begin{cases} G(t,x)G(t,x') & \text{if } n=0, \\ (\mathcal{L}_0\rhd\mathcal{L}_{n-1})(t,x,x';y) & \text{if } n\geq 1. \end{cases}$$

For $\lambda\in\mathbb{R}$, define formally

$$\mathcal{K}_\lambda(t,x,x';y) := \sum_{n=0}^\infty \lambda^{2(n+1)}\mathcal{L}_n(t,x,x';y).$$

[2] The operator in [3, Chapter 3] is more general. Indeed, by taking the spatial dimension to be $2d$, and $\theta^2(t,x)=f(\hat{x}-\hat{x}')$ where $x=(\hat{x},\hat{x}')$ with $\hat{x},\hat{x}'\in\mathbb{R}^d$, one reduces the operator in [3, Chapter 3] to the current operator.

The convergence of the above series is proved in Lemma 2.7 below. We will use the following convention for $\mathcal{K}_\lambda$:

$$\mathcal{K} := \mathcal{K}_\lambda \qquad \overline{\mathcal{K}} := \mathcal{K}_{L_\rho} \qquad \underline{\mathcal{K}} := \mathcal{K}_{l_\rho}. \tag{2.6}$$

Using these notation and conventions, we see that (2.3) can be written in the following way:

$$g(t,x,x') = J_1(t,x,x') + \lambda^2 \left(\mathcal{L}_0 \rhd g\right)(t,x,x';0), \tag{2.7}$$

which suggests, after iterations, that

$$g(t,x,x') = J_1(t,x,x') + \left(\mathcal{K} \rhd J_1\right)(t,x,x';0).$$

Theorem 2.4 For any $\mu \in \mathcal{M}_H(\mathbb{R}^d)$, SHE (1.1) has a unique (in the sense of versions) random field solution $\left\{u(t,x) : t > 0, x \in \mathbb{R}^d\right\}$ starting from μ. This solution is $L^2(\Omega)$-continuous. Moreover, the following moment estimates are true:

(1) If $\rho(u) = \lambda u$, then the two-point correlation function is equal to

$$\mathbb{E}\left[u(t,x)u(t,x')\right] = J_1(t,x,x') + \left(\mathcal{K} \rhd J_1\right)(t,x,x';0) \tag{2.8}$$

$$= \lambda^{-2} \iint_{\mathbb{R}^{2d}} \mu(\mathrm{d}z)\mu(\mathrm{d}z')\, \mathcal{K}(t, x-z, x'-z'; z'-z). \tag{2.9}$$

(2) If $|\rho(x)| \le L_\rho |x|$ for all $x \in \mathbb{R}^d$ with $L_\rho > 0$ and if $\mu \ge 0$, then

$$\mathbb{E}\left[u(t,x)u(t,x')\right] \le L_\rho^{-2} \iint_{\mathbb{R}^{2d}} \mu(\mathrm{d}z)\mu(\mathrm{d}z')\, \overline{\mathcal{K}}(t, x-z, x'-z'; z'-z). \tag{2.10}$$

(3) If ρ satisfies (1.3), then

$$\mathbb{E}\left[|u(t,x)u(t,x')|\right] \ge l_\rho^{-2} \iint_{\mathbb{R}^{2d}} \mu(\mathrm{d}z)\mu(\mathrm{d}z')\, \underline{\mathcal{K}}(t, x-z, x'-z'; z'-z). \tag{2.11}$$

In a recent article [6], an explicit expression for this kernel function $\mathcal{K}$ is obtained when $d = 1$ and $f(x) = \delta_0(x)$.

2.3 Proof of Theorem 2.4

We will first prove some lemmas. Recall the definition of the function $k(t)$ in (1.13). By the Fourier transform, this function $k(t)$ can also be rewritten in the following form

$$k(t) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}\xi) \exp\left(-\frac{\nu t}{2}|\xi|^2\right), \tag{2.12}$$

from which one can see that $t \mapsto k(t)$ is a nonincreasing function. For $t \ge 0$ and $y \in \mathbb{R}^d$, define $h_0(t,y) := 1$ and for $n \ge 1$,

$$h_n(t,y) := \int_0^t \mathrm{d}s\, h_{n-1}(s,y) k(t-s) T_{\nu/4}(t-s,y), \tag{2.13}$$

where

$$T_\nu(t,x) := \exp\left(-\frac{|x|^2}{\nu t}\right).$$

Define

$$H_\nu(t,y;\gamma) := \sum_{n=0}^{\infty} \gamma^n h_n(t,y). \tag{2.14}$$

We will use the convention that

$$h_n(t) := h_n(t,0) \quad \text{and} \quad H_\nu(t;\gamma) := H_\nu(t,0;\gamma).$$

Lemma 2.5 For $\gamma \geq 0$, it holds that

$$\limsup_{t\to\infty} \frac{1}{t} \log H_\nu(t;\gamma) \leq \theta,$$

where this constant θ can be chosen as

$$\theta := \theta(\nu,\gamma) = \inf\left\{\beta > 0 :\ \Upsilon\left(2\beta/\nu\right) < \frac{\nu}{2\gamma}\right\}. \tag{2.15}$$

Moreover, if $\Upsilon(0) < \infty$ and $\nu > 2\gamma\Upsilon(0)$, then

$$H_\nu(t;\gamma) \leq \frac{\nu}{\nu - 2\gamma\Upsilon(0)} \quad \text{for all } t \geq 0.$$

Proof Notice that for $\beta > 0$,

$$\int_0^\infty e^{-\beta t} h_n(t)\mathrm{d}t = \frac{1}{\beta}\left(\int_0^\infty e^{-\beta t} k(t)\mathrm{d}t\right)^n = \frac{1}{\beta}\left[\frac{2}{\nu}\,\Upsilon\left(\frac{2\beta}{\nu}\right)\right]^n.$$

Because $\Upsilon(\beta) \to 0$ as $\beta \to \infty$, by increasing β, we can make sure that $2\nu^{-1}\Upsilon\left(2\beta/\nu\right)\gamma < 1$. The smallest β that satisfies (2.15) gives the constant θ. When $\Upsilon(0) < \infty$, notice that

$$\lim_{t\to\infty} h_1(t) = \lim_{\beta\to 0_+} \frac{2}{\nu}\Upsilon(2\beta/\nu). \tag{2.16}$$

Hence, by the induction, $h_n(t) \leq \left[2\nu^{-1}\Upsilon(0)\right]^n$ for all $n \geq 0$. □

Even though the integrand in the definition of h_n is positive, because of the presence of t in the integrand, the following result is nontrivial (considering, for example, $\int_0^t (s(t-s))^{-2/3}\mathrm{d}s = Ct^{-1/3}$).

