Invariant measure for the nonlinear stochastic heat equation on $\mathbb{R}^d$ with no drift term

Le Chen
Auburn University

Seminário de Probabilidade e Mecânica Estatìstica

https:// spmes.impa.br/

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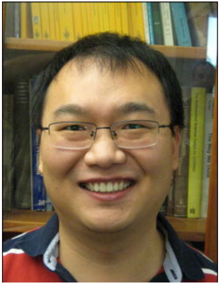
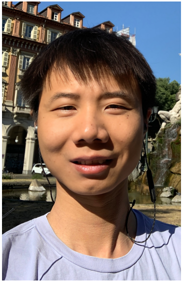
Acknowledgment

Nicholas Eisenberg



Cheng Ouyang

Samy Tindel



Panqiu Xia

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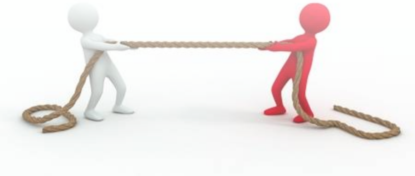
    Moment bounds in the weighted Hilbert space

Existence of invariant measure
    Main result
    Related work

Stationary limit via Gu and Li's approach

References

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u(t,x) = \lambda\, \dot{W}(t,x) u(t,x)$$



$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u$	$\dot{W} u$
Smoothing	Roughening

Centered Gaussian noise

white in time and homogeneous in space

Dalang's condition: $\Upsilon(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{\beta + |\xi|^2} < \infty$ for some hence all $\beta > 0$

Homogeneous in space

f: nonnegative & nonnegative definite

$$\mathbb{E}\left(\dot{W}(t,x)\dot{W}(s,y)\right) = \delta_0(t-s)\, f(x-y)$$

White in time

Martingale theory: Itô, Walsh, Dalang, C., ...

Nonlinear SPDE
$b(u)$

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$b(u)$

Why spatially colored noise?

1. Function-valued solution instead of singular SPDEs.
2. *Universality* relies on spatial dimension d and structure of noise.
3. Brownian polymer in a continuous random environment:
   Rovira and Tindel '05, Lacoin '11...
4. Medina, Hwa and Kardar '89:

   Random walk in a turbulent flow.

   Directed polymer:
   impurities interacting with the interface;
   anticorrelated impurities.

   Surface growth with charged ions:
   interacting via (long range) Coulomb force.

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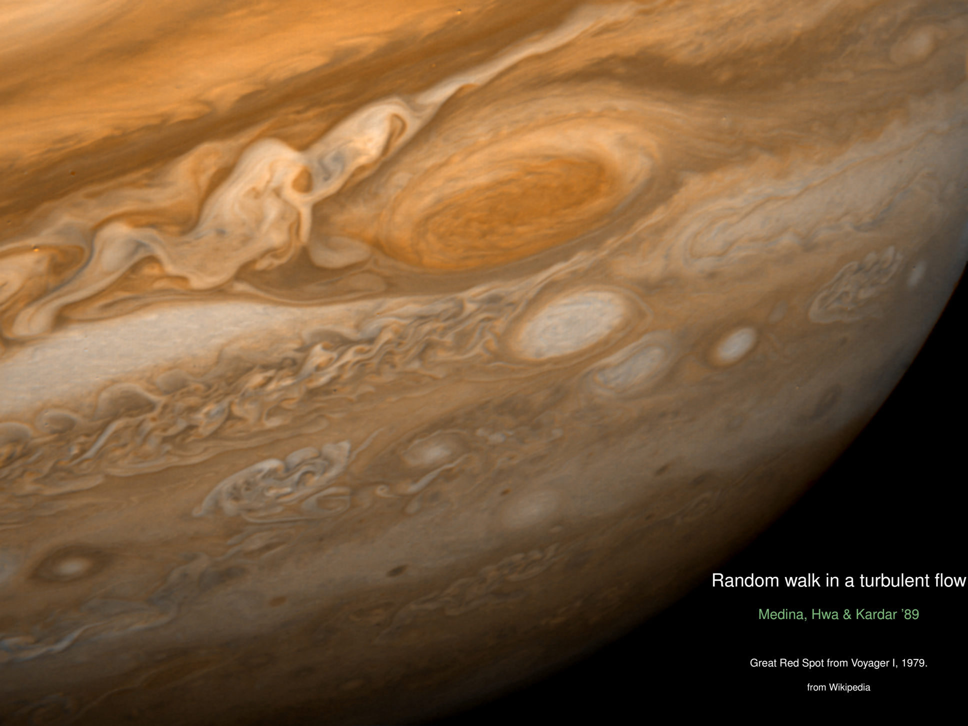

Random walk in a turbulent flow

Medina, Hwa & Kardar '89

Great Red Spot from Voyager I, 1979.

from Wikipedia

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{1}{2}\Delta\right) u(t,x) = b(u(t,x))\dot{W}(t,x), \quad x \in \mathbb{R}^d,\, t > 0, \\ u(0,\cdot) = \mu(\cdot). \end{cases}$$

1. b is Lipschitz continuous with Lipschitz constant L_b.

   $b(u) = \lambda u$: *Parabolic Anderson model.*

2. μ is *rough initial condition*, i.e., signed measure μ such that

$$\int_{\mathbb{R}^d} \exp\left(-a|x|^2\right) |\mu|(\mathrm{d}x) < \infty, \quad \text{for all } a > 0.$$

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$C_c^\infty(\mathbb{R}^d)$

1 $\quad$ $|x|^{-d/2}$

$|x|^2$ $\quad$ δ_0

$\Downarrow$

$e^{|x|^{3/2}}$ $\quad$ $|x|^{-(d+1/2)}$

$e^{|x|^3}$ $\quad$ rougher $\quad$ δ_0'

$C_c^\infty(\mathbb{R}^d)$

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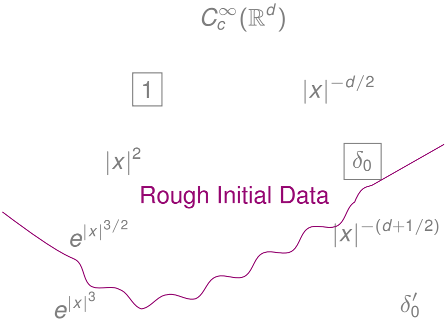

$C_c^\infty(\mathbb{R}^d)$

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Rough Initial Data

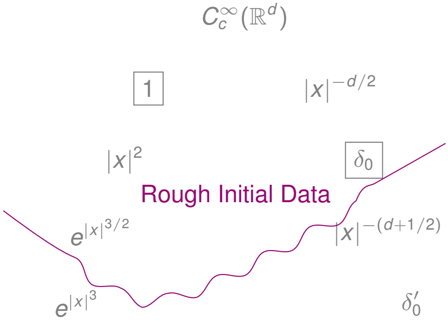
$e^{|x|^{3/2}}$ $|x|^{-(d+1/2)}$

$e^{|x|^3}$ δ_0'

$$(p_t * \mu_0)(x) < \infty \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^d$$

$$\Updownarrow$$

$$\text{RID: } \int_{\mathbb{R}^d} e^{-a|x|^2} \mu_0(\mathrm{d}x) < \infty \text{ for all } a > 0$$



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† $p_t(x) := (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$.

