Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 3. Insurance, Collars, and Other Strategies

Chapter 3. Insurance, Collars, and Other Strategies

Options can be

1. Used to insure long positions (floors)
2. Used to insure short positions (caps)
3. Written against asset positions (selling insurance)

   Covered call writing

   Covered put writing

Four positions

positions w.r.t. asset	put option	call option
long	purchased (floor)	written
short	written	purchased (cap)

Buying insurance	Selling insurance
floor = buying a put option	Covered put writing
cap = buying a call option	Covered call writing

We will work under the following setup

S&S index

index price today	\$1,000
6-month interest rate	2%
premium for 1000-strike 6-month call	\$93.809
premium for 1000-strike 6-month put	\$74.201

Insuring a long position – Floors

owning a home	owning a stock index
insuring the house	buying a put (floor)

Goal: to insure against a fall in the price of the underlying asset.

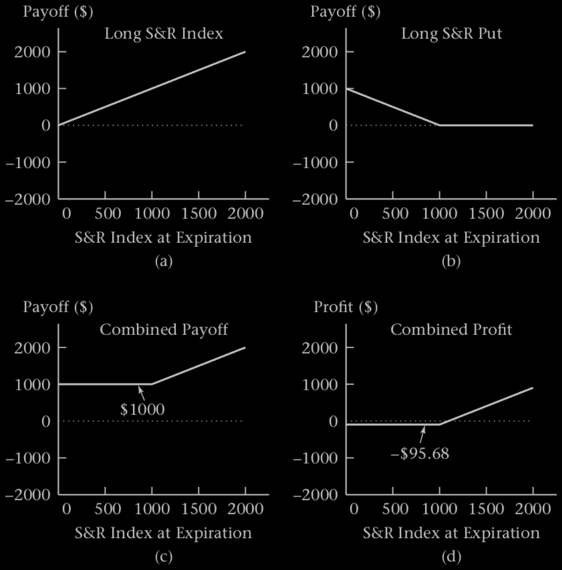
Example 3.1-1 Under the following scenario, compute the combined profit of insuring a long position via buying a put for the following S&R index.

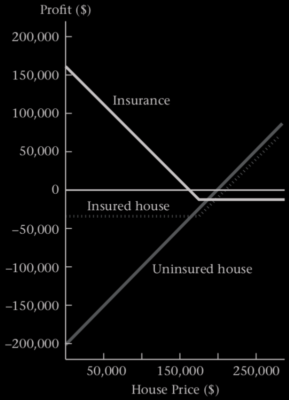
index price today	\$1,000
6-month interest rate	2%
premium for 1000-strike 6-month put	\$74.201
index price at expiration	\$900

Solution.

$$\underbrace{\$900 - \$1,000 \times 1.02}_{\text{profit on S\&R index}} + \underbrace{\$1,000 - \$900 - \$74.201 \times 1.02}_{\text{profit on put}} = -\$95.68.$$

□


Payoff ($)
Long S&R Index
2000
1000
0
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(a)
Payoff ($)
Long S&R Put
2000
1000
0
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(b)
Payoff ($)
Combined Payoff
2000
1000
$1000
0
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(c)
Profit ($)
Combined Profit
2000
1000
0
–$95.68
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(d)


Profit ($)
200,000
150,000
100,000
50,000
0
−50,000
−100,000
−150,000
−200,000
Insurance
Insured house
Uninsured house
50,000
150,000
250,000
House Price ($)

Insuring a short position – Caps

If we have a short position in the S&R index, we experience a loss when the index rises.

We can insure a short position by purchasing a call option (cap) to protect against a higher price of repurchasing the index.

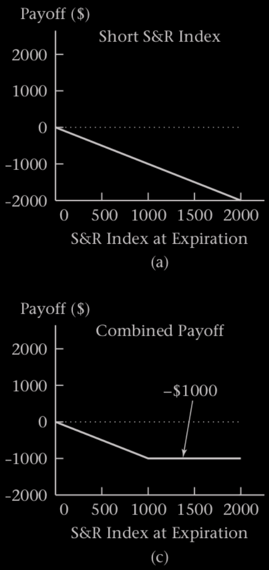
Example 3.1-2 Under the following scenario, compute the combined profit for insuring a short position via buying a call of the following S&R index.

index price today	$1,000
6-month interest rate	2%
premium for 1000-strike 6-month call	$93.809
index price at expiration	$1,100

Solution.

$$\underbrace{\$1,000 \times 1.02}_{\text{future value of short S\&R index}} - \underbrace{\$93.809 \times 1.02}_{\text{FV of premium for call}} - \underbrace{\$1,000}_{\text{exercise the call option}} = -\$75.685.$$

