

# Financial Mathematics

MATH 5870/6870<sup>1</sup>  
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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

# Chapter 10. Binomial Option Pricing: Basic Concepts

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§ 10.1 A one-period Binomial tree

§ 10.2 Constructing a Binomial tree

§ 10.3 Two or more binomial periods

§ 10.4 Put options

§ 10.5 American options

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# Binomial option pricing

The  
**binomial option pricing model**  
or  
**Cox-Ross-Rubinstein pricing model**  
assumes that

the price of the underlying asset follows a binomial distribution,

that is,

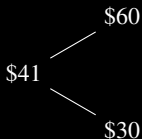
the asset price in each period can  
move only up or down by a specified amount.

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The binomial option pricing model enables us to  
**determine the price of an option,**  
given the characteristics of the stock or other underlying asset.

**Example 10.1-1** Consider an European call option on the stock of XYZ, with a \$40 strike price and one year expiration. XYZ does not pay dividends and its current price is \$41.

Assume that, in a year, the price can be either \$60 or \$30.



Can one determine the call premium?

(Let the continuously compounded risk free interest rate be 8%.)

## *Law of one price*

Positions that have the same payoff should have the same cost!

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Two portfolios (positions)

- ▶ Portfolio A: Buy one 40-strike call option.
- ▶ Portfolio B: Buy  $\Delta \in (0, 1)$  share of stock and borrow  $B$  at the risk-free rate.

These two positions should have the same cost.

**Solution.** The cost for Portfolio B at day zero is

$$\Delta \times S_0 - B.$$

and its payoff at expiration is

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} & \text{if the stock price is 30} \\ \Delta \times 60 - B \times e^{0.08} & \text{if the stock price is 60} \end{cases}$$

On the other hand, the payoff for Portfolio A should be

$$\begin{cases} 0 & \text{if the stock price is 30} \\ (60 - 40) & \text{if the stock price is 60} \end{cases}$$

By equating the two payoffs, one obtains that

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} = 0 \\ \Delta \times 60 - B \times e^{0.08} = 60 - 40 \end{cases}$$



Solution. Hence,

$$B = 20 \times e^{-0.08} \quad \text{and} \quad \Delta = 2/3.$$

Finally, since the cost of Portfolio A has to be equal to that of Portfolio B, we find the cost of Portfolio A:

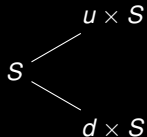
$$\Delta \times S_0 - B = \frac{2}{3} S_0 - 20 \times e^{-0.08}.$$

If we plug in  $S_0 = \$41$ , we have

$$B = \$18.462 \quad \text{and the cost is } \$ 8.871.$$

□

More generally, suppose the stock change its value over a period of time  $h$  as



Portfolio A

Payoff	$d \times S$	$u \times S$
Option	0	$u \times S - K$
Total	$C_d = 0$	$C_u = u \cdot S - K$

Portfolio B

Payoff	$d \times S$	$u \times S$
$\Delta$ share	$\Delta \cdot d \cdot S \cdot e^{\delta h}$	$\Delta \cdot u \cdot S \cdot e^{\delta h}$
$B$ bond	$Be^{rh}$	$Be^{rh}$
Total	$\Delta \cdot d \cdot S \cdot e^{\delta h} + Be^{rh}$	$\Delta \cdot u \cdot S \cdot e^{\delta h} + Be^{rh}$

For two unknowns:  $\Delta$  and  $B$ , solve:

$$\begin{cases} \Delta dS e^{\delta h} + B e^{rh} = C_d \\ \Delta uS e^{\delta h} + B e^{rh} = C_u \end{cases}$$