‡ Varadhan '68; C. & Dalang, '15; C. & Kim, '19; C. & Huang, '19...

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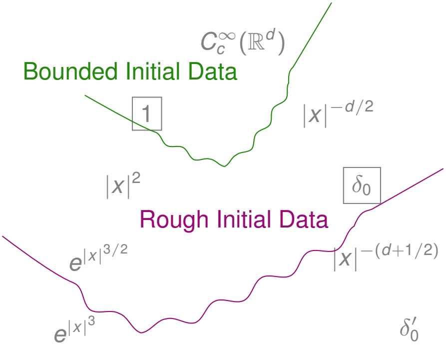

$C_c^\infty(\mathbb{R}^d)$

Bounded Initial Data

$\boxed{1}$

$|x|^{-d/2}$

$|x|^2$

$\boxed{\delta_0}$

Rough Initial Data

$e^{|x|^{3/2}}$

$|x|^{-(d+1/2)}$

$e^{|x|^3}$

δ_0'

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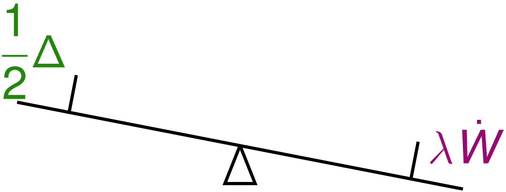
$\partial_t - \frac{1}{2}\Delta$ $u(t,x)\dot{W}(t,x)$ μ_0

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y) W(\mathrm{d}s, \mathrm{d}y)$$

$$+ (p_t * \mu_0)(x)$$

$\partial_t - \frac{1}{2}\Delta$ $u(t,x)\dot{W}(t,x)$ μ_0

Round one!



Moment Lyapunov exponents:

$$p \mapsto \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right]$$

The growth of the above mapping	$u(t,x)$
the faster	the more chaotic the more interemittent the farther from equilibrium

$$\text{(SHE)} \quad \begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}, \\ u(0,\cdot) = u_0. \end{cases}$$

For SHE on $\mathbb{R}$ with space-time white noise, many audience have contributed to the understanding of the following limit: (Bertini and Cancrini, 1995, Chen, 2015, ...)

$$\lim_{t\to\infty} t^{-1} \log \mathbb{E}\left[u(t,x)^p\right] = \frac{1}{24} p(p^2-1)\lambda^4, \qquad \text{for all } p \geq 2 \text{ and } x \in \mathbb{R},$$

$$\lim_{p\to\infty} p^{-3} \log \mathbb{E}\left[u(t,x)^p\right] = \frac{\lambda^4}{24} t, \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R}.$$

$$\text{(SWE)}\quad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \ u(t,x)\dot{W}(t,x), & t > 0, x \in \mathbb{R}, \\ u(0,\cdot) = u_0, \quad \dfrac{\partial}{\partial t} u(0,\cdot) = u_1. \end{cases}$$

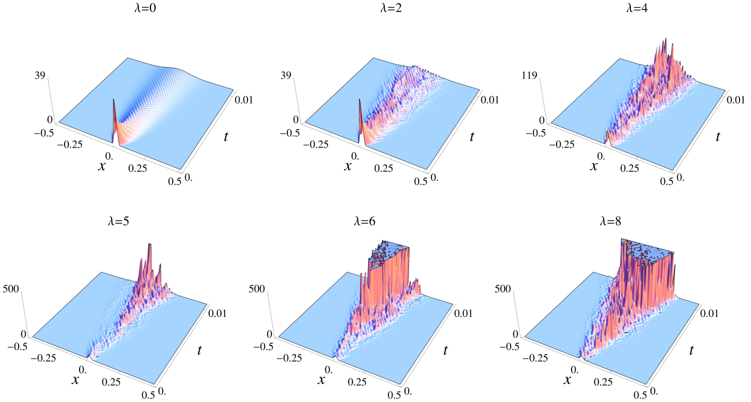
Theorem (C., Guo & Song '22)

For (SWE), if $\dot{W}$ is the space-time white noise and if $u_0 > 0$ and $u_1 \geq 0$, then

$$C_1 p^{3/2} \leq \liminf_{t\to\infty} \frac{\log \mathbb{E}[u(t,x)^p]}{t} \leq \limsup_{t\to\infty} \frac{\log \mathbb{E}[u(t,x)^p]}{t} \leq C_2 p^{3/2}, \quad p \geq 2,$$

$$C_3\, t \leq \liminf_{p\to\infty} \frac{\log \mathbb{E}[u(t,x)^p]}{p^{3/2}} \leq \limsup_{p\to\infty} \frac{\log \mathbb{E}[u(t,x)^p]}{p^{3/2}} \leq C_4\, t, \qquad t > 0.$$

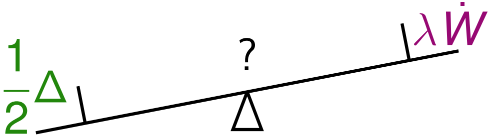
$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x) \dot{W}(t,x), \quad x \in \mathbb{R}.$$



The rate of the propagation of the tall peaks $\asymp \lambda^2$

C. & Dalang, 15.

Round two?



$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u(t,x) = \lambda\, \dot{W}(t,x) u(t,x)$$

Are there cases when

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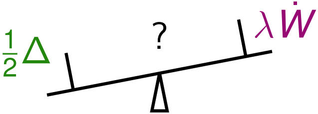
the moment Lyapunov exponents are zero?

moments are bounded in time?



