Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 13. Market-Making and Delta-Hedging

Chapter 13. Market-Making and Delta-Hedging

- Provide immediacy by standing ready to sell to buyers (at ask price) and to buy from sellers (at bid price)
- Generate inventory as needed by short-selling
- Profit by charging the bid-ask spread
- The position of a market-maker is the result of whatever order flow arrives from customers

- Proprietary trading is conceptually distinct from market-making:

  Proprietary trading: Profit by market goes up or down.

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Chapter 13. Market-Making and Delta-Hedging

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TABLE 13.1 Price and Greek information for a call option with $S = \$40$, $K = \$40$, $\sigma = 0.30$, $r = 0.08$ (continuously compounded), $T - t = 91/365$, and $\delta = 0$.

	Purchased	Written
Call price	2.7804	−2.7804
Delta	0.5824	−0.5824
Gamma	0.0652	−0.0652
Theta	−0.0173	0.0173

Example 13.2-1 Under setting of the above table,

- compute call price, Delta, Gamma and Theta.
- If stock price increases to $S = 40.75$,
  find the exact gain/loss of the market-maker.
  find the approximate gain/loss of the market-maker via Δ.
- If stock price decreases to $S = 39.25$,
  find the exact gain/loss of the market-maker.
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(Assume we liquidate the position at the same day)

Solution. Try codes/Section_13-2.nb □

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Solution. Try codes/Section_13-2.nb □

Chapter 13. Market-Making and Delta-Hedging

TABLE 13.2 Daily profit calculation over 5 days for a market-maker who delta-hedges a written option on 100 shares.

	Day 0	1	2	3	4	5
Stock ($)	40.00	40.50	39.25	38.75	40.00	40.00
Call ($)	278.04	306.21	232.82	205.46	271.04	269.27
100 × delta	58.24	61.42	53.11	49.56	58.06	58.01
Investment ($)	2051.58	2181.30	1851.65	1715.12	2051.35	2051.29
Interest ($)		−0.45	−0.48	−0.41	−0.38	−0.45
Capital gain ($)		0.95	−3.39	0.81	−3.62	1.77
Daily profit ($)		0.50	−3.87	0.40	−4.00	1.32

Example 13.3-1 Given the first line of the above table, filling all the rest entries.

Solution. Check codes/Section_13-2.nb □

Self-financing portfolio: stock moves one σ

TABLE 13.3 Daily profit calculation over 5 days for a market-maker who delta-hedges a written option on 100 shares, assuming the stock price moves up or down 1 σ each day.

	Day					
	0	1	2	3	4	5
Stock ($)	40.000	40.642	40.018	39.403	38.797	39.420
Call ($)	278.04	315.00	275.57	239.29	206.14	236.76
100 × delta	58.24	62.32	58.27	54.08	49.80	54.06
Investment ($)	2051.58	2217.66	2056.08	1891.60	1725.95	1894.27
Interest ($)		−0.45	−0.49	−0.45	−0.41	−0.38
Capital gain ($)		0.43	0.51	0.46	0.42	0.38
Daily profit ($)		−0.02	0.02	0.01	0.01	0.00

Example 13.3-2 Given the first line of the above table, filling all the rest entries.

Solution. Check codes/Section_13-2.nb □

Chapter 13. Market-Making and Delta-Hedging

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First order (in S) approximation
(with zero order in h):

$$C\left(S_{t+h}, T-(t+h)\right) \approx C(S_t, T-t) + \Delta(S_t, T-t) \times \left(S_{t+h} - S_t\right)$$

Second order in S approximation
(with zero order in h):

$$\begin{aligned} C\left(S_{t+h}, T-(t+h)\right) \approx \quad & C(S_t, T-t) + \Delta(S_t, T-t) \times \left(S_{t+h} - S_t\right) \\ & + \frac{1}{2} \times \Gamma(S_t, T-t) \times \left(S_{t+h} - S_t\right)^2 \end{aligned}$$

Delta-Gamma approximation

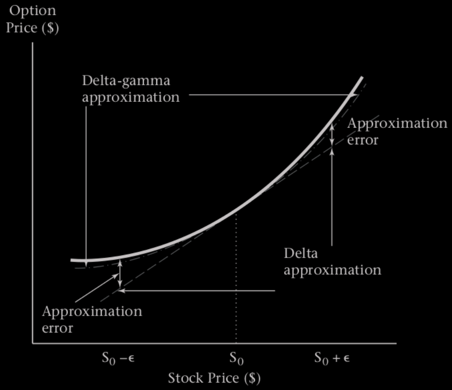
Explanations can be made
either using Taylor expansion or Δ_{average}.