Lemma 2.6 For $n \geq 0$ and $y \in \mathbb{R}^d$, all functions $t \in [0,\infty) \mapsto h_n(t,y)$ are nondecreasing.

Proof Fix $y \in \mathbb{R}^d$. The case $n = 0$ is true by definition. Suppose that it is true for n. For all $\epsilon \geq 0$, by the induction assumption,

$$\begin{aligned} h_{n+1}(t+\epsilon,y) &= \int_0^{t+\epsilon} \mathrm{d}s\, h_n(t+\epsilon-s,y)k(s)T_{\nu/4}(s,y) \\ &\geq \int_0^t \mathrm{d}s\, h_n(t+\epsilon-s,y)k(s)T_{\nu/4}(s,y) \\ &\geq \int_0^t \mathrm{d}s\, h_n(t-s,y)k(s)T_{\nu/4}(s,y) = h_{n+1}(t,y). \end{aligned}$$

This proves Lemma 2.6. □

The following lemma plays the central role in this article. Recall convention (2.6) for $\mathcal{L}_n$.

Lemma 2.7 Suppose that the correlation function f satisfies Dalang's condition (1.2). Then, for all $n \geq 1$, $t \geq 0$, $x, x', y \in \mathbb{R}^d$,

$$\mathcal{L}_n(t,x,x';y) \leq 2^n\, G(t,x)\, G(t,x')h_n(t), \tag{2.17}$$

$$\mathcal{L}_n(t,x,x';y) \geq (2\sqrt{3}\,)^{-nd} G(t,x)G(t,x')T_\nu(t,x-x')\, h_n\left(t/2,y\right), \tag{2.18}$$

Hence,

$$\mathcal{K}_\lambda(t,x,x';y) \leq \mathcal{L}_0(t,x,x')H_\nu\left(t;2\lambda^2\right), \tag{2.19}$$

$$\mathcal{K}_\lambda(t,x,x';y) \geq \mathcal{L}_0(t,x,x')T_\nu(t,x-x')H_\nu(t/2,y;(2\sqrt{3}\,)^{-d}\lambda^2). \tag{2.20}$$

Proof By definition,

$$\begin{aligned}\mathcal{L}_1(t,x,x';y) = &\int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}z\mathrm{d}z'\, G(s,z)G(s,z')f(y-(z-z'))\\ &\times G(t-s,x-z)G(t-s,x'-z').\end{aligned}$$

Notice that (see [4, Lemma 5.4])

$$G(s,z)G(t-s,x-z) = G\left(\frac{s(t-s)}{t}, z-\frac{s}{t}x\right)G(t,x)$$

and similar for the other pair. So,

$$\begin{aligned}\mathcal{L}_1(t,x,x';y) = G(t,x)G(t,x')&\int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}z\mathrm{d}z'\, f(y-(z-z'))\\ &\times G\left(\frac{s(t-s)}{t}, z-\frac{s}{t}x\right)G\left(\frac{s(t-s)}{t}, z'-\frac{s}{t}x'\right). \end{aligned}\tag{2.21}$$

Because

$$\mathcal{F}[G(t,\circ)](\xi) = \exp\left(-\frac{\nu t}{2}|\xi|^2\right),$$

the double integral over $\mathrm{d}z\mathrm{d}z'$ in (2.21) is equal to

$$\begin{aligned}&(2\pi)^{-d}\int_{\mathbb{R}^d}\hat{f}(\mathrm{d}\xi)\exp\left(\mathrm{i}\left(y-\frac{s}{t}(x-x')\right)\cdot\xi-\frac{\nu s(t-s)}{t}|\xi|^2\right)\\ =&\int_{\mathbb{R}^d}\mathrm{d}z\, f(z)G\left(\frac{2s(t-s)}{t}, z+y-\frac{s}{t}(x-x')\right).\end{aligned}\tag{2.22}$$

Now, let us prove (2.17). From (2.21) and (2.22), it is clear that

$$\mathcal{L}_1(t,x,x';y)\le(2\pi)^{-d}G(t,x)G(t,x')\int_0^t\mathrm{d}s\int_{\mathbb{R}^d}\hat{f}(\mathrm{d}\xi)\exp\left(-\frac{\nu s(t-s)}{t}|\xi|^2\right).\tag{2.23}$$

Because $s/2\le s(t-s)/t$ for $s\in[0,t/2]$, by symmetry, the above double integral is equal to

$$\begin{aligned}&2\int_0^{t/2}\mathrm{d}s\int_{\mathbb{R}^d}\hat{f}(\mathrm{d}\xi)\exp\left(-\frac{\nu s(t-s)}{t}|\xi|^2\right)\\ \le& 2\int_0^{t/2}\mathrm{d}s\int_{\mathbb{R}^d}\hat{f}(\mathrm{d}\xi)\exp\left(-\frac{\nu s}{2}|\xi|^2\right) = 2(2\pi)^d\int_0^{t/2}k(s)\mathrm{d}s\\ =& 2(2\pi)^d h_1(t/2)\le 2(2\pi)^d h_1(t),\end{aligned}$$

where in the last step we have applied Lemma 2.6. The induction step is routine. This proves (2.17) and hence (2.19).

As for the lower bound, we first prove the case $n=1$. Because f is nonnegative and

$$\begin{aligned}&G\left(\frac{2s(t-s)}{t}, z+y-\frac{s}{t}(x-x')\right)\\ \ge& 2^{-\frac{d}{2}}G\left(\frac{s(t-s)}{t}, z+y\right)T_\nu(t,x-x')\\ =& 2^{-\frac{d}{2}}(2\pi\nu s(t-s)/t)^{-d/2}e^{-\frac{|z+y|^2}{2\nu s(t-s)/s}}T_\nu(t,x-x')\\ \ge& 2^{-\frac{d}{2}}(2\pi\nu s(t-s)/t)^{-d/2}e^{-\frac{|z|^2+|y|^2}{\nu s(t-s)/s}}T_\nu(t,x-x')\\ =& 2^{-d}(\pi\nu s(t-s)/t)^{-d/2}e^{-\frac{|z|^2}{\nu s(t-s)/s}}e^{-\frac{|y|^2}{\nu s(t-s)/s}}T_\nu(t,x-x')\end{aligned}$$

$$
\begin{aligned}
&= 2^{-d} G\left(\frac{s(t-s)}{2t}, z\right) e^{-\frac{|y|^2}{\nu s(t-s)/s}} T_\nu(t, x-x') \\
&\geq 2^{-d} G\left(\frac{s(t-s)}{2t}, z\right) e^{-\frac{|y|^2}{\nu s/2}} T_\nu(t, x-x') \\
&= 2^{-d} G\left(\frac{s(t-s)}{2t}, z\right) T_\nu(t, x-x') T_{\nu/2}(s, y),
\end{aligned}
\tag{2.24}
$$

