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Parabolic Anderson model with colored noise on the torus

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We construct an intrinsic family of Gaussian noises on the d-dimensional flat torus $\mathbb{T}^d$. It is the analogue of the colored noise on $\mathbb{R}^d$ and allows us to study stochastic PDEs on the torus in the Itô sense in high dimensions. With this noise, we consider the *parabolic Anderson model* (PAM) with measure-valued initial conditions and establish some basic properties of the solution, including a sharp upper and lower bound for the moments and Hölder continuity in space and time. The study of the toy model of $\mathbb{T}^d$ in the present paper is a first step in our effort to understand how geometry and topology play a role in the behavior of stochastic PDEs on general (compact) manifolds.

Keywords: Brownian bridge; Dalang's condition; intermittency; moment asymptotics; measure-valued initial condition; moment Lyapunov exponent; stochastic heat equation on torus; theta function

1. Introduction

In this paper, we construct an intrinsic family of Gaussian noises on the d-dimensional torus $\mathbb{T}^d := [-\pi,\pi]^d$ that is *colored* in space and white in time. It is the analogue of the colored noise on $\mathbb{R}^d$, and it enables one to study, in Itô's sense, the *parabolic Anderson model* (PAM, see (1.6) below) and other *stochastic partial differential equations* (SPDEs) on $\mathbb{T}^d$ in higher dimensions. In this setting, we aim at understanding how the topology and geometry of non-Euclidean spaces influence the behavior of the solution. More specifically, let $G(t,x)$ be the heat kernel on $\mathbb{T}^d$, i.e.,

$$G(t,x) := \sum_{k\in\mathbb{Z}^d}\prod_{i=1}^{d} p\left(t, x_i + 2\pi k_i\right) \quad \text{for all } t>0 \text{ and } x=(x_1,\cdots,x_d)\in\mathbb{T}^d, \tag{1.1}$$

where we use $p(t,x) := (2\pi t)^{-1/2}e^{-x^2/(2t)}$ for $x\in\mathbb{R}$ to denote the heat kernel on $\mathbb{R}$. Sometimes we use a subscript d to denote the dimension of the space, e.g., $p_d(t,x) := (2\pi t)^{-d/2}e^{-\frac{|x|^2}{2t}}$ for $x\in\mathbb{R}^d$ where $|x| = \sqrt{x_1^2+\cdots+x_d^2}$. Alternatively, $G_d(t,x)$ in (1.1) can be expressed as

$$G_d(t,x) = \prod_{i=1}^{d} G_1(t,x_i).$$

When there is no confusion from context, this subscript d will be omitted.

Fix $\rho\ge 0$ and $\alpha>0$. The spatial covariance function of our colored noise will be given by

$$f_{\alpha,\rho}(x) := \frac{\rho}{(2\pi)^d} + \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\left(G(t,x) - \frac{1}{(2\pi)^d}\right)\mathrm{d}t, \quad \text{for } x\in\mathbb{T}^d. \tag{1.2}$$

The function $f_{\alpha,\rho}$ is an analogue of the Riesz kernel on $\mathbb{R}^d$ and is related to the spectral decomposition of the Laplacian on $\mathbb{T}^d$ (see Section 3 for more details). Moreover, it is well understood that for all $\alpha > 0$ and $\rho \geq 0$, there exists some constant $C > 0$ such that $f_{\alpha,\rho}$ admits the following estimate (see, e.g., Lemma 2.9 of Brosamler (1983))

$$\left|f_{\alpha,\rho}(x)\right| \leq \begin{cases} C & \text{if } \alpha > d/2 \\ C\left(1+\log^{-}|x|\right) & \text{if } \alpha = d/2 \\ C|x|^{-d+2\alpha} & \text{if } \alpha < d/2 \end{cases} \quad \text{for all } x \in \mathbb{T}^d, \tag{1.3}$$

where $\log^{-} t := \max(0, -\log t)$. The above estimate implies that the colored noise is smoother than the white noise. In addition, the parameter ρ controls the level of $f_{\alpha,\rho}$ while α controls its regularity. In what follows, we adopt the following convention:

$$f_\alpha(x) := f_{\alpha,0}(x) \quad \text{and} \quad f(x) := f_1(x). \tag{1.4}$$

It is known that $f(x)$, the Green's function of the Laplace operator on $\mathbb{T}^d$, is not positive. The same can be said of our covariance function $f_{\alpha,\rho}(x)$ (see Lemma 3.2 below). Throughout the paper, we use the convention that

$$G(t,x,y) = G\left(t, [\![x-y]\!]\right) \quad \text{and} \quad f_{\alpha,\rho}(x,y) = f_{\alpha,\rho}([\![x-y]\!]), \tag{1.5}$$

where for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $[\![x]\!] := ([\![x_1]\!], \cdots, [\![x_n]\!])$ with $[\![x_i]\!] = \operatorname{mod}(x_i + \pi, 2\pi) - \pi$, i.e., $[\![x_i]\!]$ is the *signed remainder*[1] of x_i when divided by π. Note that $|[\![x-y]\!]|$ is the distance between x and y on the torus, namely, $\operatorname{dist}(x,y) := |[\![x-y]\!]|$.

For the colored noise described above, we consider the following SPDE, or PAM, on $\mathbb{T}^d$

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \lambda u(t,x)\dot{W}(t,x), & (t,x) \in (0,\infty) \times \mathbb{T}^d, \\ u(0,\cdot) = \mu, \end{cases} \tag{1.6}$$

where $\dot{W}(t,x)$ is a centered Gaussian noise, white in time and colored in space, and $\lambda \neq 0$ is a constant that controls the level of the noise. We assume that the initial condition μ is a finite (nonnegative) measure on $\mathbb{T}^d$, namely,

$$C_\mu := \int_{\mathbb{T}^d} \mu(\mathrm{d}x) < \infty. \tag{1.7}$$

The solution to (1.6) is interpreted as the integral equation or the *mild solution* (see Definition 5.1 below for more details)

$$u(t,x) = J_0(t,x) + \lambda \int_0^t \int_{\mathbb{T}^d} G(t-s,x,y)u(s,y)W(\mathrm{d}s,\mathrm{d}y) \quad \text{a.s.}, \tag{1.8}$$

[1]Here we use the same convention as *Wolfram Mathematica* that the mod function always returns the positive reminder, i.e., $\operatorname{mod}(x, 2\pi) = c$ iff $c \in [0, 2\pi)$ and $x = 2\pi d + c$ for some $d \in \mathbb{Z}$. Alternatively, $[\![x]\!]$ can be equivalently expressed as $[\![x]\!] := \operatorname{mod}(x, 2\pi, -\pi)$, where $\operatorname{mod}(m,n,d)$ is the same modulo function with an offset d as that in *Wolfram Mathematica*.

where the stochastic integral is in the Itô/Walsh sense and $J_0(t,x)$ refers to the solution to the homogeneous equation, namely,

$$J_0(t,x) := \int_{\mathbb{T}^d} G(t,x,y)\mu(\mathrm{d}y). \tag{1.9}$$

Assumption 1.1 (Dalang's condition). Denote $\mathbb{Z}_*^d := \mathbb{Z}^d \setminus \{0\}$. We assume that the correlation function $f_{\alpha,\rho}(\cdot)$ satisfies the following equivalent conditions

$$\sum_{k\in\mathbb{Z}_*^d} \frac{\mathcal{F}(f_{\alpha,\rho})(k)}{|k|^2} < \infty \quad \Longleftrightarrow \quad 2(\alpha+1) > d \quad \text{(thanks to (3.2))}. \tag{1.10}$$

Remark 1.1. *Dalang's condition* usually refers to the condition on the correlation function f such that the corresponding stochastic partial differential equation (SPDE) with an additive noise has a unique solution; see Dalang (1999). For most parabolic SPDEs, Dalang's condition is usually the necessary and sufficient condition for the existence of a unique solution when the noise is of multiplicative type. In the current setting, Dalang's condition (1.10) regulates the high frequencies of $f_{\alpha,\rho}$ in the Fourier mode, which in turn controls the singularity of $f_{\alpha,\rho}(x)$ at $x = 0$ in the direct mode (see (1.3)). We also would like to mention that Dalang's condition is only a condition on α not ρ.

Remark 1.2. The solution theory of (1.6) is rather straightforward when $\alpha > d/2$ since the covariance function $f_{\alpha,\rho}$ is bounded and continuous in this case (see (1.3)). We hence assume $\alpha < d/2$ in the rest of our discussion. The case $\alpha = d/2$ can be treated in similar ways as $\alpha < d/2$, which will be left for the interested readers.

Our first main result is summarized in the following theorem:

Theorem 1.3. *Suppose that the correlation function $f_{\alpha,\rho}(\cdot)$ satisfies Dalang's condition (1.10) for some $\alpha \in (0,d/2)$. Then there exists a unique solution u to (1.6) starting from a finite measure μ on $\mathbb{T}^d$. Moreover, the solution satisfies the following properties:*

1. *For all $t > 0$ and $x, x' \in \mathbb{T}^d$, it holds that*

$$\mathbb{E}\left[u(t,x)u(t,x')\right] = J_0(t,x)J_0(t,x') + \lambda^{-2} \iint_{\mathbb{T}^{2d}} \mu(\mathrm{d}z)\mu(\mathrm{d}z')\mathcal{K}_\lambda(t-s,x,z,x',z') \tag{1.11}$$

   where $\mathcal{K}$ is defined in Definition 4.2.

2. *For all $t > 0$, $x \in \mathbb{T}^d$, and $p \geq 2$, it holds that*

$$||u(t,x)||_p \leq \sqrt{2} J_0(t,x) \left[H_{4\lambda\sqrt{p}}(t)\right]^{1/2}, \tag{1.12}$$

   where the function $H_\lambda(t)$ is defined in (3.15). Moreover, when $\lambda^2 p$ is large enough, there exists some constant $C > 0$ such that

$$||u(t,x)||_p \leq C J_0(t,x) \exp\left(C\lambda^{\max\left(\frac{4}{2(1+\alpha)-d},2\right)} p^{\max\left(\frac{2}{2(1+\alpha)-d},1\right)} t\right), \tag{1.13}$$

   for all $(t,x,p) \in (0,\infty) \times \mathbb{T}^d \times [2,\infty)$.

Remark 1.4. It is worth mentioning that our result allows measure-valued initial conditions. In particular, it includes the important cases when $\mu = \delta_0$ and $\mu = 1$. The proof of Theorem 1.3 is inspired by the method developed in Chen and Dalang (2015), Chen and Kim (2019), Chen and Huang (2019) for the *stochastic heat equation* (SHE) on $\mathbb{R}^d$. The main novelty of our approach is to observe that the Brownian bridge (starting from x and conditioned on arriving at y at time t) plays an important role. The proof of the theorem is then based on the observation that a Brownian bridge on the torus is comparable with one on an Euclidean space when t is small, whereas it is comparable to a Brownian motion on the torus when t is large; see Lemma 2.2 and Lemma 2.3 for the precise statements.

The study of the PAM in current literature usually assumes that the covariance function is positive. Unfortunately, this is not the case for $f_{\alpha,\rho}$ in general. However, one can show that it is indeed a positive function when ρ is large enough (see Lemma 3.2 below). In this case, we are able to derive the following lower bound for the second moment of u.

Theorem 1.5. *Let the spatial covariance function $f : \mathbb{T}^{2d} \to \mathbb{R}_+$ be a generic nonnegative and nonnegative definite function. Assume that f is uniformly bounded from below away from zero, i.e.,*

$$C_f := \inf_{x,x' \in \mathbb{T}^d} f(x,x') > 0, \tag{1.14}$$

then for all $\epsilon > 0$, it holds that

$$\mathbb{E}\left(u(t,x)^2\right) \geq J_0^2(t,x) + \frac{1}{2}\lambda^{-2} c_\epsilon^d C_\mu^2 \exp\left(\frac{C_f t}{2}\right), \quad \forall (t,x) \in [\epsilon,\infty) \times \mathbb{T}^d, \tag{1.15}$$

where the constants C_μ and c_ϵ are given in (1.7) and (2.10), respectively.

