# Financial Mathematics

MATH 5870/6870[1]
Fall 2021

Le Chen

lzc0090@auburn.edu

Last updated on

October 19, 2021

Auburn University
Auburn AL

[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

# Chapter 13. Market-Making and Delta-Hedging

# Chapter 13. Market-Making and Delta-Hedging

- Provide immediacy by standing ready to sell to buyers (at ask price) and to buy from sellers (at bid price)
- Generate inventory as needed by short-selling
- Profit by charging the bid-ask spread
- The position of a market-maker is the result of whatever order flow arrives from customers

- Proprietary trading is conceptually distinct from market-making:

  Proprietary trading: Profit by market goes up or down.

  Market-making: Profit by buying at the bid and selling at the ask.

# Chapter 13. Market-Making and Delta-Hedging

- Market-makers attempt to hedge the risk of their positions.
- Market-makers can control risk by Delta-hedging.
- A hedged position should earn the risk-free rate.

**TABLE 13.1** Price and Greek information for a call option with $S = \$40$, $K = \$40$, $\sigma = 0.30$, $r = 0.08$ (continuously compounded), $T - t = 91/365$, and $\delta = 0$.

| | Purchased | Written |
|---|---|---|
| Call price | 2.7804 | −2.7804 |
| Delta | 0.5824 | −0.5824 |
| Gamma | 0.0652 | −0.0652 |
| Theta | −0.0173 | 0.0173 |

Example 13.2-1 Under setting of the above table,

- compute call price, Delta, Gamma and Theta.
- If stock price increases to $S = 40.75$,
  find the exact gain/loss of the market-maker.
  find the approximate gain/loss of the market-maker via $\Delta$.
- If stock price decreases to $S = 39.25$,
  find the exact gain/loss of the market-maker.
  find the approximate gain/loss of the market-maker via $\Delta$.

(Assume we liquidate the position at the same day)

Solution. Try codes/Section_13-2.nb □

# Chapter 13. Market-Making and Delta-Hedging

# Chapter 13. Market-Making and Delta-Hedging

**TABLE 13.2** Daily profit calculation over 5 days for a market-maker who delta-hedges a written option on 100 shares.

| | Day 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Stock ($) | 40.00 | 40.50 | 39.25 | 38.75 | 40.00 | 40.00 |
| Call ($) | 278.04 | 306.21 | 232.82 | 205.46 | 271.04 | 269.27 |
| 100 × delta | 58.24 | 61.42 | 53.11 | 49.56 | 58.06 | 58.01 |
| Investment ($) | 2051.58 | 2181.30 | 1851.65 | 1715.12 | 2051.35 | 2051.29 |
| Interest ($) | | −0.45 | −0.48 | −0.41 | −0.38 | −0.45 |
| Capital gain ($) | | 0.95 | −3.39 | 0.81 | −3.62 | 1.77 |
| Daily profit ($) | | 0.50 | −3.87 | 0.40 | −4.00 | 1.32 |

Example 13.3-1 Given the first line of the above table, filling all the rest entries.

Solution. Check codes/Section_13-2.nb □

# Self-financing portfolio: stock moves one $\sigma$

**TABLE 13.3** Daily profit calculation over 5 days for a market-maker who delta-hedges a written option on 100 shares, assuming the stock price moves up or down 1 $\sigma$ each day.

| | Day | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| Stock ($) | 40.000 | 40.642 | 40.018 | 39.403 | 38.797 | 39.420 |
| Call ($) | 278.04 | 315.00 | 275.57 | 239.29 | 206.14 | 236.76 |
| 100 × delta | 58.24 | 62.32 | 58.27 | 54.08 | 49.80 | 54.06 |
| Investment ($) | 2051.58 | 2217.66 | 2056.08 | 1891.60 | 1725.95 | 1894.27 |
| Interest ($) | | −0.45 | −0.49 | −0.45 | −0.41 | −0.38 |
| Capital gain ($) | | 0.43 | 0.51 | 0.46 | 0.42 | 0.38 |
| Daily profit ($) | | −0.02 | 0.02 | 0.01 | 0.01 | 0.00 |

**Example 13.3-2** Given the first line of the above table, filling all the rest entries.

*Solution.* Check codes/Section_13-2.nb □

# Chapter 13. Market-Making and Delta-Hedging

First order (in $S$) approximation
(with zero order in $h$):

$$C\left(S_{t+h}, T-(t+h)\right) \approx C(S_t, T-t) + \Delta(S_t, T-t) \times \left(S_{t+h} - S_t\right)$$

---

Second order in $S$ approximation
(with zero order in $h$):

$$\begin{aligned} C\left(S_{t+h}, T-(t+h)\right) \approx \quad & C(S_t, T-t) + \Delta(S_t, T-t) \times \left(S_{t+h} - S_t\right) \\ & + \frac{1}{2} \times \Gamma(S_t, T-t) \times \left(S_{t+h} - S_t\right)^2 \end{aligned}$$

Delta-Gamma approximation

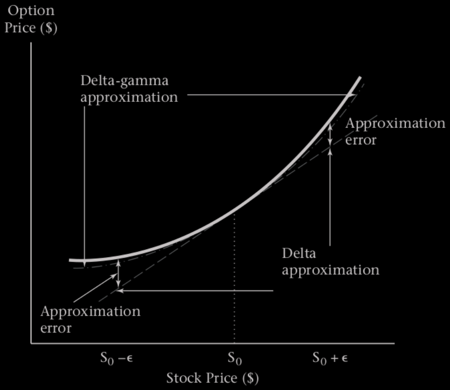
Explanations can be made
either using Taylor expansion or $\Delta_{\text{average}}$.