where we have used the fact that $s(t-s)/t \geq s/2$, which is equivalent to $s \in [0, t/2]$. we see that from (2.21) and (2.22),

$$
\begin{aligned}
\mathcal{L}_1(t, x, x'; y) \geq\ & 2^{-d} G(t, x) G(t, x') T_\nu(t, x-x') \\
& \times \int_0^{t/2} \mathrm{d}s\, T_{\nu/2}(s, y) \int_{\mathbb{R}^d} \mathrm{d}z\, f(z) G\left(\frac{s(t-s)}{2t}, z\right) \\
\geq\ & 2^{-d} G(t, x) G(t, x') T_\nu(t, x-x') \\
& \times \int_0^{t/2} \mathrm{d}s\, T_{\nu/4}(s, y) \int_{\mathbb{R}^d} \mathrm{d}z\, T_{\nu/2}(t-s, z) f(z) G\left(\frac{s(t-s)}{2t}, z\right).
\end{aligned}
$$

Because the function $z \mapsto f(z) T_{\nu/2}(t-s, z)$ is a valid correlation function, that is, it is symmetric, nonnegative, and nonnegative-definite, by taking Fourier transform and since $s(t-s)/(2t) \leq s/2$, one can see that

$$
\begin{aligned}
& \int_{\mathbb{R}^d} \mathrm{d}z\, T_{\nu/2}(t-s, z) f(z) G\left(\frac{s(t-s)}{2t}, z\right) \\
\geq & \int_{\mathbb{R}^d} \mathrm{d}z\, T_{\nu/2}(t-s, z) f(z) G\left(s/2, z\right) \\
\geq\ & 3^{-d/2} \int_{\mathbb{R}^d} \mathrm{d}z\, f(z) G(s/6, z) \\
\geq\ & 3^{-d/2} \int_{\mathbb{R}^d} \mathrm{d}z\, f(z) G(s, z) \geq 3^{-d/2} k(s).
\end{aligned}
\tag{2.25}
$$

Hence,

$$
\mathcal{L}_1(t, x, x'; y) \geq 2^{-d} 3^{-d/2} G(t, x) G(t, x') T_\nu(t, x-x') \int_0^{t/2} T_{\nu/4}(s, y) k(s) \mathrm{d}s,
$$

where the integral is equal to $h_1(t/2, y)$. Therefore, the case $n = 1$ is true.

Assume that (2.18) is true up to n. Then

$$
\begin{aligned}
\mathcal{L}_{n+1}(t, x, x'; y) =\ & \left(\mathcal{L}_0 \rhd \mathcal{L}_n\right)(t, x, x'; y) \\
\geq\ & (2\sqrt{3}\,)^{-nd} \int_0^{\frac{t}{2}} \mathrm{d}s\, h_n\left(\frac{t-s}{2}, y\right) \iint_{\mathbb{R}^{2d}} \mathrm{d}z \mathrm{d}z'\, G(s, z) G(s, z') \\
& \times G(t-s, x-z) G(t-s, x'-z') \\
& \times T_\nu(t-s, (x-z)-(x'-z')) f(y - [(x-z)-(x'-z')]) \\
\geq\ & (2\sqrt{3}\,)^{-nd} \int_0^{\frac{t}{2}} \mathrm{d}s\, h_n\left(t/2 - s, y\right) \iint_{\mathbb{R}^{2d}} \mathrm{d}z \mathrm{d}z'\, G(s, x-z) G(s, x'-z') \\
& \times G(t-s, z) G(t-s, z') T_\nu(t-s, z-z') f(y - [z-z']) \\
=\ & (2\sqrt{3}\,)^{-nd} \int_{\frac{t}{2}}^t \mathrm{d}r\, h_n\left(r - t/2, y\right) \iint_{\mathbb{R}^{2d}} \mathrm{d}z \mathrm{d}z'\, G(r, z) G(r, z')
\end{aligned}
$$

$$\times G(t-r,x-z)G(t-r,x'-z')T_\nu(r,z-z')f(y-[z-z']), \tag{2.26}$$

where we have used the fact that $s\mapsto h_n(s,y)$ is nondecreasing (Lemma 2.6). Notice that

$$T_\nu(r,z-z')\geq T_{\nu/2}(r,y-(z-z'))T_{\nu/2}(r,y).$$

By the same arguments as those in (2.21) and (2.22) with the correlation function $f(z)$ replaced by $z\mapsto f(z)T_{\nu/2}(r,z)$, the double integral $\mathrm{d}z\mathrm{d}z'$ in (2.26) becomes

$$G(t,x)G(t,x')T_{\nu/2}(r,y)\int_{\mathbb{R}^d}\mathrm{d}z\,T_{\nu/2}(r,z)f(z)G\left(\frac{2r(t-r)}{t},z+y-\frac{r}{t}(x-x')\right).$$

By (2.24), the above quantity is bounded from below by

$$2^{-d}G(t,x)G(t,x')T_\nu(t,x-x')T_{\nu/4}(r,y)\int_{\mathbb{R}^d}\mathrm{d}z\,T_{\nu/2}(r,z)f(z)G\left(\frac{r(t-r)}{2t},z\right),$$

where we have used the fact that $T_{\nu/2}(r,y)^2=T_{\nu/4}(r,y)$. Then, apply (2.25) with s replaced by $t-r$ to get

$$\begin{aligned}\mathcal{L}_{n+1}(t,x,x';y)=&\,(2\sqrt{3}\,)^{-(n+1)d}G(t,x)G(t,x')T_\nu(t,x-x')\\&\times\int_{t/2}^t\mathrm{d}s\,h_n(r-t/2,y)T_{\nu/4}(r,y)k(t-r).\end{aligned}$$

Because $r\in(t/2,t)$, $T_{\nu/4}(r,y)\geq T_{\nu/4}(t-r,y)$. Hence,

$$\begin{aligned}\int_{t/2}^t\mathrm{d}s\,h_n(r-t/2,y)T_{\nu/4}(r,y)k(t-r)&\geq\int_{t/2}^t\mathrm{d}s\,h_n(r-t/2,y)T_{\nu/4}(t-r,y)k(t-r)\\&=\int_0^{t/2}h_n(t/2-s,y)T_{\nu/4}(s,y)k(s),\end{aligned}$$

where the integral is equal to $h_{n+1}(t/2,y)$. This proves the case $n+1$ and (2.18). Finally, (2.20) is a direct consequence of (2.18). This completes the proof of Lemma 2.7. □