Compared to Theorem 1.3, Theorem 1.5 provides a matching (exponential in t) lower bound for the second moment of u. This theorem will be proved in Section 6, where one can find more discussions regarding the lower bounds of the second moment.

We believe that Theorem 1.5 holds true for all $\rho > 0$; the condition requiring ρ large enough so that $f_{\alpha,\rho}$ is positive is only a technical assumption. Indeed, if we assume in addition that the initial data μ is given by a bounded measurable function, we are able to prove the exponential lower bound for all $\rho > 0$.

Theorem 1.6. *Assume that the initial condition is given by a bounded measurable function which is also bounded below away from zero. Then under Dalang's condition (1.10), the second moment of the solution to the parabolic Anderson model satisfies the exponential lower bound for some $C, C' > 0$,*

$$\mathbb{E}\left[u(t,x)^2\right] \geq C e^{C' t},$$

when the driving noise satisfies $\rho > 0$.

The heuristics of the above theorem go as follows. Assume for a moment that $\mu \equiv 1$ so that we have the Feynman-Kac formula for the second moment:

$$\mathbb{E}\left[u(t,x)^2\right] = \mathbb{E}\left[\exp\left\{\lambda^2 \int_0^t f_{\alpha,\rho}(B_s, \widetilde{B}_s) \mathrm{d}s\right\}\right]. \tag{1.16}$$

In the above, B and $\widetilde{B}$ are two independent Brownian motions on $\mathbb{T}^d$ starting from x and $\mathbb{E}$ refers to the expectation with respect to both Brownian motions. Since $\mathbb{T}^d$ is compact, the ergodic theorem implies that the exponent in (1.16) asymptotically and almost surely becomes

$$\begin{aligned}\int_0^t f_{\alpha,\rho}(B_s,\widetilde{B}_s)\mathrm{d}s &\sim t\times\left(\frac{1}{(2\pi)^{2d}}\iint_{\mathbb{T}^{2d}} f_{\alpha,\rho}(x,y)\mathrm{d}x\mathrm{d}y\right)\\ &= \frac{\rho t}{(2\pi)^d}+\frac{t}{(2\pi)^{2d}}\iint_{\mathbb{T}^{2d}} f_\alpha(x,y)\mathrm{d}x\mathrm{d}y\\ &= \frac{\rho t}{(2\pi)^d}+\frac{t}{(2\pi)^{d}}\int_{\mathbb{T}^{d}} f_{\alpha,0}(x)\mathrm{d}x, \quad \text{as } t\uparrow\infty.\end{aligned}$$

From (1.2), we see that $\int_{\mathbb{T}^d} f_{\alpha,0}(x)\mathrm{d}x=0$. Therefore, as $t\uparrow\infty$,

$$\mathbb{E}\left[u(t,x)^2\right]\ge\left[\exp\left\{\lambda^2\mathbb{E}\int_0^t f_{\alpha,\rho}(B_s,\widetilde{B}_s)\mathrm{d}s\right\}\right]\sim\exp\left(\frac{\rho\lambda^2}{(2\pi)^d}t+o(t)\right).$$

The above argument clearly suggests that the ergodicity of the Brownian motion (due to compactness of the torus) is the main source that leads to the exponential lower bound for the second moment. As a consequence, one always observes intermittency in this situation; no phase transition takes place. However, the question becomes more delicate when $\rho=0$. In addition, Theorem 1.6 does not address the case when the initial data is rough (see Remark 6.1 below) and it only provides the second moment lower bound instead of lower bounds for the general p-th moments with $p\ge 2$ (see Remark 6.2 below). These questions will be tackled in a future work.

Finally, the Hölder regularity of the solution u is given as follows.

Theorem 1.7. *If the noise correlation satisfies Dalang's condition* (1.10) *for some* $\alpha\in(0,d/2)$, *the unique solution* u *starting from a finite measure* μ *is* β_1*–Hölder continuous in time and* β_2*–Hölder continuous in space on* $(0,\infty)\times\mathbb{T}^d$ *for all*

$$\beta_1\in\left(0,\frac{2\alpha+2-d}{4}\right)\quad \textit{and}\quad \beta_2\in\left(0,\frac{2\alpha+2-d}{2}\right).$$

This theorem will be proved in Section 7. The proof of this theorem follows similar arguments as the corresponding result for $\mathbb{R}^d$ (see Theorem 1.8 of Chen and Huang (2019)) with more complexity introduced by the fundamental solution.

There has been a growing interest in the study of SPDE (and related polymers models) on some "exotic" spaces. For example, C. Cosco, I. Seroussi and O. Zeitouni considered directed polymers on an infinite graph in Cosco, Seroussi and Zeitouni (2021). In a recent work Baudoin et al. (2023), the authors studied the PAM on Heisenberg groups. In addition, A. Mayorcas and H. Singh released a preprint Mayorcas and Singh (2023) recently studying singular SPDEs on Homogeneous Lie Groups.

The construction of the colored noise on $\mathbb{T}^d$ presented in this paper grows from a discussion between the second author and Prof. Elton Hsu during the BIRS-CMO workshop "Theoretical and Applied Stochastic Analysis" in 2018. Later, the second author was informed by Prof. Fabrice Baudoin that a fractional noise is introduced in a similar spirit on general Riemannian manifolds in Gelbaum (2014).

Our construction of the noise can be generalized to general (compact) Riemannian manifolds, thereby encompassing a large class of spaces with rich geometric and topological properties. We show that the noise covariance $f_{\alpha,\rho}$ has a straightforward Fourier series decomposition in (3.2). This is the motivation for the definition (1.2). The eigenvectors of the Laplace-Beltrami operator correspond with the Fourier series on $\mathbb{T}^d$, but we can produce a similar noise using the basis of Laplace-Beltrami eigenfunctions on a general compact manifold. This naturally raises the questions: Which specific geometric or topological properties might influence the behavior of the PAM, and how might they introduce novel features to the model? The present work can be regarded as an initial step in this direction. In particular, as elaborated in Remark 1.4 above, we observed that the behavior of the Brownian bridge $B_{t,x,y}(s)$ (starting from x and conditioned on arriving at y at time t) for small time s plays an important role. On a general compact manifold, the Brownian bridge is still comparable to a Brownian motion when t is large and s is relatively small. Whereas when t is small, $B_{t,x,y}(s)$ is concentrated around the shortest geodesic(s) connecting x and y. This observation allows us to localize our computations and to tackle the problem in direct mode without resorting to Fourier analysis. The implementation of this observation in order to study the PAM on general manifolds will be given in a subsequent work.

The rest of the paper is organized as follows. In Section 2, we prove some preliminary properties of the densities of the Brownian motion and Brownian bridge on the torus. Section 3 is dedicated to the construction of the colored noise on the torus. The main technical step in proving Theorem 1.3 is prepared in Section 4. Then we prove Theorem 1.3 in Section 5 and Theorems 1.5 and 1.6 in Section 6. Finally, the Hölder continuity of the solution claimed in Theorem 1.7 is established in Section 7. In order to improve the readability of the article, we deferred the proof of Lemma 2.1 to the supplement Chen, Ouyang and Vickery (2025).

2. Preliminary – the fundamental solution and various properties

In this section, we establish some properties of the fundamental solution.

Lemma 2.1. *The fundamental solution $G(t,x)$ satisfies*

$$C_t^d \le \frac{G(t,x)}{p_d(t,x)} \le (2C_t)^d, \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{T}^d, \tag{2.1}$$

where the constant C_t as a function of t can be expressed in the following equivalent expressions:

$$\begin{aligned} C_t &= \sum_{n=-\infty}^{\infty} e^{-\frac{2n^2\pi^2}{t}} &&= \prod_{n=1}^{\infty}\left(1-e^{-\frac{4n\pi^2}{t}}\right)\left(1+e^{-\frac{2(2n-1)\pi^2}{t}}\right)^2 \\ &= \sqrt{\frac{t}{2\pi}}\sum_{n=-\infty}^{\infty} e^{-\frac{n^2t}{2}} &&= \sqrt{\frac{t}{2\pi}}\prod_{n=1}^{\infty}\left(1-e^{-nt}\right)\left(1+e^{-\frac{(2n-1)t}{2}}\right)^2. \end{aligned} \tag{2.2}$$

Moreover, the following properties hold:

1. *C_t has the following asymptotic properties:*

$$\lim_{t\to 0} t\log\left(\log\left(C_t\right)\right) = -\pi^2 \quad \text{and} \quad \lim_{t\to\infty}\frac{1}{t}\log\left(\log\left(C_t\sqrt{\frac{2\pi}{t}}\right)\right) = -\frac{1}{2}; \tag{2.3}$$

2. *C_t satisfies the following bounds for all $t>0$:*

$$\max\left(1,\sqrt{\frac{t}{2\pi}}\right) \le C_t \le 1+\sqrt{\frac{t}{2\pi}}; \tag{2.4}$$

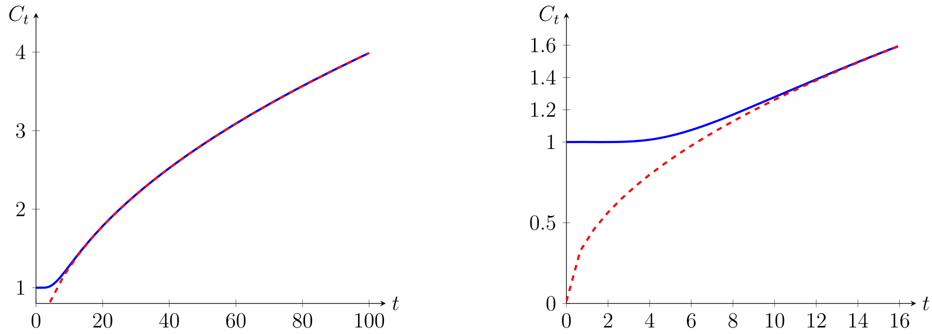

Figure 1. Some plots (solid lines) of the function C_t as given in (2.2) together with plots (dashed lines) for $\sqrt{t/(2\pi)}$. The left (resp. right) figure illustrates the small (resp. large) t behavior of C_t.

3. *For all* $t > 0$ *and* $x \in \mathbb{T}^d$, *it holds that*

$$G(t,x) \le \left(\sqrt{\frac{2\pi}{t}}\, C_t\right)^d \le \left(1 + \sqrt{\frac{2\pi}{t}}\right)^d; \tag{2.5}$$

4. *For any* $\epsilon > 0$, *there exists some constant* $\Theta_{\epsilon,d} > 0$ *such that for all* $t \ge \epsilon$,

$$\sup_{x\in\mathbb{T}^d} \left| G(t,x) - \left(\frac{1}{2\pi}\right)^d \right| \le \Theta_{\epsilon,d}\, e^{-t/2}. \tag{2.6}$$

The proof of Lemma 2.1 is given in the supplement to this paper Chen, Ouyang and Vickery (2025). The estimate in (2.6) can also be found in Theorem 2.15 of Baxter and Brosamler (1976). In Figure 1, we plot some graphs of C_t as a function of t.