## Second order in $S$ approximation (with first order in $h$):

$$
\begin{aligned}
C\left(S_{t+h}, T-(t+h)\right) \approx \quad & C(S_t, T-t) \\
& + \Delta(S_t, T-t) \times (S_{t+h} - S_t) \\
& + \frac{1}{2} \times \Gamma(S_t, T-t) \times (S_{t+h} - S_t)^2 \\
& + h \times \theta(S_t, T-t)
\end{aligned}
$$

FIGURE 13.3

Delta- and delta-gamma approximations of option price. The true option price is represented by the bold line, and approximations by dashed lines.



Example 13.4-1 Given the first column of the following table, filling the details of the rest entries:

**TABLE 13.4** Predicted option price over a period of 1 day, assuming stock price move of \$0.75, using equation (13.6). Assumes that $\sigma = 0.3$, $r = 0.08$, $T - t = 91$ days, and $\delta = 0$, and the initial stock price is \$40.

| | | | | | Option Price 1 Day Later ($h = 1$ day) | |
|---|---|---|---|---|---|---|
| | Starting Price | $\epsilon\Delta$ | $\frac{1}{2}\epsilon^2\Gamma$ | $\theta h$ | Predicted | Actual |
| $S_{t+h} = \$40.75$ | \$2.7804 | 0.4368 | 0.0183 | −0.0173 | \$3.2182 | \$3.2176 |
| $S_{t+h} = \$39.25$ | \$2.7804 | −0.4368 | 0.0183 | −0.0173 | \$2.3446 | \$2.3452 |

Solution. Working with Mathematica code... □

The value of the market-maker's investment:

$$\Delta_t S_t - C(S_t)$$

---

Market-marker's profit when the stock price changes by $\epsilon$ over a time interval $h$

$$\underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} - \underbrace{rh\,[\Delta_t S_t - C(S_t)]}_{\text{interest charge}}$$

Now replace $\underbrace{C(S_{t+h}) - C(S_t)}_{\text{Changes in value of option}}$ by its second order approximation:

$$\begin{aligned} C\left(S_{t+h}\right) - C(S_t) \approx \quad & \Delta_t \times (S_{t+h} - S_t) \\ & + \frac{1}{2} \times \Gamma_t \times (S_{t+h} - S_t)^2 \\ & + h \times \theta_t \end{aligned}$$

and $S_{t+h} - S_t$ by $\epsilon$, we see that

$$\text{Market-maker's profit}$$

$$||$$

$$\underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} \quad - \quad \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} \quad - rh \underbrace{[\Delta_t S_t - C(S_t)]}_{\text{interest charge}}$$

$$||$$

$$-\left(\frac{1}{2}\epsilon^2\Gamma_t + \theta_t h + rh\left[\Delta_t S_t - C(S_t)\right]\right)$$

We have seen that the market-maker approximately breaks even for a one-standard-deviation move in the stock:

$$\epsilon = \sigma S_t \sqrt{h} \qquad \Longleftrightarrow \qquad \epsilon^2 = \sigma^2 S_t^2 h$$

---

Finally, we see that

$$\begin{gathered}\text{Market-maker's profit} \\ \| \\ \underbrace{\Delta_t (S_{t+h} - S_t)}_{\text{Changes in value of stock}} \quad - \quad \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} \quad - rh \underbrace{[\Delta_t S_t - C(S_t)]}_{\text{interest charge}} \\ \| \\ -\left(\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + \theta_t + r\,[\Delta_t S_t - C(S_t)]\right) h\end{gathered}$$

# Chapter 13. Market-Making and Delta-Hedging

From the previous section we see that

$$\text{Market-maker's profit} = -\left(\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + \theta_t + r\left[\Delta_t S_t - C(S_t)\right]\right) h$$

If one believes that via one-standard deviation move, the market-maker's profit is approximately zero, we arrive at the Black-Scholes equation:

$$\boxed{\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + \theta_t + r\Delta_t S_t = rC(S_t)}$$

Equivalently, this can be written as a standard PDE:

$$\mathcal{L}_{\mathrm{BS}} V(t, S) = 0$$

where $V(t, S)$ refers to option (call or put) price and

$$\mathcal{L}_{\mathrm{BS}} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + r\, S \frac{\partial}{\partial S} V(t, S) - r.$$

One still needs to put the correct boundary conditions.

- Under the following assumptions:
  Underlying asset and the option do not pay dividends
  Interest rate and volatility are constant
  The stock does not make large discrete moves

- The equation is valid only when early exercise is not optimal

# Chapter 13. Market-Making and Delta-Hedging

This section will be skipped for interested students.

# Chapter 13. Market-Making and Delta-Hedging