$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta u(t,x)\right) \quad = \quad \lambda u(t,x)\dot{W}(t,x), \qquad u(0,\cdot) = u_0(\cdot)$$

$$\mathbb{E}\left(u(t,x)^2\right) \;\asymp\; H_f(t) \;\times\; \left[(p_t * u_0)(x)\right]^2$$

$u(t,x) = (p_t * u_0)(x) + \lambda \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)u(s,y)W(\mathrm{d}s,\mathrm{d}y).$

C., Kim '19, C., Huang '19'

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta u(t,x)\right) = \boxed{\lambda u(t,x)\dot{W}(t,x),} \quad \boxed{u(0,\cdot) = u_0(\cdot)}$$

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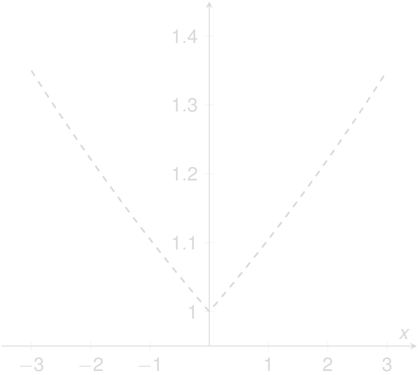
Contribution from the deterministic heat equation

$$\left(p_t * e^{\beta|\cdot|}\right)(x), \quad \beta > 0.$$

$$\left(p_t * e^{\beta|\cdot|}\right)(x) \geq \left(p_t * e^{\beta|\cdot|}\right)(0) \asymp 2e^{\frac{1}{2}\beta^2 t}, \quad t \to \infty$$

Heat injects from ∞.



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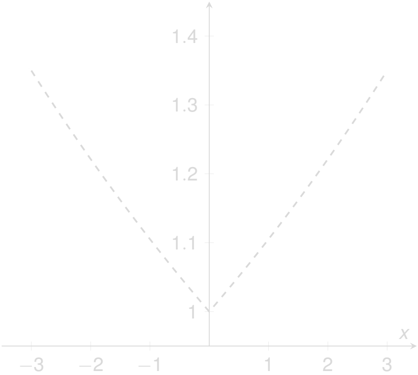

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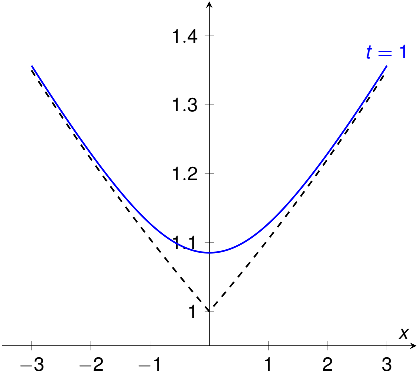

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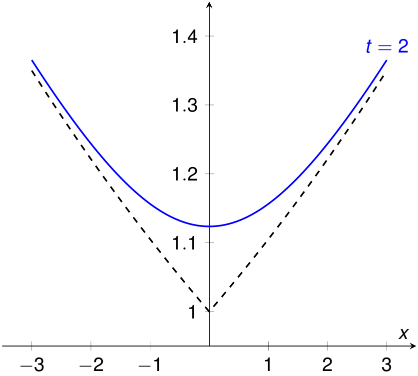

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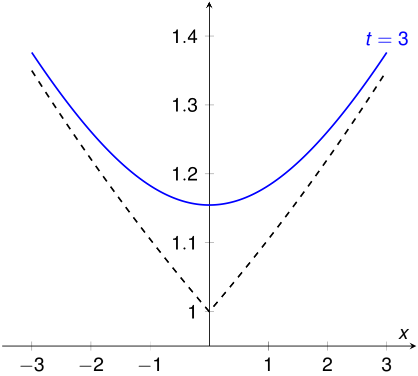

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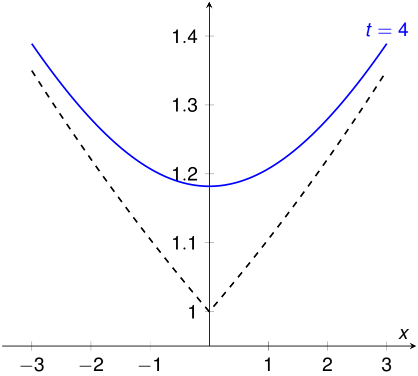

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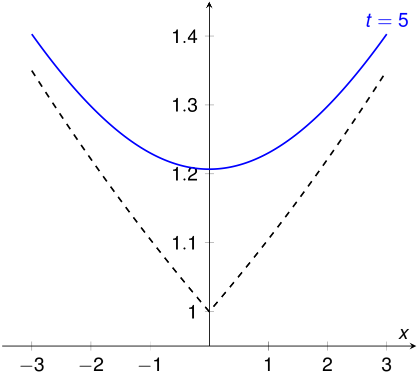

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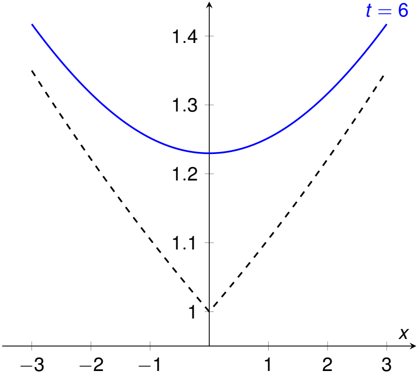

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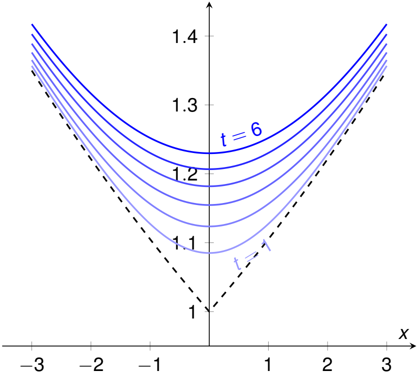
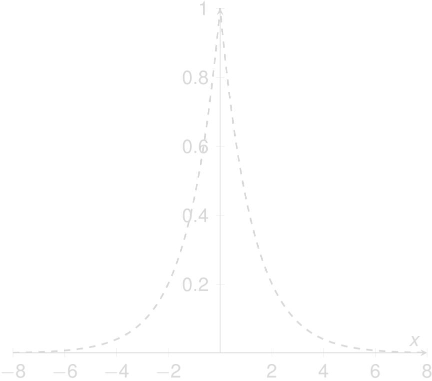


Contribution from the deterministic heat equation

$$\left(p_t * e^{\beta|\cdot|}\right)(x), \quad \beta < 0.$$

$$\left(p_t * e^{\beta|\cdot|}\right)(x) \le \left(p_t * e^{\beta|\cdot|}\right)(0) \asymp \frac{1}{|\beta|}\sqrt{\frac{2}{\pi\, t}}, \quad t \to \infty$$

Heat dissipates to $\pm\infty$.