We will need to introduce the density of the *pinned Brownian motion* or *Brownian bridge* (started at x_0 and terminating at x at time t) on $\mathbb{T}^d$:

$$G_{t,x_0,x}(s,z) := \frac{G(s,x_0,z)G(t-s,z,x)}{G(t,x_0,x)}, \quad \forall (x_0,x,z) \in \mathbb{T}^{3d},\ 0 < s < t. \tag{2.7}$$

The corresponding transition density for the Brownian bridge on $\mathbb{R}^d$ is

$$\begin{aligned} p_{t,x_0,x}(s,z) &:= \frac{p(s,x_0-z)p(t-s,z-x)}{p(t,x_0-x)} \\ &= p\left(\frac{s(t-s)}{t}, z - \left(x_0 + \frac{s}{t}(x-x_0)\right)\right), \quad \forall (x_0,x,z) \in \mathbb{R}^{3d},\ 0 < s < t. \end{aligned} \tag{2.8}$$

The following lemma states that when t is “large” and s is small, the transition density of a Brownian bridge is comparable to that of a Brownian motion on torus.

Lemma 2.2. *Fix an arbitrary* $\epsilon > 0$. *Suppose that* $t \ge \epsilon$. *Then for all* $s \in [0, t/2]$ *and* $x_0, z \in \mathbb{T}^d$, *the density for the Brownian bridge is comparable to a Gaussian on the torus,*

$$c_\epsilon^d\, G(s,x_0,z) \le G_{t,x_0,x}(s,z) \le C_\epsilon^d\, G(s,x_0,z), \tag{2.9}$$

where

$$c_\epsilon := \frac{\sqrt{\epsilon}}{2\sqrt{\pi}+\sqrt{2\epsilon}} \times \frac{1}{2\sqrt{2}} e^{-\frac{\pi^2}{2\epsilon}} \quad \textit{and} \quad C_\epsilon := 2\left(1+\sqrt{\frac{2\pi}{\epsilon}}\right) e^{\pi^2/\epsilon}. \tag{2.10}$$

Proof. We first prove the case when $d=1$. From (2.1), we see that

$$\frac{p_1(t-s,[\![z-x]\!])}{p_1(t,[\![x_0-x]\!])} \times \frac{C_{t-s}}{2C_t} \le \frac{G(t-s,z,x)}{G(t,x_0,x)} \le \frac{p_1(t-s,[\![z-x]\!])}{p_1(t,[\![x_0-x]\!])} \times \frac{2C_{t-s}}{C_t}.$$

Since $[\![x]\!] \in [-\pi,\pi]$, we see that

$$\sqrt{\frac{t-s}{t}} e^{-\frac{\pi^2}{2t}} \le \frac{p_1(t-s,[\![z-x]\!])}{p_1(t,[\![x_0-x]\!])} \le \sqrt{\frac{t-s}{t}} e^{\frac{\pi^2}{2(t-s)}}.$$

Since $t>\epsilon$ and $s\in(0,t/2)$, we see that $\epsilon/2 \le t/2 \le t-s \le t$. Hence,

$$\frac{1}{\sqrt{2}} e^{-\frac{\pi^2}{2\epsilon}} \le \frac{p_1(t-s,[\![z-x]\!])}{p_1(t,[\![x_0-x]\!])} \le e^{\pi^2/\epsilon}.$$

From (2.4), we see that

$$\frac{C_{t-s}}{C_t} \le \frac{1+\sqrt{\frac{t-s}{2\pi}}}{\sqrt{\frac{t}{2\pi}}} \le \frac{\sqrt{2\pi}+\sqrt{t}}{\sqrt{t}} \le 1+\sqrt{\frac{2\pi}{\epsilon}} \qquad \text{and}$$

$$\frac{C_{t-s}}{C_t} \ge \frac{\sqrt{\frac{t-s}{2\pi}}}{1+\sqrt{\frac{t}{2\pi}}} \ge \frac{\sqrt{\frac{t}{4\pi}}}{1+\sqrt{\frac{t}{2\pi}}} \ge \frac{\sqrt{\frac{\epsilon}{4\pi}}}{1+\sqrt{\frac{\epsilon}{2\pi}}} = \frac{\sqrt{\epsilon}}{2\sqrt{\pi}+\sqrt{2\epsilon}}.$$

Combining the above inequalities proves the case of $d=1$ with the constants given in the statement of the lemma. As for the case $d\ge 2$, one only needs to raise the above constants c_ϵ and C_ϵ to the power of d. This proves Lemma 2.2. □

The next lemma is our formal statement that when t is small, the Brownian Bridge on the torus can be compared to that on $\mathbb{R}^d$:

Lemma 2.3. *There exists a universal constant $C>0$ such that for all $t>s>0$ and $x,x_0,z\in\mathbb{T}^d$ with $z-x_0\in\mathbb{T}^d$, it holds that*

$$G_{t,x_0,x}(s,z) \le C\left(1+\sqrt{t}\right)^d \sum_{k\in\Pi^d} p_{t,x_0,x+k}(s,z),$$

where

$$\Pi := \{-2\pi, 0, 2\pi\}. \tag{2.11}$$

Note that $p_{t,x_0,x}(s,z)$ is a function on $\mathbb{R}^d$ (as apposed to a function on the torus). The condition $z-x_0\in\mathbb{T}^d$ above is simply a concise way to state that, when evaluated by $p_{t,x_0,x}(s,\cdot)$, z takes values in $[x_0^1-\pi,x_0^1+\pi]\times\cdots\times[x_0^d-\pi,x_0^d+\pi]\subset\mathbb{R}^d$ for $x_0=(x_0^1,\dots,x_0^d)$.

Proof. It suffices to prove the case $d = 1$. Fix arbitrary $t > s > 0$. Let $x, x_0, z \in \mathbb{T}$ with $z - x_0 \in \mathbb{T}$. From (2.1), we see that

$$G_{t,x_0,x}(s,z) = \frac{G(s,z-x_0)G(t-s,[\![x-z]\!])}{G(t,[\![x-x_0]\!])} \le 4\frac{C_sC_{t-s}}{C_t}\frac{p(s,z-x_0)p(t-s,[\![x-z]\!])}{p(t,[\![x-x_0]\!])},$$

where we have used the fact that $z - x_0 \in \mathbb{T}$ in the equality. By the properties of C_t given in Lemma 2.1, we have that

$$\frac{C_sC_{t-s}}{C_t} \le \frac{C_tC_t}{C_t} = C_t \le 1 + \sqrt{\frac{t}{2\pi}}.$$

Hence, for some universal constant $C > 0$,

$$G_{t,x_0,x}(s,z) \le C\left(1+\sqrt{t}\right)\frac{p(s,z-x_0)p(t-s,[\![x-z]\!])}{p(t,[\![x-x_0]\!])}.$$

In order to determine the values of the modulo functions, take into account the fact that $x, x_0, z \in [-\pi, \pi]$ and consider the following three cases:

Case I: $x - x_0 \in (\pi, 2\pi]$. In this case, as illustrated in Figure 2, we have

$$[\![x-x_0]\!] = x - x_0 - 2\pi \quad \text{and} \quad [\![x-z]\!] \in \{x - z + k : k = -2\pi, 0\}.$$

Hence,

$$\begin{aligned}\frac{p(s,z-x_0)p(t-s,[\![x-z]\!])}{p(t,[\![x-x_0]\!])} &\le \frac{p(s,z-x_0)}{p(t,x-x_0-2\pi)}\sum_{k\in\{-2\pi,0\}} p(t-s,x-z+k)\\ &= \sum_{k\in\{-2\pi,0\}}\frac{p(t,x-x_0+k)}{p(t,x-x_0-2\pi)}\times p_{t,x_0,x+k}(s,z).\end{aligned} \tag{2.12}$$

In this case, $|x - x_0 + k| \ge |x - x_0 - 2\pi|$ for $k \in \{0, -2\pi\}$. Hence, the ratio of two heat kernels in (2.12) is bounded by one. Therefore,

$$\frac{p(s,z-x_0)p(t-s,[\![x-z]\!])}{p(t,[\![x-x_0]\!])} \le \sum_{k\in\{-2\pi,0\}} p_{t,x_0,x+k}(s,z).$$

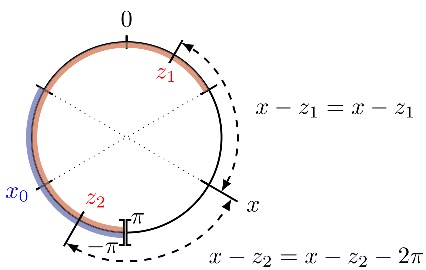

Figure 2. Illustration of Case I in the proof of Lemma 2.3. Since $x - x_0 \in (\pi, 2\pi]$ and $x, x_0 \in [-\pi, \pi]$, we see that x has to situate in $(0, \pi]$. The possible positions of x_0 are given in the blue arc. Since $z - x_0 \in [-\pi, \pi]$, the possible positions of z are highlighted in the red arc. When computing $[\![x - z]\!]$, we have two cases illustrated by z_1 and z_2.

Case II: $x - x_0 \in [-\pi, \pi]$. In this case, we have that

$$[\![x - x_0]\!] = x - x_0 \quad \text{and} \quad [\![x - z]\!] \in \{x - z + k : \ k = -2\pi, 0, 2\pi\}.$$

By the same arguments as in Case I, one can show that the ratio of two heat kernels in (2.12) is bounded by one and hence,

$$\frac{p(s, z - x_0) p\left(t - s, [\![x - z]\!]\right)}{p\left(t, [\![x - x_0]\!]\right)} \le \sum_{k \in \{-2\pi, 0, 2\pi\}} p_{t, x_0, x+k}(s, z).$$

Case III: $x - x_0 \in [-2\pi, -\pi)$. This is the symmetric case of Case I. In this case,

$$[\![x - x_0]\!] = x - x_0 + 2\pi \quad \text{and} \quad [\![x_0 - x + z]\!] \in \{x - x_0 + z + k : \ k = 0, 2\pi\}.$$

By similar arguments, we see that

$$\frac{p(s, z) p\left(t - s, [\![x_0 + z - x]\!]\right)}{p\left(t, [\![x_0 - x]\!]\right)} \le \sum_{k \in \{0, 2\pi\}} p_{t, x_0, x+k}(s, z).$$

Combining the above three cases proves Lemma 2.3 for the case $d = 1$. The generalization to the case for $d \ge 2$ is straightforward and we omit the details here. □

3. Colored noise on flat torus

In this section, we construct a family of intrinsic Gaussian noises on the flat torus $\mathbb{T}^d$ that we call *colored noise* on $\mathbb{T}^d$ that is white in time. Recall that the eigenvectors of the Laplace operator $\triangle$ are given by $\exp(ik \cdot x)$, $k \in \mathbb{Z}^d$. For any $\varphi \in L^2(\mathbb{T}^d)$, there is a unique decomposition

$$\varphi(x) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} a_k \, e^{ik \cdot x},$$

where a_k are the Fourier coefficients of φ:

$$a_k = \mathcal{F}(\varphi)(k) := (2\pi)^{-d/2} \int_{\mathbb{T}^d} \varphi(x) e^{-ik \cdot x} \mathrm{d}x, \quad \text{for all } k \in \mathbb{Z}^d.$$

In particular, $a_0 = (2\pi)^{-d/2} \int_{\mathbb{T}^d} \varphi(x) \mathrm{d}x$.

We introduce a family of Gaussian noises $\dot{W}$ on $\mathbb{T}^d$ with parameters $\alpha, \rho \ge 0$ as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space such that for any $\varphi(x)$ and $\psi(x)$ on $\mathbb{T}^d$ and $t, s > 0$, both $\dot{W}\left(1_{[0,t]}\varphi\right)$ and $\dot{W}\left(1_{[0,s]}\psi\right)$ are centered Gaussian random variables with covariance given by

$$\begin{aligned} &\mathbb{E}\left(\dot{W}\left(1_{[0,t]}\varphi\right) \dot{W}\left(1_{[0,s]}\psi\right)\right) = (s \wedge t) \langle \phi, \psi \rangle_{\alpha,\rho} \quad \text{with} \\ &\langle \phi, \psi \rangle_{\alpha,\rho} := \rho a_0 \bar{b}_0 + \sum_{k \in \mathbb{Z}^d_*} \frac{a_k \bar{b}_k}{|k|^{2\alpha}} \quad \text{and} \quad \mathbb{Z}^d_* := \mathbb{Z}^d \setminus \{0\}, \end{aligned} \tag{3.1}$$

where a_k's and b_k's are the Fourier coefficients of φ and ψ, respectively.