Second order in S approximation
(with first order in h):

$$
\begin{aligned}
C\left(S_{t+h}, T-(t+h)\right) \approx \quad & C(S_t, T-t) \\
& + \Delta(S_t, T-t) \times (S_{t+h} - S_t) \\
& + \frac{1}{2} \times \Gamma(S_t, T-t) \times (S_{t+h} - S_t)^2 \\
& + h \times \theta(S_t, T-t)
\end{aligned}
$$

FIGURE 13.3

Delta- and delta-gamma approximations of option price. The true option price is represented by the bold line, and approximations by dashed lines.



Example 13.4-1 Given the first column of the following table, filling the details of the rest entries:

TABLE 13.4 Predicted option price over a period of 1 day, assuming stock price move of \$0.75, using equation (13.6). Assumes that $\sigma = 0.3$, $r = 0.08$, $T - t = 91$ days, and $\delta = 0$, and the initial stock price is \$40.

					Option Price 1 Day Later ($h = 1$ day)	
	Starting Price	$\epsilon\Delta$	$\frac{1}{2}\epsilon^2\Gamma$	θh	Predicted	Actual
$S_{t+h} = \$40.75$	\$2.7804	0.4368	0.0183	−0.0173	\$3.2182	\$3.2176
$S_{t+h} = \$39.25$	\$2.7804	−0.4368	0.0183	−0.0173	\$2.3446	\$2.3452

Solution. Working with Mathematica code... □

The value of the market-maker's investment:

$$\Delta_t S_t - C(S_t)$$

Market-marker's profit when the stock price changes by ϵ over a time interval h

$$\underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} - \underbrace{rh\,[\Delta_t S_t - C(S_t)]}_{\text{interest charge}}$$

Now replace $\underbrace{C(S_{t+h}) - C(S_t)}_{\text{Changes in value of option}}$ by its second order approximation:

$$\begin{aligned} C\left(S_{t+h}\right) - C(S_t) \approx \quad & \Delta_t \times \left(S_{t+h} - S_t\right) \\ & + \frac{1}{2} \times \Gamma_t \times \left(S_{t+h} - S_t\right)^2 \\ & + h \times \theta_t \end{aligned}$$

and $S_{t+h} - S_t$ by ϵ, we see that

$$\begin{gathered}\text{Market-maker's profit}\\ \|\\ \underbrace{\Delta_t(S_{t+h}-S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h})-C(S_t)]}_{\text{Changes in value of option}} - rh\underbrace{[\Delta_t S_t - C(S_t)]}_{\text{interest charge}}\\ \|\\ -\left(\frac{1}{2}\epsilon^2\Gamma_t + \theta_t h + rh\left[\Delta_t S_t - C(S_t)\right]\right)\end{gathered}$$

We have seen that the market-maker approximately breaks even for a one-standard-deviation move in the stock:

$$\epsilon = \sigma S_t \sqrt{h} \qquad \Longleftrightarrow \qquad \epsilon^2 = \sigma^2 S_t^2 h$$

Finally, we see that

$$\begin{gathered}
\text{Market-maker's profit} \\
\| \\
\underbrace{\Delta_t (S_{t+h} - S_t)}_{\text{Changes in value of stock}} \quad - \quad \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} \quad - rh \underbrace{[\Delta_t S_t - C(S_t)]}_{\text{interest charge}} \\
\| \\
-\left(\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + \theta_t + r\,[\Delta_t S_t - C(S_t)] \right) h
\end{gathered}$$

Chapter 13. Market-Making and Delta-Hedging

From the previous section we see that

$$\text{Market-maker's profit} = -\left(\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + \theta_t + r\left[\Delta_t S_t - C(S_t)\right]\right) h$$

If one believes that via one-standard deviation move, the market-maker's profit is approximately zero, we arrive at the Black-Scholes equation:

$$\boxed{\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + \theta_t + r\Delta_t S_t = rC(S_t)}$$

Equivalently, this can be written as a standard PDE:

$$\mathcal{L}_{\mathrm{BS}} V(t, S) = 0$$

where $V(t, S)$ refers to option (call or put) price and

$$\mathcal{L}_{\mathrm{BS}} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + r\, S \frac{\partial}{\partial S} V(t, S) - r.$$

One still needs to put the correct boundary conditions.

- Under the following assumptions:
  Underlying asset and the option do not pay dividends
  Interest rate and volatility are constant
  The stock does not make large discrete moves

- The equation is valid only when early exercise is not optimal

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This section will be skipped for interested students.

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