Contribution from the deterministic heat equation

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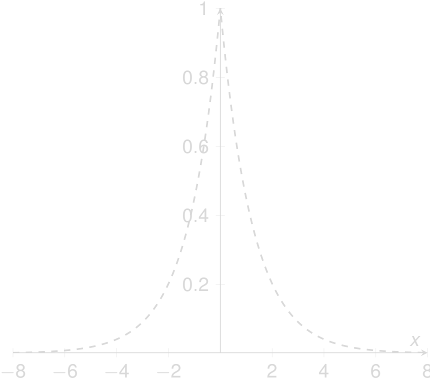

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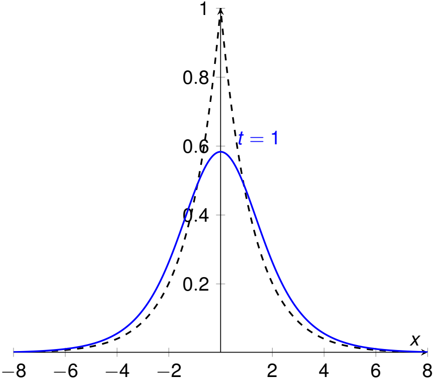

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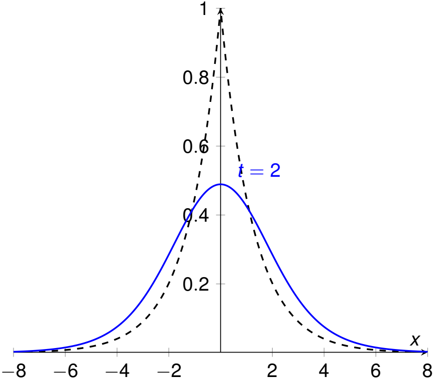

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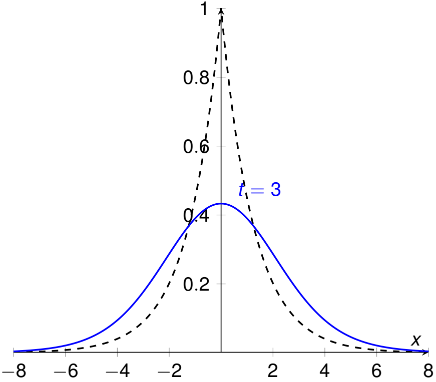

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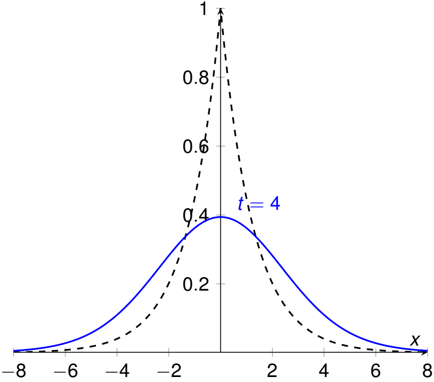

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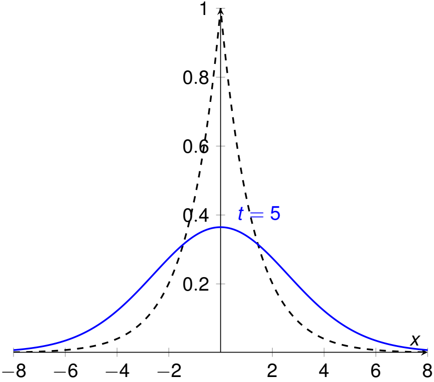

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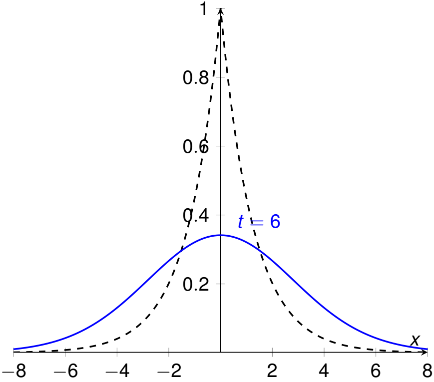

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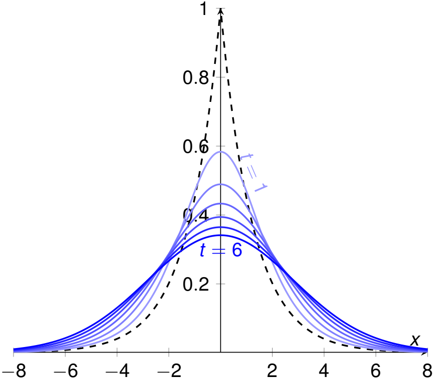
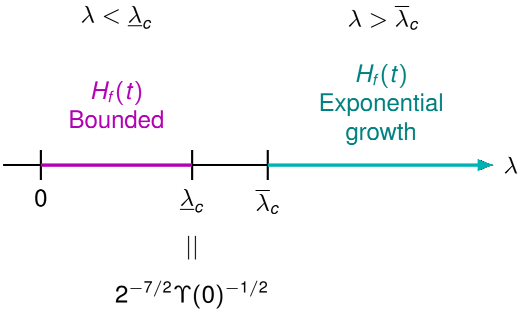


Contribution from the noise — $H_f(t)$



$\lambda < \underline{\lambda}_c$

$\lambda > \overline{\lambda}_c$

$H_f(t)$
Bounded

$H_f(t)$
Exponential
growth

λ

0

$\underline{\lambda}_c$

$\overline{\lambda}_c$

$$\underline{\lambda}_c = 2^{-7/2}\Upsilon(0)^{-1/2}$$

Theorem (C. & Kim '19)

For SHE on $\mathbb{R}^d$, the phase transition for the second moment happens iff

$$\Upsilon(0) := \lim_{\beta\to 0}\Upsilon(\beta) = (2\pi)^{-d}\int_{\mathbb{R}^d}\frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^2} < \infty.$$

$$\text{Phase transition} \quad \Longleftrightarrow \quad \Upsilon(0) := \lim_{\beta \to 0} \Upsilon(\beta) < \infty,$$

---

1. No phase transition for $d = 1$ or 2;

2. Phase transition iff

$$d \geq 3 \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{f(z)}{|z|^{d-2}} \mathrm{d}z < \infty;$$

3. Phase transition iff

$$\lim_{t \to \infty} h_1(t) < \infty, \quad \text{where} \quad h_1(t) := \mathbb{E}\left(\int_0^t f(B_t) \mathrm{d}s \right).$$

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† Strongly relies on f is both nonnegative and nonnegative definite.

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Outline

Random field

Infinitely dimensional SDE

$u : \mathbb{R}_+ \times \mathbb{R}^d \mapsto L^2(\Omega; \mathbb{R})$

$u : \mathbb{R}_+ \mapsto L^2(\Omega; H)$

$H = L^2_\rho\left(\mathbb{R}^d; \mathbb{R}\right)$

$\langle g, h \rangle_\rho := \int_{\mathbb{R}^d} g(x)h(x)\rho(x)\mathrm{d}x$

$\mathbb{E}\left(u(t,x)^2\right) < \infty$

$\mathbb{E}\left(||u(t,\cdot)||_\rho^2\right) = \int_{\mathbb{R}^d} \mathbb{E}\left(u(t,x)^2\right)\rho(x)\mathrm{d}x < \infty$

Is there an invariant measure in $L^2_\rho\left(\mathbb{R}^d; \mathbb{R}\right)$ as $t \to \infty$?