For $\rho > 0$, let $\mathcal{H}^{\alpha,\rho}$ be the completion of $L^2(\mathbb{T}^d)$ under $\langle\cdot,\cdot\rangle_{\alpha,\rho}$. Then, $(\Omega, \mathcal{H}^{\alpha,\rho}, \mathbb{P})$ gives an abstract Wiener space. When $\rho = 0$, some special care is needed in order to identify a suitable Hilbert space $\mathcal{H}^{\alpha,0}$. Let L_0^2 be the space of L^2 functions on $\mathbb{T}^d$ such that $a_0 = 0$. Denote by $\mathcal{H}_0^\alpha$ the completion of L_0^2 under $\langle\cdot,\cdot\rangle_{\alpha,\rho}$. One could have set $\mathcal{H}^{\alpha,0} = \mathcal{H}_0^\alpha$. However, when solving SPDEs and considering the mild form of the solution in (1.8), one in general would not expect $G(t-s, x-\cdot)u(s,\cdot) \in L_0^2$. Hence it is desirable to consider Wiener integrals $\dot{W}\left(1_{[0,t]}\varphi\right)$ where φ is a function on the torus such that $a_0 = (2\pi)^{-d/2}\int_{\mathbb{T}^d}\varphi(x)\mathrm{d}x \neq 0$. For this purpose, consider $\mathcal{H}_0^\alpha + \mathbb{R} := \{\varphi + c : \varphi \in \mathcal{H}_0^\alpha, \text{and } c \in \mathbb{R}\}$. We can identify $\mathcal{H}_0^\alpha + \mathbb{R}$ with $\mathcal{H}_0^\alpha$ through the equivalence relation $\sim$, in which $\varphi \sim \psi$ if $\varphi - \psi$ is a constant. Finally, we set

$$\mathcal{H}^{\alpha,0} = (\mathcal{H}_0^\alpha + \mathbb{R})/\sim .$$

With this construction, it is clear that $\dot{W}\left(1_{[0,t]}\varphi\right) = \dot{W}\left(1_{[0,t]}(\varphi + c)\right)$ for any $\varphi \in \mathcal{H}^{\alpha,0}$ and $c \in \mathbb{R}$. Throughout the rest of our discussion, we will also adopt the short-hand $\mathcal{H}^\alpha$ for $\mathcal{H}^{\alpha,0}$.

Remark 3.1. It is clear from (3.1) that $L^2(\mathbb{T}^d) \subset \mathcal{H}^{\alpha,\rho} \subset \mathcal{H}^{\beta,\rho}$ for $0 \le \alpha < \beta$. Moreover, the colored noise becomes the white noise on torus when $\rho = 1$ and $\alpha = 0$.

Recall that $f_{\alpha,\rho}$ and f_α are defined in (1.2) and (1.4) respectively.

Lemma 3.2. *Fix arbitrary $\alpha > 0$ and $\rho \ge 0$. Let $\Theta = \Theta_{1,d}$ be the constant given in (2.6). The following statements hold:*

(i) $\int_{\mathbb{T}^d} f_\alpha(x)\mathrm{d}x = 0$;
(ii) *$f_\alpha(x)$ assume both positive and negative values;*
(iii) *$f_{\alpha,\rho}(x)$ is bounded from below:*

$$f_{\alpha,\rho}(x) \ge (2\pi)^{-d/2}\left(\rho - \frac{1}{\Gamma(\alpha+1)(2\pi)^{d/2}} - (2\pi)^{d/2}2^\alpha\Theta\right), \quad \text{for all } x \in \mathbb{T}^d;$$

(iv) *$f_{\alpha,\rho}(\cdot)$ is nonnegative when $\rho \ge (2\pi)^{-d/2}\Gamma(\alpha+1)^{-1} + (2\pi)^{d/2}2^\alpha\Theta$;*
(v) *The Fourier coefficients for $f_{\alpha,\rho}$ are given by*

$$\theta_n := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{T}^d} f_{\alpha,\rho}(x)e^{-in\cdot x}\mathrm{d}x = \begin{cases} \dfrac{\rho}{(2\pi)^{d/2}} & \text{if } n = 0 \\ \dfrac{1}{|n|^{2\alpha}(2\pi)^{d/2}} & \text{if } n \in \mathbb{Z}_*^d \end{cases}; \tag{3.2}$$

(vi) *For any φ and $\psi \in \mathcal{H}^{\alpha,\rho}$, it holds that*

$$\langle\varphi,\psi\rangle_{\alpha,\rho} = \iint_{\mathbb{T}^{2d}} \varphi(x) f_{\alpha,\rho}(x,y)\psi(y)\mathrm{d}x\mathrm{d}y. \tag{3.3}$$

As a consequence, the noise introduced in (3.1) can be equivalently expressed by

$$\mathbb{E}\left(\dot{W}(1_{[0,t]}\varphi)\dot{W}(1_{[0,s]}\psi)\right) = (t\wedge s)\iint_{\mathbb{T}^{2d}} \varphi(x) f_{\alpha,\rho}(x,y)\psi(y)\mathrm{d}x\mathrm{d}y.$$

Note that the lower bounds in parts (iii) and (iv) are not optimal.

Proof. By writing

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \left(G(t,x) - (2\pi)^{-d}\right) \mathrm{d}t = \int_0^1 + \int_1^\infty = I_1 + I_2, \tag{3.4}$$

thanks to the heat kernel estimate in (2.6), one can apply the dominated convergence theorem to switch the dx- and the dt-integrals. This yields part (i). Part (ii) is an immediate consequence of part (i). As for part (iii), from (3.4), we see that

$$I_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} \left(G(t,x) - \frac{1}{(2\pi)^d}\right) \mathrm{d}t \geq -\frac{1}{\Gamma(\alpha)(2\pi)^d} \int_0^1 t^{\alpha-1} \mathrm{d}t = -\frac{1}{\Gamma(\alpha+1)(2\pi)^d},$$

where the inequality is due to the positivity of the heat kernel. As for I_2, for the constants $\Theta = \Theta_{1,d}$ given in (2.6) and $\gamma = 1/2$,

$$I_2 = \frac{1}{\Gamma(\alpha)} \int_1^\infty t^{\alpha-1} e^{-\gamma t} e^{\gamma t} \left(G(t,x) - \frac{1}{(2\pi)^d}\right) \mathrm{d}t \geq -\frac{\Theta}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\gamma t} \mathrm{d}t \geq -\Theta \gamma^{-\alpha}.$$

This proves part (iii). Part (iv) is a direct consequence of part (iii).

As for part (v), by part (i), we see that $\theta_0 = (2\pi)^{-d/2}\rho$. For $n \neq 0$, using the heat kernel represented via the eigenvectors of the Laplace operator, namely,

$$G(t,x,y) = (2\pi)^{-d} \sum_{k \in \mathbb{Z}^d} e^{-\frac{|k|^2}{2} t} e^{ik\cdot x} e^{-ik\cdot y}, \tag{3.5}$$

by an application of Fubini's theorem, we see that

$$\theta_n = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \left((2\pi)^{-d} e^{-|n|^2 t} \int_{\mathbb{T}^d} (2\pi)^{-d/2} \mathrm{d}x - 0\right) = \frac{1}{|n|^{2\alpha} (2\pi)^{d/2}}.$$

This proves part (v). Finally, denote the double integral in (3.3) by I. Then by the Plancherel theorem, we see that

$$\begin{aligned} I = \left\langle \varphi, \overline{f_{\alpha,\rho} * \psi} \right\rangle_{\mathbb{T}^d} &= \sum_{k \in \mathbb{Z}^d} \mathcal{F}(\varphi)(k) \overline{\mathcal{F}(f_{\alpha,\rho} * \psi)(k)} \\ &= \sum_{k \in \mathbb{Z}^d} \mathcal{F}(\varphi)(k) \left[(2\pi)^{d/2} \times \overline{\mathcal{F}(f_{\alpha,\rho})(k)} \times \overline{\mathcal{F}(\psi)(k)}\right] = (2\pi)^{d/2} \sum_{n \in \mathbb{Z}^d} a_n \overline{b}_n \theta_n, \end{aligned}$$

where a_n and b_n are Fourier coefficients of φ and ψ, respectively, and $\theta_n \geq 0$ are given in (3.2). This completes the proof of Lemma 3.2. □

Now we introduce some temporal functions that are determined by the noise and will appear in the Picard iterations in the proof of the main result—Theorem 1.3. Define

$$k_1(s) = k_1(s;\alpha,\rho) := \sum_{k \in \mathbb{Z}^d} \mathcal{F}(f_{\alpha,\rho})(k) e^{-s|k|^2}. \tag{3.6}$$

The next lemma gives some estimates on $k_1(s)$.

Lemma 3.3. 1. *For any* $\beta > \max(-\alpha + d/2, 0)$ *and* $s > 0$, *it holds that*

$$k_1(s;\alpha,\rho) \le \frac{\rho}{(2\pi)^{d/2}} + C_{\alpha,\beta,d}\, s^{-\beta} \quad \textit{with} \quad C_{\alpha,\beta,d} := \frac{\beta^\beta e^{-\beta}}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}_*^d} \frac{1}{|k|^{2(\alpha+\beta)}} < \infty. \tag{3.7}$$

2. *Under Dalang's condition* (1.10), *namely,* $2(\alpha+1) > d$, *for all* $t \ge 0$ *and* $\gamma > 0$, *we have that*

$$\int_0^t k_1(s;\alpha,\rho)\mathrm{d}s \le \frac{\rho t}{(2\pi)^{d/2}} + C_{\alpha,d}, \quad \textit{where} \quad C_{\alpha,d} := \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}_*^d} \frac{1}{|k|^{2\alpha+2}} < \infty, \tag{3.8}$$

and

$$\int_0^\infty e^{-\gamma s} k_1(s;\alpha,\rho)\mathrm{d}s = \frac{\rho}{(2\pi)^{d/2}}\frac{1}{\gamma} + \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}_*^d} \frac{1}{|k|^{2\alpha}\left(|k|^2+\gamma\right)} < \infty. \tag{3.9}$$

Proof. (1) From (3.2), we see that

$$\begin{aligned} k_1(s) &= \frac{\rho}{(2\pi)^{d/2}} + \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}_*^d} \frac{1}{|k|^{2\alpha}} e^{-s|k|^2} \\ &= \frac{\rho}{(2\pi)^{d/2}} + \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}_*^d} \frac{1}{|k|^{2(\alpha+\beta)}} e^{-s|k|^2+2\beta\log(|k|)}. \end{aligned} \tag{3.10}$$

Denote $g_{s,\beta}(r) := -sr^2 + 2\beta\log(r)$. By solving $g'_{s,\beta}(r) = 0$, we find that $g_{s,\beta}(r)$ is maximized at $r_0 = \sqrt{\beta/s}$, and the maximum value is equal to $g_{s,\beta}(r_0) = \beta^\beta s^{-\beta} e^{-\beta}$. The condition that $\beta > -\alpha + d/2$ implies that the remaining summation in k is finite.