Lemma 2.8 For all $\mu\in\mathcal{M}_H(\mathbb{R}^d)$ and all $t\geq 0$, x, $x'\in\mathbb{R}^d$, it holds that

$$(\mathcal{K}\rhd J_1)(t,x,x';0)=\lambda^{-2}\iint_{\mathbb{R}^{2d}}\mu(\mathrm{d}y)\mu(\mathrm{d}y')\mathcal{K}(t,x-y,x'-y';y'-y)-J_1(t,x,x'). \tag{2.27}$$

Proof By writing $J_0(t,z)$ and $J_0(t,z')$ in the integral forms and then applying the arguments in the proof of Lemma 2.7, we see that

$$\begin{aligned}(\mathcal{K}\rhd J_1)(t,x,x';0)=&\int_0^t\mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}z\mathrm{d}z'\iint_{\mathbb{R}^{2d}}\mu(\mathrm{d}y)\mu(\mathrm{d}y')\,G(s,z-y)G(s,z'-y')\\&\times f(z'-z)\mathcal{K}(t-s,x-z,x'-z';z'-z).\end{aligned}$$

By change of variables, $\hat{z}=z-y$ and $\hat{z}'=z'-y'$, and by Fubini's theorem,

$$\begin{aligned}(\mathcal{K}\rhd J_1)(t,x,x';0)=&\iint_{\mathbb{R}^{2d}}\mu(\mathrm{d}y)\mu(\mathrm{d}y')\int_0^t\mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}\hat{z}\mathrm{d}\hat{z}'\,f((y'-y)-(\hat{z}-\hat{z}'))\\&\times G(s,\hat{z})G(s,\hat{z}')\mathcal{K}(t-s,x-y-\hat{z},x'-y'-\hat{z}';(y'-y)-(\hat{z}-\hat{z}'))\\=&\iint_{\mathbb{R}^{2d}}\mu(\mathrm{d}y)\mu(\mathrm{d}y')\,(\mathcal{K}\rhd\mathcal{L}_0)(t,x-y,x'-y';y'-y).\end{aligned}$$

Then use the recursion $\mathcal{K}\rhd\mathcal{L}_0=\lambda^{-2}\mathcal{K}-\mathcal{L}_0$ to get (2.27). □

Now, we are ready to prove Theorem 2.4.

Proof of Theorem 2.4 The proof follows the same six steps as those in the proof of [4, Theorem 2.4] with some minor changes:

(1) Both proofs rely on estimates on the kernel function $\mathcal{K}$. Instead of an explicit formula as for the heat equation case (see [4, Proposition 2.2]), Lemma 2.7 ensures the finiteness and provides a bound on the kernel function $\mathcal{K}$.

(2) In the Picard iteration scheme (Steps 1–4 in the proof of [4, Theorem 2.4]), we need to check the $L^p(\Omega)$-continuity of the stochastic integral, which then guarantees that at the next step, the integrand is again in $\mathcal{P}_2$, via [4, Proposition 3.4]. The statement of [4, Proposition 3.4] is still true for $G(t,x)$ on $\mathbb{R}^d$; see [3, Proposition 2.3.13]. Note that during each iteration, the measurability is guaranteed by Proposition 2.2 (in place of [4, Proposition 3.1])

(3) In the first step of the Picard iteration scheme, the following property is useful: For all compact sets $K \subseteq [0,\infty)\times\mathbb{R}^d$,

$$\sup_{(t,x)\in K} \left(\mathcal{K} \rhd [1+J_1]\right)(t,x,x;0) < +\infty.$$

For the heat equation, this property is discussed in [4, Lemma 3.9]. Here, Lemma 2.8 gives the desired result with minimal requirements on the initial data. This property, together with the calculation of the upper bound on the function $\mathcal{K}$ in Lemma 2.7, guarantees that all the $L^p(\Omega)$-moments of $u(t,x)$ are finite. This property is also used to establish uniform convergence of the Picard iteration scheme, hence $L^p(\Omega)$-continuity of $(t,x)\mapsto I(t,x)$.

(4) Moment formula (2.8) is clear from the Picard iterations. Formula (2.9) is due to Lemma 2.8.

As for (2.10), we only need to consider the nonlinear case. By (1.7), the function $g(t,x,x') = \mathbb{E}\left[u(t,x)u(t,x')\right]$ satisfies (2.3) with "=", λ, and $\mathcal{K}$ replaced by "$\le$", L_ρ, and $\overline{\mathcal{K}}$, respectively.

Similarly, for the lower bound (2.11), thanks to (1.3), the above g function satisfies (2.3) with "=" and λ replaced by "$\ge$" and l_ρ, respectively. Hence, this integral inequality is solved by (2.11), that is, by (2.9) with "=" and λ replaced by "$\ge$" and l_ρ, respectively. With this, we complete the proof of Theorem 2.4. □

3 Conditions for Phase Transitions: Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. We need to prove three lemmas first.

Lemma 3.1 Fix $a>0$. Let $c=\nu\pi/(2a^2)$. Then for all $t\in[0,\infty)\times\mathbb{R}^d$,

$$\int_{[-a,a]^d} G(t,y)\mathrm{d}y \ge (1+c\,t)^{-d/2},$$

and

$$\int_0^t \mathrm{d}s \int_{[-a,a]^d} G(s,y)\mathrm{d}y \ge \begin{cases} 2c^{-1}\left(\sqrt{ct+1}-1\right) & \text{if } d=1,\\ c^{-1}\log(1+c\,t) & \text{if } d=2,\\ 2\left[c(d-2)\right]^{-1}\left(1-(1+c\,t)^{1-d/2}\right) & \text{if } d\ge 3.\end{cases}$$

Proof We only need to prove the case where $d=1$. Notice that

$$\int_{-a}^{a} G(t,y)\mathrm{d}y = 2\Phi\left(\frac{a}{\sqrt{\nu t}}\right)-1,$$

where $\Phi(x)$ is the distribution of the standard normal distribution. Denote

$$F(t) := \sqrt{1+\frac{\nu\pi}{2a^2}t}\left[2\Phi\left(\frac{a}{\sqrt{\nu t}}\right)-1\right].$$

Clearly, $F(0)=1$. By l'Hospital's rule, $\lim_{t\to\infty} F(t)=1$. By studying $F'(t)$, one can show that for some $t_0>0$, $F(t)$ is nondecreasing over $[0,t_0]$ and nonincreasing over $[t_0,\infty]$. Therefore, $F(t)\ge 1$. The rest calculations follow from Example 4. $\square$

Lemma 3.2 For all $y\in\mathbb{R}^d$, we have

$$\lim_{t\to\infty} h_1(t)<\infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} h_1(t,y)<\infty.$$

Proof Because $h_1(t,y)\le h_1(t)$, the "if" part is clear. On the other hand, for any $\epsilon\in (0,t)$,

$$h_1(t,y)\ge\int_\epsilon^t \mathrm{d}s\, k(s)T_{\nu/4}(s,y)\ge T_{\nu/4}(\epsilon,y)\left[h_1(t)-h_1(\epsilon)\right].$$

This proves Lemma 3.2. $\square$

As we mentioned in the introduction section, finding some nontrivial lower bounds for the second moment is the most challenging task of this article. The following function gives this nontrivial lower bounds:

$$H_\nu^*(t,y;\gamma) := \sum_{n=0}^\infty \gamma^n h_1(t/n,y)^n.$$

Lemma 3.3 The following statements hold:

(1) For all $t\ge 0$, $y\in\mathbb{R}^d$, and $\gamma>0$, $H_\nu(t,y;\gamma)\ge H_\nu^*(t,y;\gamma)$.