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Admissible weight functions

$H = L^2_\rho\left(\mathbb{R}^d\right)$

Definition (Tessitore & Zabczyk' 98)

A function $\rho : \mathbb{R}^d \mapsto \mathbb{R}$ is called an *admissible weight function* if it is

1. strictly positive,
2. bounded,
3. continuous,
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5. for all $T > 0$, there exists a constant $C_\rho(T)$ such that

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Moreover, if $\widehat{\rho}$ is another admissible weight such that $\int_{\mathbb{R}^d} \frac{\rho(\xi)}{\widehat{\rho}(\xi)} \mathrm{d}\xi < \infty$, then

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Examples of admissible weight functions

It is easy to show that the following weight functions are admissible:

$$\begin{cases} \rho(x) = \exp(-a|x|) & a > 0, \\ \rho(x) = \left(1 + |x|^a\right)^{-1} & a > d. \end{cases}$$

Proposition (C. & Eisenberg' 22)

Set

$$\rho_b(x) := \exp\left(-|x|^b\right), \quad x \in \mathbb{R}^d, \quad \text{with } b > 0.$$

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Let $\mathcal{M}_1(H)$ be the space of probability measure on $(H, \mathcal{B}(H))$.

Let $\mathcal{L}\left(u(t,\cdot;\mu)\right) \in \mathcal{M}_1(H)$ denote the law of $u(t,\cdot)$ starting from μ at $t = 0$.

Step 1. To obtain the tightness of the law of $u(t,\cdot)$, we need to establish:

$$\limsup_{t\to\infty} \mathbb{E}\left(||u(t,\cdot)||_\rho^2\right) < \infty$$

Step 2. Construct an invariant measure via the *Krylov-Bogoliubov theorem*:

$$\eta(A) = \lim_{n\to\infty} \frac{1}{T_n} \int_{t_0}^{T_n+t_0} \mathscr{L}(u(t,\cdot;\mu))(A)\mathrm{d}t,$$

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Outline

Introduction/Background

- Weighted Hilbert space and Krylov-Bogoliubov theorem
- Moment bounds in the weighted Hilbert space

Existence of invariant measure

- Main result
- Related work

Stationary limit via Gu and Li's approach

References

Theorem (C. & Eisenberg 24, Moment bounds)

Assume that

- (i) $\rho \in L^1(\mathbb{R}^d; \mathbb{R}_+)$;
- (ii) *The rough initial condition* μ *satisfies*

$$\mathcal{G}_\rho\left(t; |\mu|\right) := \int_{\mathbb{R}^d} \left[(p_t * |\mu|)(x)\right]^2 \rho(x)\, \mathrm{d}x < \infty, \quad \forall t > 0;$$

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$$\Upsilon\left(0\right) < \infty \quad \textit{and} \quad L_b < \underline{\lambda}_c := 2^{-7/2}\Upsilon(0)^{-1/2}.$$

Then for some constant $C > 0$, *which does not depend on* t,

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$$\Upsilon(0) < \infty \quad \text{and} \quad L_b < \underline{\lambda}_c := 2^{-7/2}\Upsilon(0)^{-1/2}.$$

Then for some constant $C > 0$, *which does not depend on* t,

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Theorem (C. & Eisenberg 24, Moment bounds)

Assume that

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Plan

- Introduction/Background
  - Weighted Hilbert space and Krylov-Bogoliubov theorem
  - Moment bounds in the weighted Hilbert space
- Existence of invariant measure
  - Main result
  - Related work
- Stationary limit via Gu and Li's approach
- References

Outline

Theorem (C. & Eisenberg 24, Invariant measure)

Assume that

(i) ρ *is admissible; and*

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Theorem (C. & Eisenberg 24, Invariant measure – continued)

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Then we have that

(a) *for any* $\tau > 0$,

$$\{\mathcal{L}u(t,\circ;\mu)\}_{t\geq\tau} \quad \textit{is tight},$$

i.e., for any $\epsilon \in (0,1)$, *there exists a compact set* $\mathscr{K} \subset L^2_\rho(\mathbb{R}^d)$ *such that*

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(b) *there exists a nontrivial invariant measure for SHE.*

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Weak path comparison;
Stochastic comparison

$$\Upsilon(0) < \infty$$

Phase transition/moment
pointwise bounded

$$\Upsilon_\alpha(\beta) < \infty$$

Hölder continuity;
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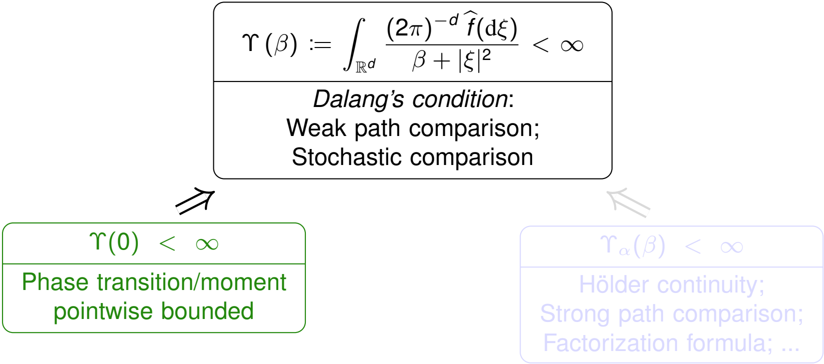

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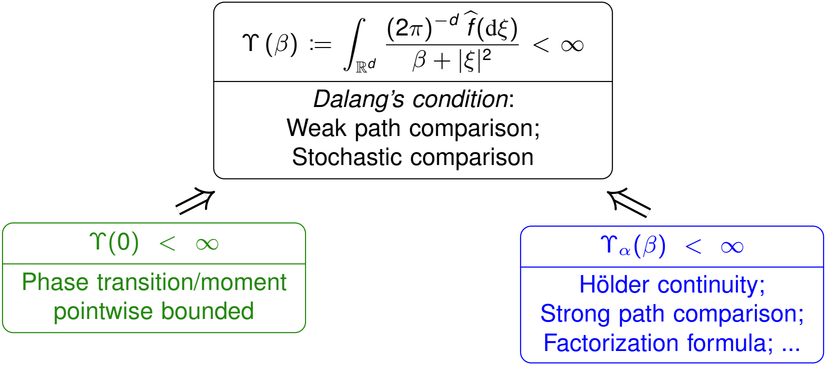