(2) From the expression for $k_1(s)$ in (3.10), one easily sees that $k_1(s)$ is non-increasing and nonnegative. Moreover, $k_1(s)$ is integrable at $s = 0$ because from (3.10),

$$\begin{aligned} 0 \le \int_0^t k_1(s)\mathrm{d}s &\le \frac{\rho t}{(2\pi)^{d/2}} + \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}_*^d} \frac{1}{|k|^{2\alpha}} \int_0^\infty \mathrm{d}s\, e^{-s|k|^2} \\ &= \frac{\rho t}{(2\pi)^{d/2}} + \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}_*^d} \frac{1}{|k|^{2\alpha+2}} < \infty, \end{aligned}$$

where the last inequality is due to Dalang's condition (1.10). This proves (3.8). The equality in (3.9) can be proved in the same way. This proves Lemma 3.3. □

Let f_α^* denote the Riesz kernel on $\mathbb{R}^d$ and $\widehat{f_\alpha^*}$ be its Fourier transform, i.e.,

$$f_\alpha^*(x) := |x|^{-d+2\alpha} \quad \text{and} \quad \widehat{f_\alpha^*}(\xi) = c_{d,\alpha}|\xi|^{-2\alpha}, \quad \text{for all } x \text{ and } \xi \in \mathbb{R}^d. \tag{3.11}$$

Similar to (3.6), define

$$k_2(s;\alpha) = k_2(s) := \int_{\mathbb{R}^d} \widehat{f_\alpha^*}(\xi) \exp\left(-\frac{s|\xi|^2}{2}\right) \mathrm{d}\xi = C_{d,\alpha}\, s^{\alpha-d/2}, \tag{3.12}$$

where the last equality is an easy exercise (see, e.g., Example 1.2 of Chen and Kim (2019)). Hence, Dalang's condition (1.10) ensures the integrability of $k_2(s)$ at $s=0$ and

$$\int_0^\infty e^{-\gamma s} k_2(s;\alpha)\mathrm{d}s = C'_{d,\alpha}\gamma^{-\alpha+\frac{d}{2}-1} \quad \text{for all } \gamma>0. \tag{3.13}$$

Define $h_0(t;\alpha,\rho):=1$ and inductively for $n\geq 1$,

$$h_{n+1}(t;\alpha,\rho)=h_{n+1}(t):=\int_0^t h_n(t-s)\big(k_1(s)+k_2(s)+1\big)\mathrm{d}s. \tag{3.14}$$

The following lemma can be proved in the same way as Lemma 2.6 of Chen and Kim (2019) with $k(s)$ there replaced by $k_1(s)+k_2(s)+1$.

Lemma 3.4. *All functions $h_n(\cdot)$, $n\geq 1$, defined in (3.14) are nondecreasing on $\mathbb{R}_+$.*

For any $\lambda\neq 0$, define

$$H_\lambda(t):=\sum_{n=0}^\infty \lambda^{2n}h_n(t). \tag{3.15}$$

Lemma 3.5. *Suppose that Dalang's condition (1.10) holds. For all $\lambda\neq 0$, it holds that*

$$\limsup_{t\to\infty}\frac{1}{t}\log H_\lambda(t)\leq \gamma_0(\lambda):=\inf\left\{\gamma:\ \lambda^2\Theta_\gamma<1\right\}<\infty, \tag{3.16}$$

where

$$\Theta_\gamma:=\frac{\rho}{(2\pi)^{d/2}}\frac{1}{\gamma}+\frac{1}{(2\pi)^{d/2}}\sum_{k\in\mathbb{Z}_*^d}\frac{1}{|k|^{2\alpha}\left(|k|^2+\gamma\right)}+C_{d,\alpha}\gamma^{-(\alpha+1-d/2)}+\gamma^{-1}<\infty. \tag{3.17}$$

Moreover, when λ is large enough,

$$\gamma_0(\lambda)\lesssim\lambda^{\max\left(\frac{4}{2(1+\alpha)-d},2\right)}. \tag{3.18}$$

One may check Lemma 2.5 of Chen and Kim (2019) or Lemma A.1 of Balan and Chen (2018) (see also Foondun and Khoshnevisan (2013)) for similar accounts.

Proof. For any $\gamma>0$,

$$\int_0^\infty e^{-\gamma t}H_\lambda(t)\mathrm{d}t=\sum_{n=0}^\infty\lambda^{2n}\left[\int_0^\infty e^{-\gamma t}\left(k_1(s)+k_2(s)+1\right)\mathrm{d}t\right]^n=\sum_{n=0}^\infty\lambda^{2n}\Theta_\gamma^n, \tag{3.19}$$

where from (3.9) and (3.13), we obtain the expression of Θ_γ in (3.17). Thanks to Dalang's condition (1.10), Θ_γ is finite. Because $\Theta_\gamma\downarrow 0$ as $\gamma\uparrow\infty$, we see that $\gamma_0(\lambda)$ in (3.16) is well-defined and is finite.

It is not easy to compute $\gamma_0(\lambda)$ since the dependence on γ in Θ_γ is implicit. Instead, we can obtain an estimate of γ_0 using (3.7). Let $k_1'(s)$ be the upper bound of $k_1(s)$ given in(3.7), namely,

$$k_1^\dagger(s)=C_{\rho,\alpha,\beta,d}\left(1+s^{-\beta}\right),\quad \text{for}\quad \beta\in\left(\max\left(-\alpha+d/2,0\right),1\right).$$

Accordingly, we define $h_n^\dagger(t)$, $H_\lambda^\dagger(t)$, $\Theta_\gamma^\dagger$, and $\gamma_0^\dagger(\lambda)$. It is clear that

$$H_\lambda(t) \le H_\lambda^\dagger(t) \quad \text{for all } t \ge 0 \quad \text{and} \quad \gamma_0(\lambda) \le \gamma_0^\dagger(\lambda).$$

Now, $\Theta_\gamma^\dagger$ has a more explicit expression:

$$\begin{aligned}\Theta_\gamma^\dagger &= \frac{\rho}{(2\pi)^{d/2}}\frac{1}{\gamma} + C_{\rho,\alpha,\beta,d}\left(\gamma^{-1} + \Gamma(1-\beta)\gamma^{-1+\beta}\right) + C_{d,\alpha}\gamma^{-(\alpha+1-d/2)} + \gamma^{-1} \\ &= C'_{\rho,\alpha,\beta,d}\left(\frac{1}{\gamma} + \frac{1}{\gamma^{\alpha+1-d/2}} + \frac{1}{\gamma^{1-\beta}}\right) \asymp \frac{1}{\gamma^{1-\beta}}, \quad \text{as } \gamma \to \infty,\end{aligned}$$

where the asymptotic form is due to the fact that

$$\max\left(-\alpha + \frac{d}{2}, 0\right) < \beta < 1 \quad \Longleftrightarrow \quad 0 < 1-\beta < \min\left(\alpha + 1 - \frac{d}{2}, 1\right).$$

Therefore, when λ is large enough we have

$$\gamma_0^\dagger(\lambda) = \inf\left\{\gamma : \lambda^2\Theta_\gamma^\dagger < 1\right\} \lesssim \lambda^{\frac{2}{1-\beta}}.$$

Finally, replacing β by $\max(-\alpha + d/2, 0)$ in the above upper bound completes the proof of Lemma 3.5. □

4. Resolvent kernel function $\mathcal{K}_\lambda$

Let us first introduce some functions:

Definition 4.1. For $h, w : \mathbb{R}_+ \times \mathbb{T}^{4d} \to \mathbb{R}$, define the space-time convolution operator "$\triangleright$" by

$$(h \triangleright w)(t, x_0, x, x_0', x') := \int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, h(t-s, z, x, z', x')\, w\left(s, x_0, z, x_0', z'\right) f_{\alpha,\rho}(z, z').$$

Definition 4.2. Formally define the functions $\mathcal{L}_n : \mathbb{R}_+ \times \mathbb{T}^{4d} \to \mathbb{R}_+$ recursively by

$$\mathcal{L}_n\left(t, x_0, x, x_0', x'\right) := \begin{cases} G(t, x_0, x)\, G\left(t, x_0', x'\right) & \text{if } n = 0, \\ (\mathcal{L}_0 \triangleright \mathcal{L}_{n-1})\left(t, x_0, x, x_0', x'\right) & \text{otherwise,}\end{cases} \tag{4.1}$$

and for $\lambda \ne 0$, define the resolvent $\mathcal{K} : \mathbb{R}_+ \times \mathbb{T}^{4d} \to \mathbb{R}_+$ by

$$\mathcal{K}_\lambda\left(t, x_0, x, x_0', x'\right) := \sum_{n=0}^{\infty} \lambda^{2n}\mathcal{L}_n\left(t, x_0, x, x_0', x'\right). \tag{4.2}$$

The aim of this section is to prove the following proposition, which shows the well-posedness of $\mathcal{L}_n$ and $\mathcal{K}_\lambda$ and provides some estimates at the same time:

Proposition 4.3. *There exists a constant $C_{\alpha,\rho,d} > 0$ such that for all $t > 0$, $x, x_0, x', x_0' \in \mathbb{T}^d$, $\lambda \neq 0$, and $n \geq 1$, it holds that*

$$\mathcal{L}_n(t, x_0, x, x_0', x') \leq C^n_{\alpha,\rho,d}\, G(t, x_0, x)G(t, x_0', x')h_n(t), \tag{4.3}$$

and, by denoting $C_* := \lambda\, C^{1/2}_{\alpha,\rho,d}$,

$$\mathcal{K}_\lambda\left(t, x_0, x, x_0', x'\right) \leq G(t, x_0, x)G(t, x_0', x')H_{C_*}(t) < \infty. \tag{4.4}$$

Proof. Notice that we can use the bridge density to rewrite $\mathcal{L}_1$ as follows:

$$\begin{aligned}\mathcal{L}_1(t, x_0, x, x_0', x') =& G(t, x_0, x)G(t, x_0', x')\left(\int_0^{t/2} \mathrm{d}s + \int_{t/2}^{t} \mathrm{d}s\right) \\ &\times \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, G_{t,x_0,x}(s, z)G_{t,x_0',x'}(s, z')f_{\alpha,\rho}(z, z') \\ =:& G(t, x_0, x)G(t, x_0', x')\left(I_1 + I_2\right).\end{aligned} \tag{4.5}$$

By the symmetry of the Brownian bridge, we only need to estimate I_1:

$$\mathcal{L}_1(t, x_0, x, x_0', x') = 2G(t, x_0, x)G(t, x_0', x')I_1. \tag{4.6}$$

The proof below consists of five steps. Fix an arbitrary $\epsilon > 0$ and we will divide the proof into two cases corresponding to large time $t \geq \epsilon$ and small time $t < \epsilon$ in the first two steps, and the value of ϵ will be determined in the third step of the proof. We will use C to denote a generic constant that may change values at each appearance.