□



Payoff ($)
Short S&R Index
2000
1000
0
−1000
−2000
0 500 1000 1500 2000
S&R Index at Expiration
(a)
Payoff ($)
Long S&R Call
2000
1000
0
−1000
−2000
0 500 1000 1500 2000
S&R Index at Expiration
(b)
Payoff ($)
Combined Payoff
–$1000
2000
1000
0
−1000
−2000
0 500 1000 1500 2000
S&R Index at Expiration
(c)
Profit ($)
Combined Profit
–$75.68
2000
1000
0
−1000
−2000
0 500 1000 1500 2000
S&R Index at Expiration
(d)

Selling insurance

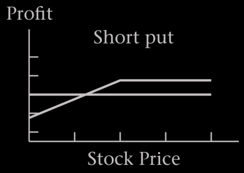
For every insurance buyer there must be an insurance seller

Strategies used to sell insurance

- Covered writing (option overwriting or selling a covered call) is writing an option when there is a corresponding long position in the underlying asset.
- Naked writing is writing an option when the writer does not have a position in the asset.

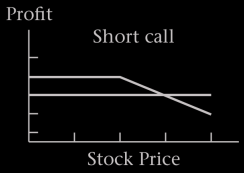
Covered call writing

Long position of the asset + Sell a call option $\sim$



Covered put writing

Short position of the asset + Sell a put option $\sim$



Covered call writing

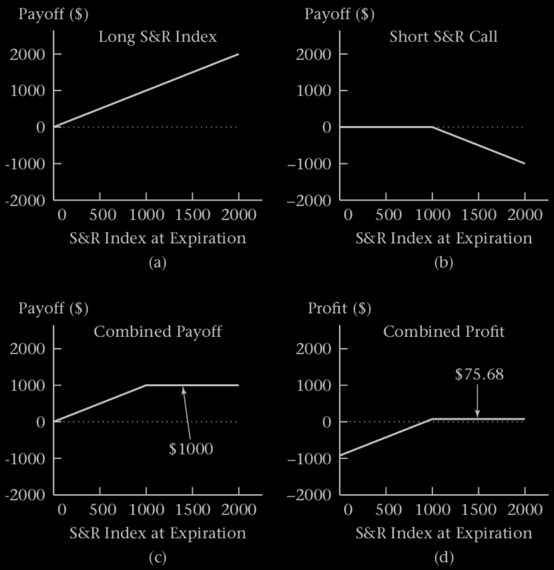
Example 3.1-3 Under the following scenario, compute the combined profit for writing a covered call for S&R index.

index price today	\$1,000
6-month interest rate	2%
premium for 1000-strike 6-month call	\$93.809
index price at expiration	\$1,100

Solution.

$$\underbrace{\$1,100 - \$1,000 \times 1.02}_{\text{profit on S\&R index}} + \underbrace{\$1,000 - \$1,100 + \$93.809 \times 1.02}_{\text{profit on written call}} = \$75.68.$$

□



Payoff ($)
Long S&R Index
2000
1000
0
-1000
-2000
0
500
1000
1500
2000
S&R Index at Expiration
(a)
Payoff ($)
Short S&R Call
2000
1000
0
−1000
−2000
0
500
1000
1500
2000
S&R Index at Expiration
(b)
Payoff ($)
Combined Payoff
2000
1000
0
-1000
-2000
$1000
0
500
1000
1500
2000
S&R Index at Expiration
(c)
Profit ($)
Combined Profit
2000
1000
0
−1000
−2000
$75.68
0
500
1000
1500
2000
S&R Index at Expiration
(d)

Covered put writing

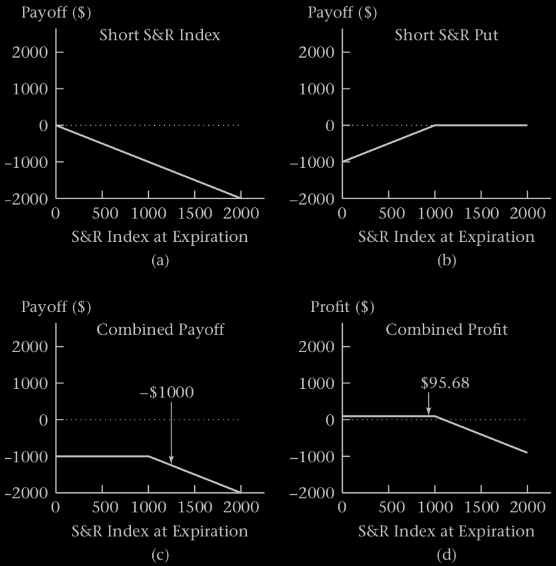
Example 3.1-4 Under the following scenario, compute the combined profit for writing a covered put for S&R index.

index price today	\$1,000
6-month interest rate	2%
premium for 1000-strike 6-month put	\$74.201
index price at expiration	\$900

Solution.

$$\underbrace{\$1,000 \times 1.02 - \$900}_{\text{profit on selling S\&R index}} + \underbrace{\$900 - \$1,000 + \$74.201 \times 1.02}_{\text{profit on written put}} = \$95.685.$$