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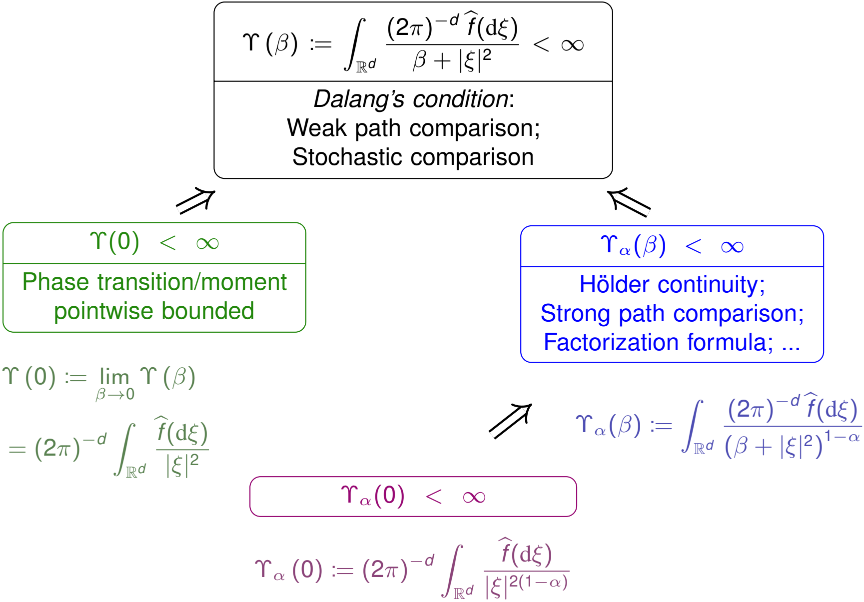

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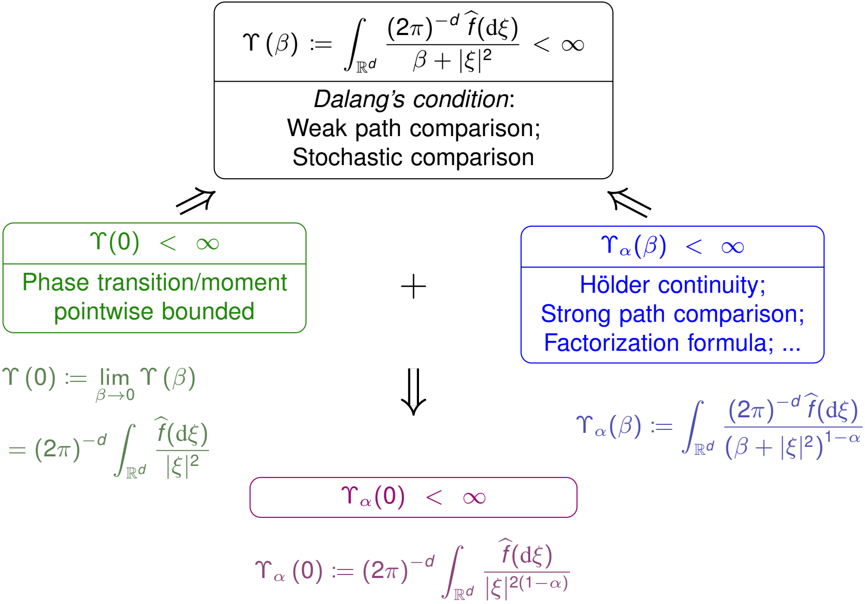

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Existence of
invariant measure

Recall the *Bessel kernel* (see, e.g., Section 1.2.2 of Grafakos '14)

$$f_s(x) := \mathcal{F}^{-1}\left[\frac{1}{(1+|\xi|^2)^{s/2}}\right](x), \quad s > 0,\ x \in \mathbb{R}^d.$$

1. $f_s(x) > 0$ for all $x \in \mathbb{R}^d$ and $||f_s||_{L^1(\mathbb{R}^d)} = 1$;
2. there exists a constant $C_{s,d} > 0$ such that $f_s(x) \le C_{s,d} e^{-\frac{|x|}{2}}$ for $|x| \ge 2$;
3. there exists a constant $c_{s,d} > 0$ such that

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4. When $s > d$, this is the *Matérn class* of correlation functions.

Recall the *Bessel kernel* (see, e.g., Section 1.2.2 of Grafakos '14)

$$f_s(x) := \mathcal{F}^{-1}\left[\frac{1}{(1+|\xi|^2)^{s/2}}\right](x), \quad s>0,\ x\in\mathbb{R}^d.$$

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Corr. fun.	$f_s(x)$	$\frac{1}{(1+\lvert x\rvert^2)^{s/2}}$
Spectral den.	$\frac{1}{(1+\lvert\xi\rvert^2)^{s/2}}$	$f_s(\xi)$
$\Upsilon_\alpha(0)$	$\frac{\Gamma\left(\frac{d}{2}-1+\alpha\right)\Gamma\left(\frac{s-d}{2}+1-\alpha\right)}{2^d\pi^{d/2}\,\Gamma(d/2)\,\Gamma(s/2)}$	$\frac{\Gamma(1-\alpha)\,\Gamma(\alpha-1+s/2)}{2^{2d}\pi^{3d/2}\Gamma(d/2)\Gamma(s/2)}$
$\Upsilon_\alpha(0)<\infty$	$s>d-2(1-\alpha)>0$	$s\wedge d>2(1-\alpha)>0$
$\Upsilon(0)$	$\frac{\Gamma\left(\frac{2+s-d}{2}\right)}{2^{d-1}\pi^{d/2}(d-2)\Gamma(s/2)}$	$\frac{2^{1-2d}\pi^{-3d/2}}{(s-2)\Gamma(d/2)}$
$\Upsilon(0)<\infty$	$s>d-2>0$	$s\wedge d>2$

Corr. fun. Spectral den.	$f_s(x)$ $\frac{1}{(1+\lvert\xi\rvert^2)^{s/2}}$	$\frac{1}{(1+\lvert x\rvert^2)^{s/2}}$ $f_s(\xi)$
$\Upsilon_\alpha(0)$ $\Upsilon_\alpha(0)<\infty$	$\frac{\Gamma\left(\frac{d}{2}-1+\alpha\right)\Gamma\left(\frac{s-d}{2}+1-\alpha\right)}{2^d\pi^{d/2}\,\Gamma(d/2)\,\Gamma(s/2)}$ $s>d-2(1-\alpha)>0$	$\frac{\Gamma(1-\alpha)\,\Gamma(\alpha-1+s/2)}{2^{2d}\pi^{3d/2}\Gamma(d/2)\Gamma(s/2)}$ $s\wedge d>2(1-\alpha)>0$
$\Upsilon(0)$ $\Upsilon(0)<\infty$	$\frac{\Gamma\left(\frac{2+s-d}{2}\right)}{2^{d-1}\pi^{d/2}(d-2)\Gamma(s/2)}$ $s>d-2>0$	$\frac{2^{1-2d}\pi^{-3d/2}}{(s-2)\Gamma(d/2)}$ $s\wedge d>2$