Step 1. The case for $\mathcal{L}_1$ with $t \geq \epsilon$. In this case, we are going to use the fact that the fundamental solution G is bounded above and below uniformly for all $t \geq \epsilon$. To this end, first observe that by Lemma 3.2, there exists a $\rho_* > 0$ large enough such that $f_{\alpha,\rho_*}(x, y) \geq 0$ for all $x, y \in \mathbb{T}^d$. Let ρ_* be the smallest such ρ_*, i.e.,

$$\rho_* := \inf\left\{\rho :\ f_{\alpha,\rho}(x, y) \geq 0,\ \forall x, y \in \mathbb{T}^d\right\}.$$

It is readily seen that

$$0 \leq |f_{\alpha,\rho}(x, y)| \leq f_{\alpha,\widehat{\rho}}(x, y), \quad \text{with } \widehat{\rho} := \rho \vee \rho_*. \tag{4.7}$$

Since $t \geq \epsilon$, we can apply Lemma 2.2 to yield that

$$\begin{aligned}I_1 = |I_1| =& \left|\int_0^{t/2} \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, G_{t,x_0,x}(s, z)G_{t,x_0',x'}(s, z')f_{\alpha,\rho}(z, z')\right| \\ \leq& C_\epsilon^d \int_0^{t/2} \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, G(s, x_0, z)G(s, x_0', z')|f_{\alpha,\rho}(z, z')| \\ \leq& C_\epsilon^d \int_0^{t/2} \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, G(s, x_0, z)G(s, x_0', z')f_{\alpha,\widehat{\rho}}(z, z') =: C_\epsilon^d\, I_*,\end{aligned}$$

where we have applied (4.7) in the second inequality. We can evaluate I_* using the Fourier series representations of G and $f_{\alpha,\widehat{\rho}}$ to see that

$$
\begin{aligned}
|I_*| =& \left| \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} \int_0^{t/2} \mathrm{d}s\, e^{2ik\cdot(x_0-x_0')} \mathcal{F}\left(f_{\alpha,\widehat{\rho}}\right)(k) e^{-s|k|^2} \right| \\
\leq& \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} \int_0^{t/2} \mathrm{d}s\, \mathcal{F}\left(f_{\alpha,\widehat{\rho}}\right)(k) e^{-s|k|^2} \\
=& \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} \int_0^{t/2} \mathrm{d}s \left[\mathcal{F}\left(f_{\alpha,\rho}\right)(k) + \mathcal{F}\left(\frac{\widehat{\rho}-\rho}{(2\pi)^d}\right)(k) \right] e^{-s|k|^2} \\
=& \frac{1}{(2\pi)^{d/2}} \int_0^{t/2} \left[k_1(s) + (\widehat{\rho}-\rho)\right] \mathrm{d}s.
\end{aligned}
$$

Therefore, by enlarging $t/2$ to t in the above integral and taking into account of the expression C_ϵ in (2.10), we have proved that when $t \geq \epsilon$,

$$
\mathcal{L}_1(t,x_0,x,x_0',x') \leq C\left(1+\frac{1}{\sqrt{\epsilon}}\right)^d e^{d\pi^2/\epsilon} G(t,x_0,x) G(t,x_0',x') \int_0^t \left[k_1(s) + (\widehat{\rho}-\rho)\right] \mathrm{d}s. \tag{4.8}
$$

Step 2. The case for $\mathcal{L}_1$ with $t < \epsilon$. In this case, we embed the torus in $\mathbb{R}^d$; then we apply the techniques of Chen and Kim (2019) to achieve the upper bound. To this end, recall that the Riesz kernel f_α^* and its Fourier transform $\widehat{f_\alpha^*}$ on $\mathbb{R}^d$ are specified in (3.11).

From (1.5), we can see that the integrand for I_1 given in (4.5) are 2π-periodic in each component of z and z'. Hence, we can equivalently integrate over the domains: $(z,z') \in \mathbb{T}^d(x_0) \times \mathbb{T}^d(x_0')$, where

$$
\mathbb{T}^d(x) := [x_1-\pi, x_1+\pi] \times \cdots \times [x_d-\pi, x_d+\pi] \quad \text{for all } x = (x_1,\cdots,x_d) \in \mathbb{R}^d.
$$

Hence,

$$
I_1 = \int_0^{t/2} \mathrm{d}s \int_{\mathbb{T}^d(x_0')} \mathrm{d}z' \int_{\mathbb{T}^d(x_0)} \mathrm{d}z\, G_{t,x_0,x}(s,z) G_{t,x_0',x'}(s,z') f_{\alpha,\rho}(z,z').
$$

Since both $z-x_0$ and $z'-x_0'$ in the above integral belong to $\mathbb{T}^d$, we can apply Lemma 2.3 to see that

$$
I_1 \leq C(1+t)^d \sum_{k,k'\in\Pi^d} \int_0^{t/2} \mathrm{d}s \int_{\mathbb{T}^d(x_0')} \mathrm{d}z' \int_{\mathbb{T}^d(x_0)} \mathrm{d}z \times p_{t,x_0,x+k}(s,z) p_{t,x_0',x'+k'}(s,z') f_{\alpha,\rho}(z,z'),
$$

where we recall that $\Pi := \{-2\pi, 0, 2\pi\}$. By (1.3), we see that for some universal constant $C > 0$,

$$
f_{\alpha,\rho}(z,z') = f_{\alpha,\rho}(\llbracket z-z' \rrbracket) \leq C \sum_{k''\in\Pi^d} f_{\alpha,\rho}^*(z-z'+k'').
$$

Thanks to the nonnegativity of the integrand, we can apply the above inequality to I_1 and extend the integration domain from $\mathbb{T}^d(x_0) \times \mathbb{T}^d(x_0')$ to $\mathbb{R}^{2d}$ to yield that

$$
I_1 \leq C(1+t)^d \sum_{k,k',k''\in\Pi^d} \int_0^{t/2} \mathrm{d}s\, (1+s)^d \iint_{\mathbb{R}^{2d}} \mathrm{d}z'\mathrm{d}z\, f_{\alpha,\rho}^*(z-z'+k'')
$$

$$\times p_{t,x_0,x+k}(s,z)\, p_{t,x_0',x'+k'}(s,z')\,.$$

We have reduced the problem from $\mathbb{T}^d$ to $\mathbb{R}^d$. Now we can apply the Plancherel theorem to the above integral to obtain

$$\begin{aligned} I_1 = |I_1| \leq & C(1+t)^d \int_0^{t/2} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\xi\, e^{-\frac{s(t-s)}{t}|\xi|^2} \widehat{f_\alpha^*}(\xi) \\ \leq & C(1+t)^d \int_0^{t/2} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\xi\, e^{-\frac{s}{2}|\xi|^2} \widehat{f_\alpha^*}(\xi) = C(1+t)^d \int_0^{t/2} \mathrm{d}s\, k_2(s), \end{aligned}$$

where the second inequality is due to the fact that $s/2 \leq s(t-s)/t$ when $s \in [0, t/2]$. Therefore, we have proved that when $t \in (0, \epsilon)$,

$$\mathcal{L}_1(t, x_0, x, x_0', x') \leq C\,(1+\epsilon)^d\, G(t, x_0, x) G(t, x_0', x') \int_0^t k_2(s) \mathrm{d}s. \tag{4.9}$$

Note that we use the condition $t < \epsilon$ only in the last step to bound the factor $(1+t)^d$ by $(1+\epsilon)^d$.

Step 3. Determination of ϵ for the estimate of $\mathcal{L}_1$. Combining the estimates in both (4.8) and (4.9), we see that

$$\mathcal{L}_1(t, x_0, x, x_0', x') \leq C\,(C_{1,\epsilon} + C_{2,\epsilon}) G(t, x_0, x) G(t, x_0', x') \int_0^t [k_1(s) + k_2(s) + 1]\, \mathrm{d}s, \tag{4.10}$$

for all $t > 0$, where

$$C_{1,\epsilon} := \left(1 + \frac{1}{\sqrt{\epsilon}}\right)^d e^{d\pi^2/\epsilon} \quad \text{and} \quad C_{2,\epsilon} := (1+\epsilon)^d\,.$$

It is clear that both $C_{i,\epsilon}$ are continuous functions of $\epsilon > 0$. Since $C_{1,\epsilon}$ is monotone decreasing while $C_{2,\epsilon}$ is monotone increasing, there exists an unique $\epsilon_0 > 0$ that minimizes the sum of the two, namely $\epsilon_0 := \arg\min \left(C_{1,\epsilon} + C_{2,\epsilon}\right)$. Finally, one can choose the constant $C_{\alpha,\rho,d}$ in (4.3) to be $C_{1,\epsilon_0} + C_{2,\epsilon_0}$ (up to another factor of generic constant). This completes the proof of (4.3) in the case of $n = 1$.

Step 4. Upper bounds for $\mathcal{L}_n$. Suppose now (4.1) holds for $n-1$. By Definition 4.2 and the induction assumption, we have

$$\begin{aligned} \mathcal{L}_n = & \int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z' G(t-s, z, x) G(t-s, z', x') \mathcal{L}_{n-1}(s, x_0, z, x_0', z') f_{\alpha,\rho}(z, z') \\ \leq & C^{n-1} G(t, x_0, x) G(t, x_0', x') \\ & \times \int_0^t \mathrm{d}s\, h_{n-1}(s) \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z' G_{t,x_0,x}(s, z) G_{t,x_0',x'}(s, z') f_{\alpha,\rho}(z, z'). \end{aligned} \tag{4.11}$$

Using the fact that $h_n(t)$ is nondecreasing (see Lemma 3.4) and the symmetry of Brownian bridge, we see that

$$\mathcal{L}_n\left(t, x_0, x, x_0', x'\right) \leq 2C^{n-1} G(t, x_0, x) G(t, x_0', x') I_n(t),$$

where

$$I_n(t) := \int_0^{t/2} \mathrm{d}s\, h_{n-1}(t-s) \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, G_{t,x,x_0}(s,z) G_{t,x',x_0'}(s,z') f_{\alpha,\rho}(z,z').$$

Now, one can carry out the same three steps as the proof of Proposition 4.3 to bound $I_n(t)$:

$$\begin{aligned} I_n(t) \leq & C \int_0^{t/2} h_{n-1}(t-s)\,(k_1(s)+k_2(s)+1)\,\mathrm{d}s \\ \leq & C \int_0^{t} h_{n-1}(t-s)\,(k_1(s)+k_2(s)+1)\,\mathrm{d}s = C h_n(t). \end{aligned}$$

This completes the proof of (4.3).

Step 5. Upper bound for the resolvent $\mathcal{K}$. Finally, the estimate of $\mathcal{K}$ in (4.4) is a direct consequence of the estimate of $\mathcal{L}_n$ in (4.3) and the definition of $H_\lambda(t)$ in (3.15). This completes the whole proof of Proposition 4.3. □

5. Proof of Theorem 1.3

Now we are ready to introduce the mild solution to (1.6) and establish its well-posedness. Let $\dot{W}$ be the centered and spatially homogeneous Gaussian noise introduced above, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{W_t(A); t \geq 0, A \in \mathscr{B}(\mathbb{T}^d)\}$ be the martingale measure associated to the noise $\dot{W}$ in the sense of Walsh Walsh (1986), where $\mathscr{B}(\mathbb{T}^d)$ refers to the Borel σ-algebra on $\mathbb{T}^d \subseteq \mathbb{R}^d$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the underlying augmented filtration generated by $\dot{W}$

$$\mathcal{F}_t = \sigma\{W_s(A) : 0 \leq s \leq t, A \in \mathscr{B}(\mathbb{T})\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the σ-field generated by all $\mathbb{P}$-null sets in $\mathcal{F}$.

Definition 5.1. A process $u = \{u(t,x);\, t>0, x \in \mathbb{T}^d\}$ is called *a random field solution* or the *mild solution* to (1.6) if:

(i) u is adapted, i.e., for each $t > 0$ and $x \in \mathbb{T}^d$, $u(t,x)$ is $\mathcal{F}_t$-measurable;
(ii) u is jointly measurable with respect to $\mathscr{B}((0,\infty) \times \mathbb{T}^d) \times \mathcal{F}$;
(iii) for each $t > 0$ and $x \in \mathbb{T}^d$, it holds that

$$\mathbb{E}\left[\int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}y\,\mathrm{d}y'\, G(t-s,x,y)u(s,y)f(y,y')G(t-s,x,y')u(s,y')\right] < \infty; \tag{5.1}$$

(iv) for each $t > 0$ and $x \in \mathbb{T}^d$, u satisfies (1.8) a.s. for all $(t,x) \in (0,\infty) \times \mathbb{T}^d$.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The existence and uniqueness in $L^2(\Omega)$, as well as the two-point correlation estimates of the solution to (1.6), can be established from the standard Picard iteration. More precisely, with Proposition 4.3, we can carry out the same six steps as those in the proof of Theorem 2.4 in

Section 3.3 of Chen and Dalang (2015); One may also check the proofs of Theorems 1.4 and 1.5 of Candil, Chen and Lee (2024).