(2) For all $a>0$ and $y\in\mathbb{R}^d$, if $\gamma\ge e/h_1(a,y)$, then

$$H_\nu^*(t,y;\gamma)\ge\frac{e^{t/a}-1}{e-1},\quad \text{for all } t\ge 0.$$

(3) If $\lim_{t\to\infty} h_1(t)=\infty$, then for all $\gamma>0$ and all $y\in\mathbb{R}^d$, we have

$$H_\nu^*(t,y;\gamma)\ge\frac{e^{t/a}-1}{e-1},\quad \text{for all } t\ge 0,$$

where $a>0$ is the value such that $h_1(a,y)=e/\gamma$.

(4) For $t>0$ fixed, the function $y\mapsto H_\nu(t,y;\gamma)$ is radial and nonincreasing in the sense that for all $x,y\in\mathbb{R}^d$, $H_\nu(t,x;\gamma)=H_\nu(t,y;\gamma)$ if $|x|=|y|$ and $H_\nu(t,x;\gamma)\le H_\nu(t,y;\gamma)$ if $|x|\ge|y|$. The same is true for $H_\nu^*(t,y;\gamma)$.

Proof (1) This is because

$$h_n(t,y)\ge h_1(t/n,y)^n,\quad \text{for } n\in\mathbb{N}, \tag{3.1}$$

which can be proved by induction. Indeed, it holds trivially for $n=1$. Suppose (3.1) holds for n. Now by the induction assumption and the fact that $t\mapsto h_1(t,y)$ is nondecreasing (Lemma 2.6), we see that

$$h_{n+1}(t,y)=\int_0^t \mathrm{d}s\, h_n(t-s,y)k(t-s)T_{\nu/4}(t-s,y)$$

$$\begin{aligned}
&\geq \int_0^t \mathrm{d}s\, h_1\left(\frac{t-s}{n}, y\right)^n k(t-s) T_{\nu/4}(t-s, y) \\
&\geq \int_0^{\frac{t}{n+1}} \mathrm{d}s\, h_1\left(\frac{t-\frac{t}{n+1}}{n}, y\right)^n k(t-s) T_{\nu/4}(t-s, y) \\
&= h_1\left(\frac{t}{n+1}, y\right)^n \int_0^{\frac{t}{n+1}} \mathrm{d}s\, k(t-s) T_{\nu/4}(t-s, y) \\
&= h_1(t/(n+1), y)^{n+1}.
\end{aligned}$$

(2) Fix $a > 0$ and $y \in \mathbb{R}^d$. Note that $h_1(t, y)$ is nondecreasing. So, when $h_1(a, y) > e/\gamma$,

$$\sum_{n=0}^{\infty} \gamma^n h_1(t/n, y)^n \geq \sum_{n=0}^{t/a} \gamma^n h_1(t/n, y)^n \geq \sum_{n=0}^{t/a} \gamma^n h_1(a, y)^n \geq \frac{e^{\lfloor t/a \rfloor + 1} - 1}{e-1} \geq \frac{e^{t/a} - 1}{e-1}.$$

(3) Fix arbitrary $\gamma > 0$ and $y \in \mathbb{R}^d$. One can find $a > 0$ such that $h_1(a, y) = e/\gamma$. Then apply the same arguments as those in (2).

(4) Fix $t > 0$. It is clear that both functions $T_\nu(t, y)$ and $h_0(t, y)$ are radial and nonincreasing in y. Then by induction, one can easily show that the same is true for all functions $y \mapsto h_n(t, y)$, $n \geq 1$, which implies the same property for both functions $H_\nu(t, y; \gamma)$ and $H_\nu^*(t, y; \gamma)$. $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 (1) We start with the case where $\Upsilon(0) < \infty$. From (2.23), we know that

$$\begin{aligned}
\mathcal{L}_1(t, x, x'; y) &\leq (2\pi)^{-d}\, G(t, x) G(t, x') \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}\xi) \exp\left(-\frac{\nu s}{4}|\xi|^2\right) \\
&= \frac{4}{\nu(2\pi)^d} G(t, x) G(t, x') \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{|\xi|^2}.
\end{aligned}$$

Denote $\theta := \frac{4}{\nu(2\pi)^d}\Upsilon(0)$. Hence, by induction,

$$\mathcal{L}_n(t, x, x'; y) \leq \theta^n G(t, x) G(t, x')$$

and if $L_\rho^2 \theta < 1$, that is,

$$L_\rho \leq 2^{-1}(2\pi)^{d/2} \nu^{1/2} \Upsilon(0)^{-1/2} =: \underline{\lambda}_c,$$

then

$$\mathcal{K}(t, x, x'; y) \leq G(t, x) G(t, x') \frac{1}{1 - \theta L_\rho^2}.$$

By (2.10), for all $\mu \in \mathcal{M}_H(\mathbb{R}^d)$ with $\mu \geq 0$,

$$||u(t, x)||_2^2 \leq J_0^2(t, x) \frac{1}{1 - \theta L_\rho^2}.$$

As μ satisfies (1.15), for all $\beta > 0$,

$$J_0(t, x) \leq \left(\sup_{y \in \mathbb{R}^d} G(t, x-y) e^{\beta|y|}\right) \int_{\mathbb{R}^d} e^{-\beta|y|} \mu(\mathrm{d}y).$$

Notice that

$$G(t, x-y) e^{\beta|y|} \leq \prod_{i=1}^d \frac{1}{\sqrt{2\pi\nu t}} \exp\left(-\frac{(x_i - y_i)^2}{2\nu t} + \beta|y_i|\right)$$

$$\leq (2\pi\nu t)^{-d/2} \exp\left(\frac{\beta^2 \nu d}{2} t + \beta \sum_{i=1}^{d} |x_i|\right).$$

Therefore,

$$\sup_{x\in\mathbb{R}^d} \overline{m}_2(x) \leq \frac{\beta^2\nu d}{2} \qquad \text{for all } \beta > 0,$$

which implies that $\sup_{x\in\mathbb{R}^d} \overline{m}_2(x) = 0$. This proves the first case in (1.10).