Corr. fun.	$f_{s_1}(x) + \widehat{f}_{s_2}(x)$	$\sim \begin{cases} \lvert x\rvert^{s_1-d} & \lvert x\rvert \to 0, \\ \lvert x\rvert^{-s_2} & \lvert x\rvert \to \infty, \end{cases}$
Spectral den.	$\widehat{f}_{s_1}(\xi) + f_{s_2}(\xi)$	$\sim \begin{cases} \lvert \xi\rvert^{s_2-d} & \lvert \xi\rvert \to 0, \\ \lvert \xi\rvert^{-s_1} & \lvert \xi\rvert \to \infty. \end{cases}$
$\Upsilon_\alpha(0)$	$\frac{\Gamma\left(\frac{d}{2} - 1 + \alpha\right)\Gamma\left(\frac{s_1-d}{2} + 1 - \alpha\right)}{2^d \pi^{d/2}\,\Gamma(d/2)\,\Gamma(s_1/2)} + \frac{\Gamma(1-\alpha)\,\Gamma\left(\alpha - 1 + \frac{s_2}{2}\right)}{2^{2d}\pi^{3d/2}\,\Gamma(d/2)\,\Gamma(s_2/2)}$	
$\Upsilon_\alpha(0) < \infty$	$0 < d - 2(1-\alpha) < s_1$ and $0 < 2(1-\alpha) < s_2$	
$\Upsilon(0)$	$\frac{\Gamma(d/2 - 1)\,\Gamma((s_1 - d)/2 + 1)}{2^d \pi^{d/2}\,\Gamma(d/2)\,\Gamma(s_1/2)} + \frac{\Gamma(s_2/2 - 1)}{2^{2d}\pi^{3d/2}\,\Gamma(d/2)\,\Gamma(s_2/2)}$	
$\Upsilon(0) < \infty$	$0 < d - 2 < s_1$ and $s_2 > 2$	

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Spectral den.	$\widehat{f}_{s_1}(\xi)+f_{s_2}(\xi)$	$\sim \begin{cases} \lvert \xi\rvert^{s_2-d} & \lvert \xi\rvert \to 0, \\ \lvert \xi\rvert^{-s_1} & \lvert \xi\rvert \to \infty. \end{cases}$
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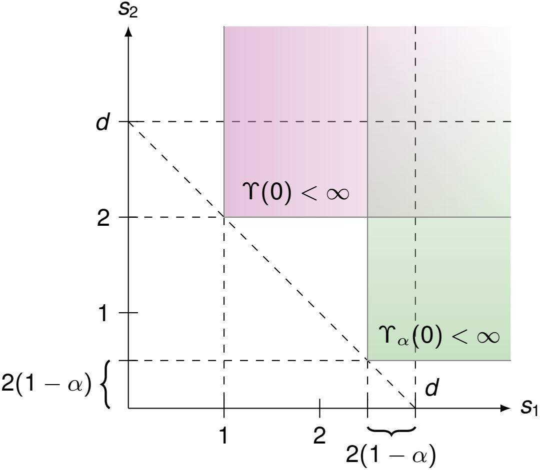
s_2
d
2
1
$2(1-\alpha)$
$\Upsilon(0) < \infty$
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d
s_1
1
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Outline

- Introduction/Background
  - Weighted Hilbert space and Krylov-Bogoliubov theorem
  - Moment bounds in the weighted Hilbert space
- Existence of invariant measure
  - Main result
  - Related work
- Stationary limit via Gu and Li's approach
- References

Among others, Tessitore and Zabczyk '98 requires that $d \geq 3$ and

$$L_b^{-2} > \frac{\Gamma(d/2-1)2^{d/2-2}}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \left(\left|\mathcal{F}(\sqrt{\widehat{f}}\ )\right| * \left|\mathcal{F}(\sqrt{\widehat{f}}\ )\right|\right)(\xi)|\xi|^{2-d}\mathrm{d}\xi,$$

which should be compared with ours:

$$L_b < 2^{-7/2}\Upsilon(0)^{-1/2}.$$

When $\mathcal{F}(\sqrt{\widehat{f}}\ ) \geq 0$, then one can remove the absolute value to show that:

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However, one can find examples when the absolute values are essential:

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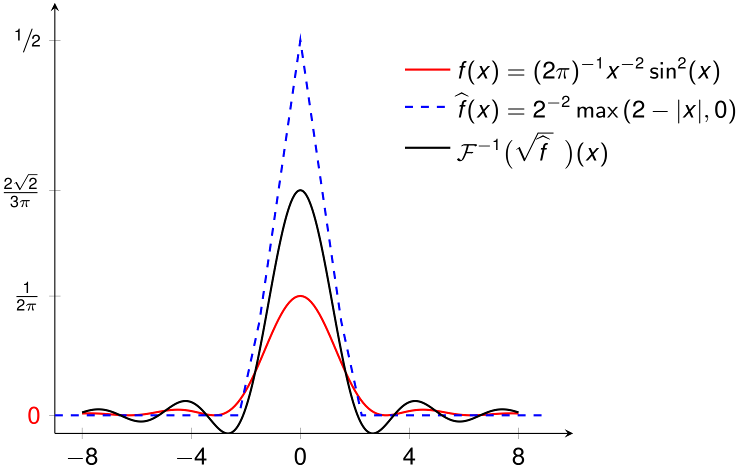
$1/2$
$\frac{2\sqrt{2}}{3\pi}$
$\frac{1}{2\pi}$
0
-8
-4
0
4
8
$f(x) = (2\pi)^{-1} x^{-2} \sin^2(x)$
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E.g.,

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Using the smoothing effect of the heat kernel, our conditions on initial data:

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The drift pointing to the origin helps to cancel the growth of the moments.

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where $Z(\cdot)$ is a stationary random field.

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1. $u_0(x) = 1 + g(x)$ with $g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$,
2. $d \geq 3$,
3. $f(0) < \infty$,
4. $L_b < \beta_0$,

then

$$u(t, \cdot) \Rightarrow Z(\cdot) \quad \text{in } C(\mathbb{R}^d), \text{ as } t \to \infty,$$

where $Z(\cdot)$ is a stationary random field.