It remains to prove the p-th moment bounds in (1.12). One can follow the same strategy as in the proof of Theorem 1.7 of Chen and Huang (2019) to establish (1.12); see the proof of part (ii) of Theorem 1.4 in Section 5.1 of Candil, Chen and Lee (2024) for another presentation. In essence, let $u_n(t,x)$ be the immature solutions in the Picard iterations, namely $u_0(t,x) = J_0(t,x)$, and for $n \geq 1$,

$$u_n(t,x) = J_0(t,x) + \lambda \int_0^t \int_{\mathbb{T}^d} G(t-s,x,y) u_{n-1}(s,y) W(\mathrm{d}s, \mathrm{d}y).$$

An application of the *Burkholder-Davis-Gundy inequality* (BDG inequality) shows that their p-th moments satisfy the following integral inequality

$$\begin{aligned} ||u_{n+1}(t,x)||_p^2 \leq 2J_0^2(t,x) + 8p\lambda^2 \int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}y\mathrm{d}y' f_{\alpha,\rho}(y,y')\, G(t-s,x,y)\, ||u_n(s,y)||_p \\ \times G(t-s,x,y')\, ||u_n(s,y')||_p. \end{aligned} \tag{5.2}$$

Then one can show that

$$g_n(t,x) := \sqrt{2} J_0(t,x) \left(\sum_{k=0}^{n} \left[16p\lambda^2\right]^k h_k(t) \right)^{1/2}$$

is a super solution to (5.2), namely, $||u_n(t,x)||_p \leq g_n(t,x)$. Then one can show that for (t,x) fixed, $\{u_n(t,x) : n \geq 0\}$ is a Cauchy sequence in $L^p(\Omega)$. Finally, the memorization relation holds in the limit:

$$||u(t,x)||_p = \lim_{n\to\infty} ||u_n(t,x)||_p \leq \lim_{n\to\infty} g_n(t,x) = \sqrt{2} J_0(t,x) \left(\sum_{k=0}^{\infty} \left[16p\lambda^2\right]^k h_k(t) \right)^{1/2},$$

where $h_k(t)$ are given in (3.14). This completes the sketch of the proof of (1.12). We refer the interested readers to the proof of Theorem 1.7 of Chen and Huang (2019) or the proof of part (ii) of Theorem 1.4 in Section 5.1 of Candil, Chen and Lee (2024) for more details.

Finally, when $\lambda^2 p$ is large enough, one can obtain (1.13) from the estimate of $\gamma_0(\lambda)$ in (3.18). This completes the whole proof of Theorem 1.3. □

6. Lower bound for the second moment

Proof of Theorem 1.5. Fix an arbitrary $\epsilon > 0$. Under condition (1.14), thanks to Lemma 2.2, one can derive the corresponding lower bound in step 1 of the proof of Proposition 4.3. In particular,

$$\begin{aligned} \mathcal{L}_1(t,x_0,x,x_0',x') &\geq c_\epsilon^d \int_0^{t/2} \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, G(s,x_0,z) G(s,x_0',z') f(z,z') \\ &\geq c_\epsilon^d C_f \int_0^{t/2} \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z'\, G(s,x_0,z) G(s,x_0',z') = c_\epsilon^d C_f \frac{t}{2}, \end{aligned}$$

whenever $t \geq \epsilon$, where the constant c_ϵ is given in (2.10).

A lower bound of $\mathcal{L}_2$ can be derived similarly:

$$\begin{aligned}
&\mathcal{L}_2\left(t,x_0,x,x_0',x'\right)\\
&=\int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z' G(s,z,x)G(s,z',x')\mathcal{L}_1(t-s,x_0,z,x_0',z')f(z,z')\\
&\geq \int_0^{\frac{t}{2}} \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z' G(s,z,x)G(s,z',x')\mathcal{L}_1(t-s,x_0,z,x_0',z')f(z,z')\\
&\geq \frac{c_\epsilon^d C_f}{2}\int_0^{\frac{t}{2}} \mathrm{d}s\,(t-s) \iint_{\mathbb{T}^{2d}} \mathrm{d}z\mathrm{d}z' G(s,z,x)G(s,z',x')\, C_f\\
&= \frac{c_\epsilon^d C_f^2}{2}\frac{1}{2}\left(t^2-\left(\frac{t}{2}\right)^2\right) \geq \frac{c_\epsilon^d C_f^2}{2}\frac{1}{2}\frac{t^2}{2}.
\end{aligned}$$

Finally, an induction argument together with the elementary relation $t^n-(t/2)^n \geq t^n/2$ gives us that, uniformly for all $x_0,x,x_0',x' \in \mathbb{T}^d$,

$$\mathcal{L}_n\left(t,x_0,x,x_0',x'\right) \geq \frac{c_\epsilon^d}{2}\frac{1}{n!}\left(\frac{C_f t}{2}\right)^n,$$

and hence, for any $\lambda \neq 0$,

$$\mathcal{K}_\lambda\left(t,x_0,x,x_0',x'\right) \geq \frac{c_\epsilon^d}{2}\exp\left(\frac{C_f t}{2}\right).$$

Then an application of (1.11) proves Theorem 1.5. □

When the initial condition is given by a bounded measurable function, one has the Feynman-Kac representation for second moment of the solution to the parabolic Anderson model; see Proposition 4.3 of Hu and Nualart (2009) for the case of $\mathbb{R}^d$ or Carmona and Molchanov (1994) for the case of $\mathbb{Z}^d$. Taking advantage of this Feynman-Kac representation, we are able to prove the exponential lower bound of the second moment for all $\rho > 0$. Before proceeding, we would like to make two remarks:

Remark 6.1. We would like to highlight that, instead of bounded initial conditions, the Feynman-Kac representations for the second moment of the parabolic Anderson model on $\mathbb{R}^d$, starting from **rough initial conditions**, have also been explored in the literature. Here, "rough initial conditions" refer specifically to cases where the initial condition μ is a signed Borel measure satisfying:

$$\int_{\mathbb{R}^d} e^{-a|x|^2}|\mu|(\mathrm{d}x) < \infty \quad \text{for all } a > 0.$$

For more details, see (Chen, Hu and Nualart, 2017, Theorem 2.1), (Huang, Lê and Nualart, 2017a, Proposition 4.4), and (Huang, Lê and Nualart, 2017b, Proposition 4.3).

Remark 6.2. With the Feynman-Kac representations for the moments, one should be able to carry out the combinatorial arguments using the Feynman diagrams to derive the lower bounds for the general p-th moments, $p \geq 2$, that match the corresponding upper bounds in (1.13). This approach has been recently applied in cases of the whole space $\mathbb{R}^d$; see Hu and Wang (2024) and Chen, Guo and Song (2024). We plan to explore this topic in future research.

Proof of Theorem 1.6. Without loss of generality, we may assume the initial condition is the constant one. The case where the initial conditions are bounded away from zero can be reduced to this case. The Feynman-Kac formula for the second moment then has the following form:

$$\mathbb{E}\left[u(t,x)^2\right] = \mathbb{E}_x\left[\exp\left\{\int_0^t f_{\alpha,\rho}(B_s,\widetilde{B}_s)\mathrm{d}s\right\}\right], \tag{6.1}$$

where B_s and $\widetilde{B}_s$ are two independent Brownian motions on torus both starting from x, and $\mathbb{E}_x$ refers to the expectation with respect to both Brownian motions. We apply Jensen's inequality to obtain that

$$\mathbb{E}\left[u(t,x)^2\right] \gtrsim \left[\exp\left\{\int_0^t \mathbb{E}_x\left(f_{\alpha,\rho}\left(B_s,\widetilde{B}_s\right)\right)\mathrm{d}s\right\}\right]. \tag{6.2}$$

Now recall the definition of $f_{\alpha,\rho}$ in (1.4), the spectral representation of the heat kernel $G(t,x,y)$ in (3.5). We use the fact that

$$\mathbb{E}_x\left((2\pi)^{-d/2}e^{ik\cdot(B_s-\tilde{B}_s)}\right) = e^{-|k|^2 s},$$

to simplify the right hand-side of (6.2) as follows,

$$\begin{aligned}\left[\int_0^t \mathbb{E}_x\left(f_{\alpha,\rho}\left(B_s,\tilde{B}_s\right)\right)\mathrm{d}s\right] &= \frac{\rho t}{(2\pi)^d} + \frac{2^\alpha}{(2\pi)^d}\sum_{k\in\mathbb{Z}_*^d}|k|^{-2\alpha}\int_0^t e^{-|k|^2 s}\mathrm{d}s \\ &= \frac{\rho t}{(2\pi)^d} + \frac{2^\alpha}{(2\pi)^d}\sum_{k\in\mathbb{Z}_*^d}|k|^{-2\alpha-2}\left[1-e^{-|k|^2 t}\right] \sim \frac{\rho t}{(2\pi)^d}, \quad \text{as } t\uparrow\infty.\end{aligned}$$

The proof is thus completed. □

The exponential lower bound suggests full intermittency for the solution. As explained in the introduction, the exponential growth of the second moment is due to ergodicity of Brownian motion on a compact manifold. The time average should converge to the space average,

$$\frac{1}{t}\int_0^t f_{\alpha,\rho}(B_s-\widetilde{B}_s)\mathrm{d}s \to \int_{\mathbb{T}^d} f_{\alpha,\rho}(x)\mathrm{d}x = \frac{\rho}{(2\pi)^d},$$

in lim sup at a rate of $\sqrt{\log\log t}$ (see Brosamler (1983)). It is clear that our argument above is quite generic and does not depend on the fact that the state space considered here is a torus (as opposed to a general compact manifold).

It is well known that there is a phase transition in the intensity parameter λ (in terms of the growth rate of the second moment) in $\mathbb{R}^d$ for $d\geq 3$ (see, e.g., Chen and Kim (2019)). This is a result of the fact that Brownian motion is transient when $d\geq 3$. We do not see this phase transition on the torus as long as $\rho>0$ regardless of the value of α: the second moment also blows up exponentially in time. However, it is not clear what happens when $\rho=0$.

7. Hölder continuity

We first establish the following lemma, the proof of which is similar to Lemma 3.1 of Chen and Huang (2019).

Lemma 7.1. *There exists some universal constant $C > 0$ such that for all $\beta \in (0, 1]$, $x, y \in \mathbb{T}^d$, and $t' \geq t > 0$,*

$$|G(t,x) - G(t',x)| \leq Ct^{-\beta/2} G(2t',x)\,(t'-t)^{\beta/2} \quad \textit{and} \tag{7.1}$$

$$|G(t,x) - G(t,y)| \leq Ct^{-\beta/2} \,[G(2t,x) + G(2t,y)]\, \mathrm{dist}(x,y)^{\beta}, \tag{7.2}$$

where we recall that $\mathrm{dist}(x,y) = |\,[\![x-y]\!]\,|$.