As for the second case in (1.10), by the lower bound for the second moment in (2.11) and that for $\mathcal{K}$ in (2.20) and part (1) of Lemma 3.3, we see that

$$\begin{aligned}\mathbb{E}\left[u(t,x)^2\right] \geq l_\rho^{-2} \iint_{\mathbb{R}^{2d}} \mu(\mathrm{d}z)\mu(\mathrm{d}z')\mathcal{L}_0(t, x-z, x-z') \\ \times T_\nu(t, z-z') H_\nu\left(\frac{t}{2}, z-z'; (2\sqrt{3})^{-d} l_\rho^2\right).\end{aligned}$$

Because $\mu \geq 0$ and $\mu \neq 0$, there exists $a > 0$ such that $\int_{[-a,a]^d} \mu(\mathrm{d}z) > 0$. Hence, we can restrict the above double integral from $\mathbb{R}^{2d}$ to $[-a,a]^{2d}$. Because of the radial and nonincreasing property of both functions $x \mapsto T_\nu(t,x)$ and $x \mapsto H_\nu(t,x;\gamma)$ (the latter one is due to part (4) of Lemma 3.3), we see that for all $z, z' \in [-a,a]^d$,

$$T_\nu(t, z-z') = \exp\left(-\frac{|z-z'|^2}{\nu t}\right) \geq \exp\left(-\frac{4da^2}{\nu t}\right)$$

and

$$H_\nu\left(\frac{t}{2}, z-z'; (2\sqrt{3})^{-d} l_\rho^2\right) \geq H_\nu\left(\frac{t}{2}, 2\vec{a}; (2\sqrt{3})^{-d} l_\rho^2\right),$$

where $\vec{a} = (a, \ldots, a) \in \mathbb{R}^d$. Moreover, we can bound $\mathcal{L}_0$ from below in the following way:

$$\begin{aligned}\mathcal{L}_0(t, x-z, x-z') &\geq (2\pi\nu t)^{-d} \exp\left(-\frac{2|x|^2 + |z|^2 + |z'|^2}{\nu t}\right) \\ &= 2^{-d}\mathcal{L}_0(t/2, z, z') T_\nu(t/2, x).\end{aligned}$$

Therefore, combining these lower bounds shows that

$$\mathbb{E}\left[u(t,x)^2\right] \geq l_\rho^{-2} \left[\int_{[-a,a]^d} \mu(\mathrm{d}z) G(t/2, z)\right]^2 H_\nu\left(\frac{t}{2}, 2\vec{a}; (2\sqrt{3})^{-d} l_\rho^2\right) \exp\left(-\frac{4da^2 + 2|x|^2}{\nu t}\right). \tag{3.2}$$

Finally, one can replace the above H_ν by H_ν^* because of part (1) of Lemma 3.3. For the cases when l_ρ is sufficiently large, one can apply part (2) of Lemma 3.3 to conclude that $\inf_{x\in\mathbb{R}^d} \underline{m}_2(x) > 0$.

(2) When $\Upsilon(0) = \infty$, the moment bound (2.8), Lemma 2.7, and part (3) of Lemma 3.3 together imply that $\inf_{x\in\mathbb{R}^d} \underline{m}_2(x) > 0$. The statement $m_1(x) \equiv 0$ is due to (1.7).

(3) The equivalence between (1.11) and (1.14) is due to (2.16). The implication "(1.12)⇒(1.14)" is because that

$$\lim_{t\to\infty} h_1(t) = (2\pi)^{-d} \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}\xi) \exp\left(-\frac{\nu t}{2}|\xi|^2\right) = \frac{2}{\nu(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{|\xi|^2}.$$

On the other hand, if $d \leq 2$, then Lemma 3.1 implies that (1.14) fails. This proves the implication "(1.14)⇒(1.12)". This completes the whole proof of Theorem 1.3. □

4 Intermittency Fronts: Proof of Theorem 1.4

We first prove the following lemma.

Lemma 4.1 If $\mu \geq 0$ satisfies (1.19) for some $\beta > 0$, then we have

$$J_0^2(t,x) \leq C^2(2\pi\nu t)^{-d} \exp\left(-\frac{2\beta}{\sqrt{d}}|x| + \nu\beta^2 t\right),$$

where $C = \int_{\mathbb{R}^d} e^{\beta|x|}\mu(\mathrm{d}x)$.

Proof Notice that

$$\frac{|y_1| + \cdots + |y_d|}{\sqrt{d}} \leq |y| = \sqrt{y_1^2 + \cdots + y_d^2} \leq |y_1| + \cdots + |y_d|.$$

By the same arguments as the proof of [4, Lemma 4.4] with β replaced by $\beta/\sqrt{d}$,

$$\begin{aligned} J_0^2(t,x) &\leq C^2(2\pi\nu t)^{-d} \prod_{i=1}^{d} \exp\left(-\frac{2\beta}{\sqrt{d}}|x_i| + \frac{\nu\beta^2}{d}t\right) \\ &\leq C^2(2\pi\nu t)^{-d} \exp\left(-\frac{2\beta}{\sqrt{d}}|x| + \nu\beta^2 t\right). \end{aligned}$$

□

Now, we prove Theorem 1.4.

Proof of Theorem 1.4 We first prove the upper bound. By (2.9) and (2.19),

$$||u(t,x)||_2^2 \leq L_\rho^{-2} J_0^2(t,x) \exp(\theta t),$$

where $\theta := \theta(\nu, L_\rho)$ is defined in (2.15). Hence, by Lemma 4.1, for $\alpha > 0$,

$$\sup_{|x|>\alpha t} ||u(t,x)||_2^2 \leq L_\rho^{-2}(2\pi\nu t)^{-d} \exp\left(-\frac{2\beta}{\sqrt{d}}\alpha t + \nu\beta^2 t + \theta t\right),$$

where $C := \int_{\mathbb{R}^d} e^{\beta|x|}\mu(\mathrm{d}x)$. Now, the exponential growth rate are

$$-\frac{2\beta}{\sqrt{d}}\alpha t + \nu\beta^2 t + \theta t < 0 \quad \Longleftrightarrow \quad \alpha > \frac{\sqrt{d}}{2}\left(\nu\beta + \frac{\theta}{\beta}\right),$$

which proves the upper bound.