---

1. Our weak limit is in $L^2_\rho(\mathbb{R}^d)$.
2. Can one classify initial conditions with respect to possible stationary limit?
3. Extended to Singular SPDEs Gerolla, Hairer & Li, '23.

Theorem (C., Ouyang, Tindel, Xia, '24+)

Suppose that

1. *The rough initial condition μ is such that*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} |(p_t * \mu)(x)| < \infty. \qquad (\star)$$

2. *(Condition for phase transition/weak disorder) the spectral measure $\widehat{f}$ and λ satisfy*

$$\Upsilon(0) < \infty \quad \text{and} \quad \lambda < \underline{\lambda}_c := 2^{-7/2}\Upsilon(0)^{-1/2}.$$

Then there exists a random field $Z = \{Z(x)\}_{x\in\mathbb{R}^d}$ s.t. $Z \in L^2_\rho(\mathbb{R}^d)$ a.s. &

$$u(t,\cdot) \xrightarrow{(d)} Z, \quad \text{as } t\to\infty, \quad \text{in } L^2_\rho(\mathbb{R}^d),$$

for all $\rho \in L^1(\mathbb{R}^d;\mathbb{R}_+)$ [†]*.*

[†] ρ does not need to be admissible.

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Phase transition/Weak disorder $\quad\iff\quad \Upsilon(0) := \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^2} < \infty$

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1. No phase transition for $d = 1$ or 2;
2. Phase transition iff

$$d \geq 3 \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{f(z)}{|z|^{d-2}} \mathrm{d}z < \infty;$$

3. Phase transition iff

$$\lim_{t\to\infty} h_1(t) < \infty, \quad \text{where} \quad h_1(t) := \mathbb{E}\left(\int_0^t f(B_t)\mathrm{d}s\right).$$

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† f may both have heavy tail and blow up at zero.

Earlier results, Rovira & Tindel '05, Lacoin '11,..., require $f(0) < \infty$.

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Theorem (C., Ouyang, Tindel, Xia, '24+ (Continued...))

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Moreover, suppose u_1 and u_2 are two solutions to SHE starting from μ_1 and μ_2, respectively. Assume that both μ_i satisfy ($\star$). Let Z_1 and Z_2 be the respective limiting random fields given in part 1. Then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \left|\left(p_t * (\mu_1 - \mu_2)\right)(x)\right| = 0 \quad \Longrightarrow \quad Z_1 \overset{(d)}{=} Z_2 \, .$$

Perturbation condition [†]: Let $\mu = \mu_0 + \mu_1$ such that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \left|\left(p_t * \mu_1\right)(x)\right| = 0.$$

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Examples of perturbations μ_1 s.t.

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \left|\left(p_t * \mu_1\right)(x)\right| = 0.$$

1. $\mu_1 = \delta_0$. $\notin L^\infty(\mathbb{R}^d)$
2. $\mu_1(\mathrm{d}x) = |x|^{-\alpha}\mathrm{d}x$ for $x \in (0, d)$. $\notin L^1(\mathbb{R}^d) \cup L^\infty(\mathbb{R}^d)$
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Plan

Chen, L., & Eisenberg, N. (2024).Invariant Measures for the Nonlinear Stochastic Heat Equation with No Drift Term. *J. Theoret. Probab.*, *37*(2), 1357–1396. https://doi.org/10.1007/s10959-023-01302-4

Chen, L., Ouyang, C., Tindel, S., & Xia, P. (2024).On ergodic properties of stochastic PDEs. *Preprint arXiv:2412.03521*. http://arXiv.org/abs/2412.03521

Main references:

Chen, L., & Huang, J. (2019).Comparison principle for stochastic heat equation on $\mathbb{R}^d$. *Ann. Probab.*, *47*(2), 989–1035. https://doi.org/10.1214/18-AOP1277

Chen, L., & Kim, K. (2019).Nonlinear stochastic heat equation driven by spatially colored noise: Moments and intermittency. *Acta Math. Sci. Ser. B (Engl. Ed.)*, *39*(3), 645–668. https://doi.org/10.1007/s10473-019-0303-6

Gu, Y., & Li, J. (2020).Fluctuations of a nonlinear stochastic heat equation in dimensions three and higher. *SIAM J. Math. Anal.*, *52*(6), 5422–5440. https://doi.org/10.1137/19M1296380

Tessitore, G., & Zabczyk, J. (1998).Invariant measures for stochastic heat equations. *Probab. Math. Statist.*, *18*(2, Acta Univ. Wratislav. No. 2111), 271–287.

Reference for continuous polymer and colored noise:

Lacoin, H. (2011). Influence of spatial correlation for directed polymers. *Ann. Probab.*, *39*(1), 139–175. https://doi.org/10.1214/10-AOP553

Medina, E., Hwa, T., Kardar, M., & Zhang, Y. C. (1989). Burgers' equation with correlated noise: Renormalization-group analysis and applications to directed polymers and interface growth. *Phys. Rev. A (3)*, *39*(6), 3053–3075. https://doi.org/10.1103/PhysRevA.39.3053

Rovira, C., & Tindel, S. (2005). On the Brownian-directed polymer in a Gaussian random environment. *J. Funct. Anal.*, *222*(1), 178–201. https://doi.org/10.1016/j.jfa.2004.07.017

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* References are produced from *SPDEs-Bib*: https://github.com/chenle02/SPDEs-Bib
* Download the bib file: https://github.com/chenle02/SPDEs-Bib/blob/main/All.bib

* Sources: **MathSciNet**, **APS**, and **arXiv**.

References for SHE with extra dissipativity:

Assing, S., & Manthey, R. (2003).Invariant measures for stochastic heat equations with unbounded coefficients. *Stochastic Process. Appl.*, *103*(2), 237–256. https://doi.org/10.1016/S0304-4149(02)00211-9

Brzeniak, Z., & Gatarek, D. (1999).Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces. *Stochastic Process. Appl.*, *84*(2), 187–225. https://doi.org/10.1016/S0304-4149(99)00034-4

Cerrai, S. (2003).Stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Probab. Theory Related Fields*, *125*(2), 271–304. https://doi.org/10.1007/s00440-002-0230-6

Eckmann, J.-P., & Hairer, M. (2001).Invariant measures for stochastic partial differential equations in unbounded domains. *Nonlinearity*, *14*(1), 133–151. https://doi.org/10.1088/0951-7715/14/1/308

Hairer, M., & Mattingly, J. C. (2004).Ergodic properties of highly degenerate 2D stochastic Navier-Stokes equations. *C. R. Math. Acad. Sci. Paris*, *339*(12), 879–882. https://doi.org/10.1016/j.crma.2004.09.035

Thank you!