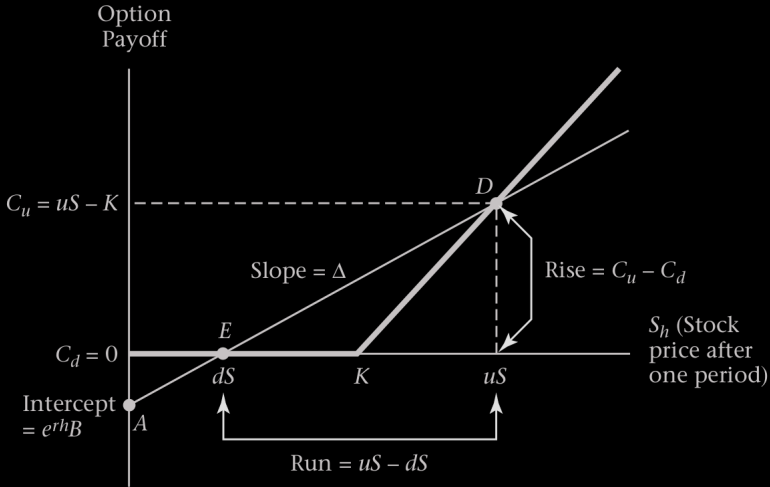
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Set  $S_h$  be either  $dS$  or  $uS$  and

$C_h$  be either  $C_u$  or  $C_d$ .

Plot  $S_h$  (x-axis) versus  $C_h$  (y-axis).

$$\Delta S_h e^{\delta h} + B e^{rh} = C_h$$



$$\Delta = e^{-\delta h} \frac{C_h - C_d}{S(u-d)} \quad \text{and} \quad B = e^{-rh} \frac{uC_d - dC_u}{u-d}$$

$$\Delta S + B = e^{-rh} \left( C_u \underbrace{\frac{e^{(r-\delta)h} - d}{u-d}}_{:=p^*} + C_d \underbrace{\frac{u - e^{(r-\delta)h}}{u-d}}_{:=1-p^*} \right)$$

$p^*$  the **risk-neutral probability** of  
an increase in the stock price.

## Arbitraging a mispriced option

**Example 10.1-2** Find arbitrage opportunities in Example 10.1-1 with

- ▶ the option price being overpriced with \$9.00;
- ▶ the option price being underpriced with \$8.25,

instead of the risk-neutral pricing \$8.871.

**Solution.** One can buy the synthetic option which cost \$8.25 and sell the real one by earning \$9.00. Hence, the present value of the profit is

$$\$9 - \$8.871 = \$0.129.$$



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$$u = e^{(r-\delta)h + \sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}}$$

- ▶  $r$ : continuously compounded annual interest rate.
- ▶  $\delta$ : continuously dividend yield.
- ▶  $\sigma$ : annual volatility.
- ▶  $h$ : the length of a binomial period in years.

# Continuously Compounded Returns

$$r_{t,t+h} = \ln (S_{t+h} / S_t)$$

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

$$r_{t,t+nh} = \sum_{i=1}^n r_{t+(i-1)h,t+ih}$$

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Go over 3 examples on p. 301

# Volatility

The **volatility** of an asset is the standard deviation of continuously compounded returns.

- ▶ A year is divided into  $n$  periods (say,  $n = 12$ ) of length  $h = 1/n$ .
- ▶ Let  $\sigma^2$  be the annual continuously compounded return.
- ▶ Assuming that the continuously compounded returns are independent and identically distributed
- ▶ We have

$$\sigma^2 = 12 \times \sigma_{\text{monthly}}^2$$

and

$$\sigma_h = \sigma \sqrt{h} \quad \text{or} \quad \sigma = \frac{\sigma_h}{\sqrt{h}}.$$

## Constructing $u$ and $d$

With no volatility

$$S_{t+h} = F_{t,t+h} = S_t e^{(r-\delta)h}$$

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With volatility

$$uS_t = F_{t,t+h} e^{+\sigma\sqrt{h}}$$

$$dS_t = F_{t,t+h} e^{-\sigma\sqrt{h}}$$

⇓

$$u = e^{(r-\delta)h + \sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}}$$

# Estimating Historical Volatility

TABLE 10.1

Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 7/7/2010 to 9/8/2010.