Now, we consider the lower bound. Fix a constant $a > 0$ such that $\int_{[-a,a]^d} \mu(\mathrm{d}z) > 0$. Denote $\kappa := (2\sqrt{3}\,)^{-d}$. Following the notation in the proof of Theorem 1.3, from (3.2) we see that for $\alpha > 0$,

$$\sup_{|x|\geq\alpha t} ||u(t,x)||_2^2 \geq l_\rho^{-2}\left[\int_{[-a,a]^d} G(t/2,z)\mu(\mathrm{d}z)\right]^2 \exp\left(-\frac{2\alpha^2 t}{\nu} - \frac{4da^2}{\nu t}\right) H_\nu\left(t/2, 2\vec{a}; \kappa l_\rho^2\right),$$

which implies that

$$\liminf_{t\to+\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \log ||u(t,x)||_2^2 \geq -\frac{2\alpha^2}{\nu} + \liminf_{t\to\infty} \frac{1}{t} \log H_\nu\left(t/2, 2\vec{a}; \kappa l_\rho^2\right).$$

Therefore, by part (1) of Lemma 3.3,

$$\underline{\lambda}(2) \geq \left(\nu \liminf_{t\to\infty} \frac{1}{t} \log H_\nu^*\left(t, 2\vec{a}; \kappa l_\rho^2\right)\right)^{1/2}.$$

Then apply Lemma 3.3 for the above limit. This completes the proof of Theorem 1.4. □

Appendix

In Appendix 1, we will compute the functions $k(t)$ and $h_1(t)$ defined in (1.13), and also the lower bound for the second moment Lyapunov exponents for some specific kernel functions that satisfy Dalang's condition (1.2). In Appendix 2, we will show that the asymmetric convolution $\rhd$ is associative.

1 Appendix: Examples

Example 1.2 (Riesz kernels) Suppose $f(z) = |z|^{-\alpha}$ with $\alpha \in (0, 2\wedge d)$. Then,

$$k(t) = (2\pi\nu t)^{-d/2}\int_0^\infty \exp\left(-\frac{r^2}{2\nu t}\right) r^{-\alpha+d-1}\frac{\pi^{d/2}}{\Gamma(1+d/2)}\mathrm{d}r = C_{\alpha,\nu,d}\, t^{-\alpha/2},$$

with $C_{\alpha,\nu,d} := \frac{\nu^{-\alpha/2}}{2^{1+\alpha/2}}\frac{\Gamma((d-\alpha)/2)}{\Gamma(1+d/2)}$, and

$$h_1(t) = C^*_{\alpha,\nu,d}\, t^{1-\alpha/2}, \qquad \Upsilon(\beta) = C'_{\alpha,\nu,d}\beta^{-1+\alpha/2},$$

for some constants $C^*_{\alpha,\nu,d} = \frac{\nu^{-\alpha/2}}{2^{\alpha/2}(2-\alpha)}\frac{\Gamma((d-\alpha)/2)}{\Gamma(1+d/2)}$ and $C'_{\alpha,\nu,d} > 0$. By induction,

$$h_n(t) = C^n_{\alpha,\nu,d}\frac{t^{n(1-\alpha/2)}\Gamma(1-\alpha/2)^n}{\Gamma(n(1-\alpha/2)+1)}, \quad \text{for all } n\geq 0,$$

and hence

$$H_\nu(t;\lambda^2) = E_{1-\alpha/2,1}\left(\lambda^2 C_{\alpha,\nu,d}\,\Gamma(1-\alpha/2)t^{1-\alpha/2}\right),$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function with two parameters

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \Re\alpha > 0,\ \beta\in\mathbb{C},\ z\in\mathbb{C};$$

see, for example, [22]. The following asymptotic expansions are useful: As $|z|\to\infty$,

$$E_{\alpha,\beta}(z) \sim \frac{1}{\alpha}\exp\left(z^{1/\alpha}\right) - \sum_{k=1}^\infty \frac{z^{-k}}{\Gamma(\beta-\alpha k)}, \quad \text{if } 0<\alpha<2 \text{ and } |\arg z| < \alpha\pi/2, \tag{A.1}$$

we have

$$\lim_{t\to\infty}\frac{1}{t}\log H_\nu(t;\lambda^2) = [C_{\alpha,\nu,d}\,\Gamma(1-\alpha/2)]^{\frac{2}{2-\alpha}}\ \lambda^{\frac{4}{2-\alpha}}.$$

By Lemma 3.3,

$$\liminf_{t\to\infty}\frac{1}{t}\log H^*_\nu(t;\lambda^2) \geq C^\star_{\alpha,\nu,d}\lambda^{\frac{4}{2-\alpha}}, \qquad \text{with} \quad C^\star_{\alpha,\nu,d} = \left(\frac{C^*_{\alpha,\nu,d}}{e}\right)^{\frac{2}{2-\alpha}}.$$

The power of λ, which is $4/(2-\alpha)$, in both upper and lower bounds recovers the result by Foondun et al [15, Theorem 1.7]. The solution is always fully intermittent, that is, there is no phase transition for the second moment.

Example 1.3 (Ornstein-Uhlenbeck-type kernels) Suppose $f(z) = \exp(-|z|^\alpha)$ for $\alpha \in (0, 2]$. The case when $\alpha = 2$ has a closed form as follows:

$$k(t) = (2\pi\nu t)^{-d/2}\int_0^t \exp\left(-\frac{r^2}{2\nu t} - r^2\right) r^{+d-1}\frac{\pi^{d/2}}{\Gamma(1+d/2)}\mathrm{d}r = d^{-1}(1+2\nu t)^{-d/2},$$

and

$$h_1(t) = \begin{cases} \nu^{-1}\left(\sqrt{2\nu t+1}-1\right) & \text{if } d=1,\\ (4\nu)^{-1}\log(1+2\nu t) & \text{if } d=2,\\ [\nu(d-2)d]^{-1}\left(1-(1+2\nu t)^{1-d/2}\right) & \text{if } d\geq 3,\end{cases}$$

and

$$\Upsilon(\beta) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\pi^{d/2} e^{-|\xi|^2/2}}{\beta + |\xi|^2} \mathrm{d}\xi = d^{-1} 2^{-d} e^{\beta/2} \beta^{\frac{d}{2}-1} \Gamma\left(1 - \frac{d}{2}, \frac{\beta}{2}\right),$$

where $\Gamma(\nu, x) := \int_x^\infty t^{\nu-1} e^{-t} \mathrm{d}t$ is the incomplete Gamma function. Among the three equivalent conditions (1.11), (1.12), and (1.14), it is immediate to see from condition (1.12) that phase transition happens if and only if $d \geq 3$ because of the fast decay of $f(x)$ at ∞ and the boundedness of $f(x)$ at zero. One can also check this from condition (1.14). When $d = 1$ or 2, $h_1(t) \to \infty$ as $t \to \infty$. However, when $d \geq 3$, $h_1(t) \to (\nu(d-2)d)^{-1} < \infty$ as $t \to \infty$. Hence, phase transition occurs if and only if $d \geq 3$.