Proof. In the proof, we use C to denote a generic constant which may change its value at each appearance. Choose and fix an arbitrary $\beta \in (0,1]$. We start with (7.2). From the mean value theorem, we have for some $\xi := \xi(x,y) \in \mathbb{T}^d$ such that

$$\begin{aligned}
|G(t,x) - G(t,y)| &= |\nabla G(t,\xi)| \times |[\![x-y]\!]| \\
&= \left[\sum_{k\in\mathbb{Z}^d} (2\pi t)^{-d/2} \frac{|\xi + 2\pi k|}{t} e^{-\frac{|\xi+2\pi k|^2}{2t}}\right] \times \mathrm{dist}(x,y) \\
&\leq \left[\frac{C}{\sqrt{t}} \sum_{k\in\mathbb{Z}^d} (4\pi t)^{-d/2} e^{-\frac{|\xi+2\pi k|^2}{4t}}\right] \times \mathrm{dist}(x,y) \\
&= \frac{C}{\sqrt{t}} G(2t,\xi)\,\mathrm{dist}(x,y) \leq \frac{C}{\sqrt{t}}\,[G(2t,x) + G(2t,y)]\,\mathrm{dist}(x,y),
\end{aligned}$$

where the last inequality can be obtained via scaling arguments (see, e.g., the proof of Lemma 3.1 of Chen and Huang (2019)). We can apply the above inequality and (2.5) to see that

$$\begin{aligned}
|G(t,x) - G(t,y)| &= |G(t,x) - G(t,y)|^{\beta} |G(t,x) - G(t,y)|^{1-\beta} \\
&\leq \frac{C}{t^{\beta/2}}\,[G(2t,x) + G(2t,y)]^{\beta d}\, \mathrm{dist}(x,y)^{\beta}\,[G(2t,x) + G(2t,y)]^{(1-\beta)d} \\
&= \frac{C}{t^{\beta/2}}\,[G(2t,x) + G(2t,y)]\,\mathrm{dist}(x,y)^{\beta},
\end{aligned}$$

which proves that (7.2). To prove (7.1), observe that

$$\begin{aligned}
|G(t,x) - G(t',x)| \leq \Bigg| &\sum_{k\in\mathbb{Z}^d} (2\pi t)^{-d/2} e^{-\frac{|x+2\pi k|^2}{2t}} - (2\pi t')^{-d/2} e^{-\frac{|x+2\pi k|^2}{2t}} \\
&+ (2\pi t')^{-d/2} e^{-\frac{|x+2\pi k|^2}{2t}} - (2\pi t')^{-d/2} e^{-\frac{|x+2\pi k|^2}{2t'}} \Bigg| \\
\leq\ & t^{d/2} \left| (t')^{-d/2} - t^{d/2} \right| G(t,x) + \left| \sum_{k\in\mathbb{Z}^d} (2\pi t')^{-d/2} \left(e^{-\frac{|x+2\pi k|^2}{2t}} - e^{-\frac{|x+2\pi k|^2}{2t'}} \right) \right| \\
=:\ & I_1 + I_2.
\end{aligned}$$

Following inequality (3.3) in Chen and Huang (2019), we find that $I_1 \le Ct^{-\beta/2}|t'-t|^{\beta/2}G(t,x)$. As for I_2, from the mean value theorem, we can deduce that for some $\xi \in [t,t']$,

$$
\begin{aligned}
I_2 &= \sum_{k\in\mathbb{Z}^d} (2\pi t')^{-d/2} \frac{|x+2\pi k|^2}{(2\xi)^2}\,|t'-t|\,e^{-\frac{|x+2\pi k|^2}{2\xi}} \le C \sum_{k\in\mathbb{Z}^d} (2\pi t')^{-d/2} \frac{1}{2\xi}\,|t'-t|\,e^{-\frac{|x+2\pi k|^2}{4\xi}} \\
&\le C \sum_{k\in\mathbb{Z}^d} (2\pi t')^{-d/2} \frac{1}{2\xi}\,|t'-t|\,e^{-\frac{|x+2\pi k|^2}{4\xi}} \le C \frac{|t'-t|}{t} G(2t',x).
\end{aligned}
$$

Together with the fact that $I_2 \le G(t',x)$, we see that

$$
I_2 = I_2^{\beta/2} I_2^{1-\beta/2} \le C \left[\frac{|t'-t|}{t} G(2t',x)\right]^{\beta/2} G(t',x)^{1-\beta/2} \le Ct^{-\beta/2}|t'-t|^{\beta/2} G(2t',x).
$$

Combining the bounds for I_1 and I_2 proves (7.1). □

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. We need only control the Hölder modulus of

$$
I(t,x) := \iint_{(0,t]\times\mathbb{T}^d} G(t-s,x,y)\lambda u(s,y) W(\mathrm{d}s,\mathrm{d}y).
$$

Fix an arbitrary $n \ge 1$. Following Chen and Huang (2019), we need to compute the p-th moment increments

$$
\|I(t,x) - I(t',x')\|_p^2 \le C\left[I_1(t,x,x') + I_2(t,t',x') + I_3(t,t',x)\right],
$$

for $t,t' \in [1/n,n]$ and $x,x' \in \mathbb{T}^d$ with $t' > t$, where I_1, I_2, and I_3 are defined as follows:

$$
\begin{aligned}
I_1(t,x,x') &:= \int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}y_1\mathrm{d}y_2\, f_{\alpha,\rho}(y_1,y_2)\,|G(t-s,x,y_1) - G(t-s,x',y_1)| \\
&\qquad \times |G(t-s,x,y_2) - G(t-s,x',y_2)|\, \|u(s,y_1)\|_p\, \|u(s,y_2)\|_p\,; \\
I_2(t,t',x') &:= \int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}y_1\mathrm{d}y_2\, f_{\alpha,\rho}(y_1,y_2)\,|G(t-s,x',y_1) - G(t'-s,x',y_1)| \\
&\qquad \times |G(t-s,x',y_2) - G(t'-s,x',y_2)|\, \|u(s,y_1)\|_p\, \|u(s,y_2)\|_p\,; \\
I_3(t,t',x') &:= \int_t^{t'} \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}y_1\mathrm{d}y_2\, f_{\alpha,\rho}(y_1,y_2) G(t'-s,x',y_1) \\
&\qquad \times G(t'-s,x',y_2)\, \|u(s,y_1)\|_p\, \|u(s,y_2)\|_p\,.
\end{aligned}
$$

In the following, we use C_n to denote a generic constant that may depend on n and may change its value at each occurrence.

To control I_1, from the p-th moment formula (1.12), we see that

$$\begin{aligned} I_1(t,x,x') \leq & C_n \int_0^t \mathrm{d}s \iint_{\mathbb{T}^{2d}} \mathrm{d}y_1 \mathrm{d}y_2\, f_{\alpha,\rho}(y_1,y_2) \iint_{\mathbb{T}^{2d}} \mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2) \\ & \times G(s,y_1,z_1)\,|G(t-s,x,y_1)-G(t-s,x',y_1)| \\ & \times G(s,y_2,z_2)\,|G(t-s,x,y_2)-G(t-s,x',y_2)|\,. \end{aligned} \tag{7.3}$$

We can then apply (7.2) to obtain that for all $\beta \in (0,1)$,

$$\begin{aligned} & |G(t-s,x,y_1)-G(t-s,x',y_1)| \\ & \quad = |G\left(t-s, [\![x-y_1]\!]\right) - G\left(t-s,[\![x'-y_1]\!]\right)| \\ & \quad \leq C\left[G\left(2(t-s),[\![x-y_1]\!]\right) + G\left(2(t-s),[\![x'-y_1]\!]\right)\right] \frac{\mathrm{dist}(x,x')^\beta}{(t-s)^{\beta/2}} \\ & \quad = C\left[G\left(2(t-s),x,y_1\right) + G\left(2(t-s),x',y_1\right)\right] \frac{\mathrm{dist}(x,x')^\beta}{(t-s)^{\beta/2}}, \end{aligned}$$

where we have used the fact that

$$\mathrm{dist}([\![x-y_1]\!]\,,[\![x'-y_1]\!]) = |[\![\, [\![x-y_1]\!] - [\![x'-y_1]\!]\,]\!]| = |[\![x-x']\!]| = \mathrm{dist}(x,x')\,.$$

Hence,

$$\begin{aligned} & G(s,y_1,z_1)\,|G(t-s,x,y_1)-G(t-s,x',y_1)| \\ & \quad \leq CG(2s,y_1,z_1)\left[G(2(t-s),x,y_1)+G(2(t-s),x',y_1)\right] \frac{\mathrm{dist}(x,x')^\beta}{(t-s)^{\beta/2}} \\ & \quad = C\frac{\mathrm{dist}(x,x')^\beta}{(t-s)^{\beta/2}}\left[G(2t,x,z_1)G_{2t,x,z_1}(2s,y_1) + G(2t,x',z_1)G_{2t',x',z_1}(2s,y_1)\right], \end{aligned}$$

where we have used the density for the pinned Brownian motion; see (2.7). Therefore, by plugging the above upper bound and the corresponding one with (y_1,z_1) replaced by (y_2,z_2) back to (7.3), we obtain four terms in the expansion:

$$I_1(t,x,x') \leq \sum_{k=1}^4 I_{1,k}(t,x,x').$$

For $I_{1,1}$, we have

$$\begin{aligned} I_{1,1}(t,x,x') \leq & C_n\, \mathrm{dist}(x,x')^{2\beta}\, J_0(2t,x)J_0(2t,x) \int_0^t \mathrm{d}s\, \frac{1}{(t-s)^\beta} \\ & \times \iint_{\mathbb{T}^{2d}} \mathrm{d}y_1\mathrm{d}y_2\, f_{\alpha,\rho}(y_1,y_2) G_{2t,x,z_1}(2s,y_1)\, G_{2t,x,z_1}(2s,y_2)\,. \end{aligned}$$

Now an application of Lemma 2.3 shows that

$$I_{1,1}(t,x,x') \le C_n \operatorname{dist}(x,x')^{2\beta} J_0(2t,x) J_0(2t,x) \int_0^t \mathrm{d}s \, \frac{1}{(t-s)^\beta} \times \iint_{\mathbb{T}^{2d}} \mathrm{d}y_1 \mathrm{d}y_2 \, f_{\alpha,\rho}(y_1,y_2) \sum_{k,k'\in\Pi^d} p_{2t,x,z_1+k}(2s,y_1)\, p_{2t,x,z_1+k'}(2s,y_2).$$

By the same arguments as Step 2 of the proof of Theorem 1.3, we obtain that

$$\begin{aligned} I_{1,1}(t,x,x') \le & C_n \operatorname{dist}(x,x')^{2\beta} J_0^2(2t,x) \int_0^t \mathrm{d}s \, \frac{1}{(t-s)^\beta} \iint_{\mathbb{R}^{2d}} \mathrm{d}y_1 \mathrm{d}y_2 \, f^*_{\alpha,\rho}(y_1-y_2+k'') \\ & \times \sum_{k,k',k''\in\Pi^d} p_{2t,x,z_1+k}(2s,y_1)\, p_{2t,x,z_1+k'}(2s,y_2) \\ \le & C_n \operatorname{dist}(x,x')^{2\beta} J_0^2(2t,x) \int_0^t \frac{k_2(s)}{s^\beta} \mathrm{d}s \\ = & C_n \operatorname{dist}(x,x')^{2\beta} J_0^2(2t,x) \int_0^t s^{\alpha-d/2-\beta} \mathrm{d}s, \end{aligned}$$

where the last step is due to (3.12). Therefore, provided that $\beta < 1+\alpha-d/2 \in (0,1)$ (recall that $\alpha \in (0,d/2)$), we have

$$I_{1,1}(t,x,x') \le C_n \operatorname{dist}(x,x')^{2\beta} J_0^2(2t,x).$$

The computations for the $I_{1,2}, I_{1,3}, I_{1,4}$ are similar. Combining all these bounds, we conclude that

$$I_1(t,x,x') \le C_n \left(J_0(2t,x) + J_0(2t,x')\right)^2 \operatorname{dist}(x,x')^{2\beta}.$$

The proof for the time increments, namely, I_2 and I_3, is similar, which will be omitted here. □

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Supplementary Material

Supplement to "Parabolic Anderson model with colored noise on the torus" (DOI: 10.3150/24-BEJ1838SUPP; .pdf). In the supplement Chen, Ouyang and Vickery (2025) we prove Lemma 2.1 regarding the comparability of the heat kernel $G(t,x)$ on $\mathbb{T}^d$ and the Gaussian density $p(t,x)$.

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