Date	S&P 500		IBM	
	Price	$\ln(S_t/S_{t-1})$	Price	$\ln(S_t/S_{t-1})$
7/7/2010	1060.27		127	
7/14/2010	1095.17	0.03239	130.72	0.02887
7/21/2010	1069.59	-0.02363	125.27	-0.04259
7/28/2010	1106.13	0.03359	128.43	0.02491
8/4/2010	1127.24	0.01890	131.27	0.02187
8/11/2010	1089.47	-0.03408	129.83	-0.01103
8/18/2010	1094.16	0.00430	129.39	-0.00338
8/25/2010	1055.33	-0.03613	125.27	-0.03238
9/1/2010	1080.29	0.02338	125.77	0.00398
9/8/2010	1098.87	0.01705	126.08	0.00246
Standard deviation	0.02800		0.02486	
Standard deviation $\times \sqrt{52}$	0.20194		0.17926	

- ▶ Volatility computation should exclude dividend.
- ▶ But since dividends are small and infrequent; the standard deviation will be similar whether you exclude dividends or not when computing the standard deviation.

## One-period Example with a Forward Tree

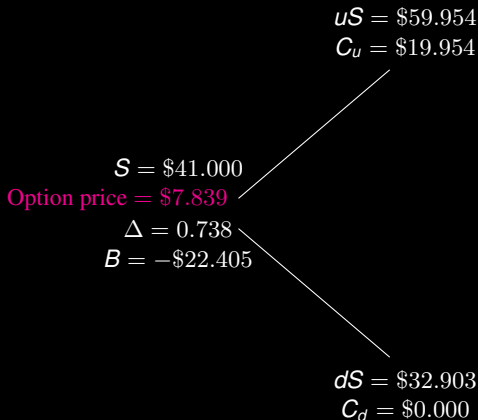
**Example 10.2-1** Consider a European call option on a stock, with a \$40 strike and 1 year to expiration. The stock does not pay dividends, and its current price is \$41. Suppose the volatility of the stock is 30%. The continuously compounded risk-free interest rate is 8%.

Use these inputs to calculate the followings:

1. the final stock prices  $uS$  and  $dS$
2. the final option values  $C_u$  and  $C_d$
3.  $\Delta$  and  $B$
4. the option price:  $\Delta S + B$ .

Solution. In summary:

$$S = 41, K = 40, r = 0.08, \delta = 0, \sigma = 0.30, h = 1.$$



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## Questions

- ▶ How to handle more than one binomial period?
- ▶ How to price put options?
- ▶ How to price American options?



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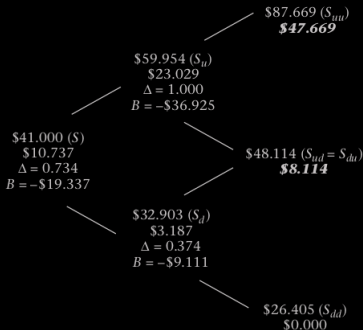
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FIGURE 10.4

Binomial tree for pricing a European call option; assumes  $S = \$41.00$ ,  $K = \$40.00$ ,  $\sigma = 0.30$ ,  $r = 0.08$ ,  $T = 2.00$  years,  $\delta = 0.00$ , and  $h = 1.000$ . At each node the stock price, option price,  $\Delta$ , and  $B$  are given. Option prices in **bold italic** signify that exercise is optimal at that node.



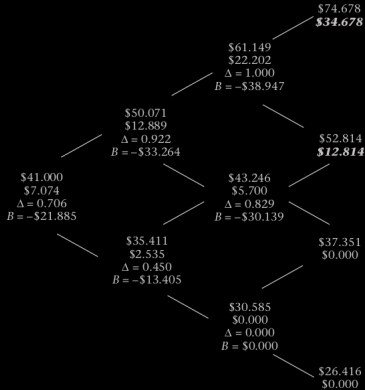
## Some observations:

- ▶ The option price is greater for the 2-year than for the 1-year option
- ▶ The option was priced by working **backward** through the binomial tree.
- ▶ The option's  $\Delta$  and  $B$  are different at different nodes. At a given point in time,  $\Delta$  increases to 1 as we go further into the money
- ▶ Permitting early exercise would make no difference. At every node prior to expiration, the option price is greater than  $S-K$ ; hence, we would not exercise even if the option had been American.

Dividing the time to expiration into more periods allows us to generate a more realistic tree with a larger number of different values at expiration.