Example 1.4 (Brownian motion case) When $f(z) \equiv 1$, the noise reduces to a space-independent noise. In this case,

$$k(t) \equiv 1, \qquad h_1(t) = t, \qquad \Upsilon(\beta) = (2\pi)^{-d} \beta^{-1},$$

and by (2.15) and Lemma 3.3,

$$\lim_{t\to\infty} \frac{1}{t} \log H_\nu(t; \lambda^2) = \frac{\lambda^2}{(2\pi)^d} \quad \text{and} \quad \liminf_{t\to\infty} \frac{1}{t} \log H_\nu^*(t; \lambda^2) \geq \frac{\lambda^2}{e}.$$

Because $h_1(t) \to \infty$ as $t \to$ or because $\Upsilon(\beta) \to \infty$ as $\beta \to 0_+$, we see that there is no phase transition in this case. The system is always fully intermittent.

Example 1.5 (Space-time white noise case) When $d = 1$ and $f = \delta_0$, we have

$$k(t) = \frac{1}{\sqrt{2\pi\nu t}}, \qquad h_1(t) = \sqrt{\frac{2t}{\pi\nu}}, \qquad \Upsilon(\beta) = \frac{1}{2\sqrt{\beta}},$$

and by (2.15) and Lemma 3.3,

$$\lim_{t\to\infty} \frac{1}{t} \log H_\nu(t; \lambda^2) = \frac{\lambda^4}{2\nu} \quad \text{and} \quad \liminf_{t\to\infty} \frac{1}{t} \log H_\nu^*(t; \lambda^2) \geq \frac{2\lambda^4}{\pi\nu e^2}.$$

Clearly, there is no phase transition in this case.

Example 1.6 (Lower bound for $d = 1, 2$) When $f(x) \geq 1_{[-a,a]^d}(x)$ for some $a > 0$ and $d = 1, 2$, then by Lemmas 3.1 and 3.3,

$$\liminf_{t\to\infty} \frac{1}{t} \log H_\nu^*(t; \lambda^2) \geq \frac{\nu\pi}{2a^2} \left[\left(1 + \frac{\nu\pi e}{4a^2\lambda^2}\right)^2 - 1\right]^{-1} \to \frac{\lambda^2}{e} \quad \text{as } \lambda \to \infty \text{ if } d = 1,$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \log H_\nu^*(t; \lambda^2) \geq \frac{\nu\pi}{2a^2} \left[\exp\left(\frac{\nu\pi e}{2a^2\lambda^2}\right) - 1\right]^{-1} \to \frac{\lambda^2}{e} \quad \text{as } \lambda \to \infty \text{ if } d = 2.$$

2 Associative property of the convolution "$\rhd$"

Lemma 2.7 Let h, w, and g be three real-valued functions defined on $[0, \infty) \times \mathbb{R}^{3d}$. Suppose that $(h \rhd (w \rhd g))(t, x, x'; y)$ and $((h \rhd w) \rhd g)(t, x, x'; y)$ are well defined, where $t \geq 0$, x, x', and $y \in \mathbb{R}^d$. Then

$$(h \rhd (w \rhd g))(t, x, x'; y) = ((h \rhd w) \rhd g)(t, x, x'; y).$$

Proof By definition,

$$\begin{aligned}
&(h \rhd (w \rhd g))(t, x, x'; y) \\
=& \int_0^t \mathrm{d}s_1 \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_1' \, h(t - s_1, x - z_1, x' - z_1'; y - (z_1 - z_1'))
\end{aligned}$$

$$
\begin{aligned}
&\times (w \triangleright g)(s_1, z_1, z_1'; y) f(y - (z_1 - z_1')) \\
= &\int_0^t \mathrm{d}s_1 \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_1' \, h(t - s_1, x - z_1, x' - z_1'; y - (z_1 - z_1')) f(y - (z_1 - z_1')) \\
&\times \int_0^{s_1} \mathrm{d}s_2 \iint_{\mathbb{R}^{2d}} \mathrm{d}z_2 \mathrm{d}z_2' \, w(s_1 - s_2, z_1 - z_2, z_1' - z_2'; y - (z_2 - z_2')) \\
&\times g(s_2, z_2, z_2'; y) f(y - (z_2 - z_2')).
\end{aligned}
$$

Then by change of variables

$$
\begin{aligned}
&\hat{s}_1 = t - s_2 && \hat{z}_1 = x - z_2 && \hat{z}_1' = x' - z_2', \\
&\hat{s}_2 = t - s_1 && \hat{z}_2 = x - z_1 && \hat{z}_2' = x' - z_1',
\end{aligned}
$$

we see that

$$
\begin{aligned}
&(h \triangleright (w \triangleright g))(t, x, x'; y) \\
= &\int_0^t \mathrm{d}\hat{s}_1 \iint_{\mathbb{R}^{2d}} \mathrm{d}\hat{z}_1 \mathrm{d}\hat{z}_1' \, g(t - \hat{s}_1, x - \hat{z}_1, x' - \hat{z}_1'; y) f(y - [(x - \hat{z}_1) - (x - \hat{z}_1')]) \\
&\times \int_0^{\hat{s}_1} \mathrm{d}\hat{s}_2 \iint_{\mathbb{R}^{2d}} \mathrm{d}\hat{z}_2 \mathrm{d}\hat{z}_2' \, w(\hat{s}_1 - \hat{s}_2, \hat{z}_1 - \hat{z}_2, \hat{z}_1' - \hat{z}_2'; y - [(x - \hat{z}_1) - (x' - \hat{z}_1')]) \\
&\times h(\hat{s}_2, \hat{z}_2, \hat{z}_2'; y - [(x - \hat{z}_2) - (x' - \hat{z}_2')]) f(y - [(x - \hat{z}_2) - (x' - \hat{z}_2')]) \\
= &\int_0^t \mathrm{d}\hat{s}_1 \iint_{\mathbb{R}^{2d}} \mathrm{d}\hat{z}_1 \mathrm{d}\hat{z}_1' \, g(t - \hat{s}_1, x - \hat{z}_1, x' - \hat{z}_1'; y) f(y - [(x - \hat{z}_1) - (x - \hat{z}_1')]) \\
&\times (h \triangleright w)(\hat{s}_1, \hat{z}_1, \hat{z}_1'; y - [(x - \hat{z}_1) - (x - \hat{z}_1')]) \\
= &((h \triangleright w) \triangleright g)(t, x, x'; y).
\end{aligned}
$$

This completes the proof of Lemma 2.7. □

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