**FIGURE 10.5**

Binomial tree for pricing a European call option; assumes  $S = \$41.00$ ,  $K = \$40.00$ ,  $\sigma = 0.30$ ,  $r = 0.08$ ,  $T = 1.00$  years,  $\delta = 0.00$ , and  $h = 0.333$ . At each node the stock price, option price,  $\Delta$ , and  $B$  are given. Option prices in **bold italic** signify that exercise is optimal at that node.



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We compute put option prices using the same stock price tree and in almost the same way as call option prices

The only difference with a European put option occurs at expiration  
Instead of computing the price as

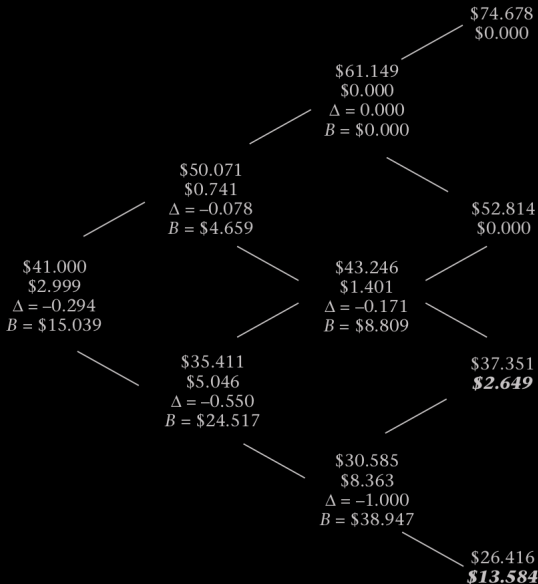
$$\max(0, S - K)$$

we use

$$\max(0, K - S)$$

**FIGURE 10.6**

Binomial tree for pricing a European put option; assumes  $S = \$41.00$ ,  $K = \$40.00$ ,  $\sigma = 0.30$ ,  $r = 0.08$ ,  $T = 1.00$  years,  $\delta = 0.00$ , and  $h = 0.333$ . At each node the stock price, option price,  $\Delta$ , and  $B$  are given. Option prices in *bold italic* signify that exercise is optimal at that node.



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At each node we use the following formula to compute the price:

$$P(S, K, t) = \max \left( K - S, e^{-rh} [P(uS, K, t+h)p^* + P(dS, K, t+h)(1-p^*)] \right)$$

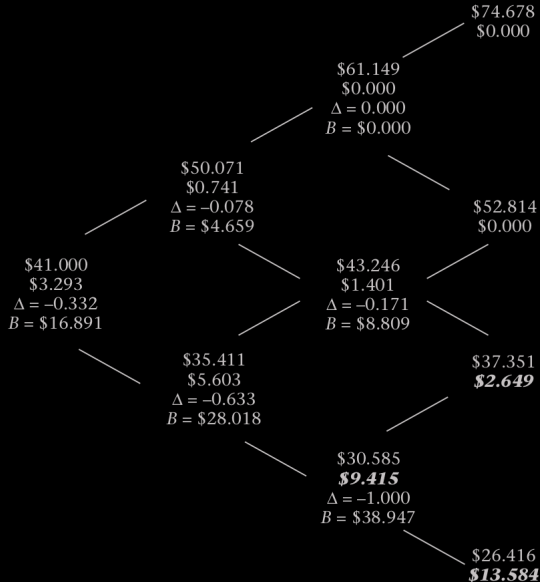
$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

Or simply

$$P(S, K, t) = \max(K - S, \Delta S + B)$$

**FIGURE 10.7**

Binomial tree for pricing an American put option; assumes  $S = \$41.00$ ,  $K = \$40.00$ ,  $\sigma = 0.30$ ,  $r = 0.08$ ,  $T = 1.00$  years,  $\delta = 0.00$ , and  $h = 0.333$ . At each node the stock price, option price,  $\Delta$ , and  $B$  are given. Option prices in *bold italic* signify that exercise is optimal at that node.



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This section is left for motivated students.

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Problems: 10.1, 10.2, 10.3, 10.6, 10.7, 10.8, 10.10, 10.12.

Due Date: TBA