□


Payoff ($)
Short S&R Index
2000
1000
0
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(a)
Payoff ($)
Short S&R Put
2000
1000
0
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(b)
Payoff ($)
Combined Payoff
2000
1000
–$1000
0
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(c)
Profit ($)
Combined Profit
2000
1000
$95.68
0
–1000
–2000
0
500
1000
1500
2000
S&R Index at Expiration
(d)

Chapter 3. Insurance, Collars, and Other Strategies

It is possible to mimic a long forward position on an asset by

buying a call + selling a put,

with each option having the same strike price and expiration time.

||

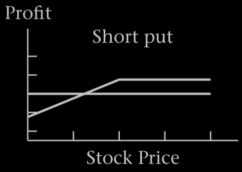
A synthetic forward

Example 3.2-1 Working with the S&R index. Suppose that

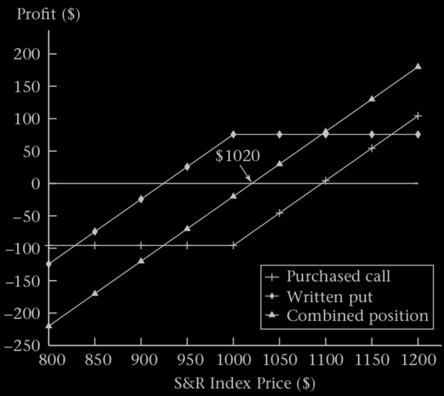
6-month interest rate	2%
premium for 1000-strike 6-month call	\$93.809
premium for 1000-strike 6-month put	\$74.201

Draw profit digram for the combined position of a purchased call with a written put, namely,

+



Solution.



Profit ($)
200
150
100
50
0
−50
−100
−150
−200
−250
800
850
900
950
1000
1050
1100
1150
1200
S&R Index Price ($)
$1020
Purchased call
Written put
Combined position

□

A synthetic long forward contract

We pay the net option premium

We pay the strike price

The actual forward

We pay zero premium

We pay the forward price

Basic Assumption

The net cost of buying the index using options

must equal

the net cost of buying the index using a forward contract.

NO ARBITRAGE!

The Put-Call parity equation

$$\mathrm{Call}(K, T) - \mathrm{Put}(K, T) = \mathrm{PV}\left(F_{0,T} - K\right)$$

- K: strike price
- T: expiration date
- $\mathrm{Call}(\cdot, \circ)$: the premium for call.
- $\mathrm{Put}(\cdot, \circ)$: the premium for put.
- $F_{0,T}$: the forward price at time T if one enters at time 0 into a long forward position.
- $\mathrm{PV}(\cdot)$: the present value function.

Example 3.2-2 Check Example 3.2-1 to see if the put-call parity equation is satisfied.

Solution. We need to check:

$$\$93.809 - \$74.201 \stackrel{?}{=} \text{PV}(\$1,000 \times 1.02 - \$1,000)$$

Clearly, LHS $= \$19.61$. On the other hand, the RHS is equal to

$$\begin{aligned}\text{PV}(\$1,000 \times 1.02 - \$1,000) &= \text{PV}\left(1,000 \times (1.02 - 1)\right) \\ &= \text{PV}\left(1,000 \times 0.02\right) \\ &= \frac{1,000 \times 0.02}{1.02} \\ &= \$19.61.\end{aligned}$$

Hence, the put-call parity equation is satisfied. □

$$\text{Call}(K,T) - \text{Put}(K,T) = \text{PV}\left(F_{0,T} - K\right)$$

$$\Updownarrow$$

$$\text{PV}\left(F_{0,T}\right) + \text{Put}(K,T) = \text{Call}(K,T) + \text{PV}\left(K\right)$$

Buying the index and buying the put

generate the same payoff as

buying the call and buying a zero-coupon bond (i.e. lending) PV(K)

$$\text{Call}(K, T) - \text{Put}(K, T) = \text{PV}\left(F_{0,T} - K\right)$$

$$\Updownarrow$$

$$\text{PV}\left(F_{0,T}\right) - \text{Call}(K, T) = \text{PV}\left(K\right) - \text{Put}(K, T)$$

Writing a covered call

has the same profit as

lending $\text{PV}(K)$ and selling a put

$$\text{Call}(K,T) - \text{Put}(K,T) = \text{PV}\left(F_{0,T}\right) - \text{PV}\left(K\right)$$

Revisit four positions in Section 3.1

Position	Meaning	equivalent to
Inuring a long position (floors)	Index + Put	Bound + Call
Inuring a short position (caps)	−Index + Call	−Bound + Put
Covered call writing	Index − Call	Bound − Put
Covered put writing	−Index − Put	− Bound − Call

Chapter 3. Insurance, Collars, and Other Strategies

It is always possible

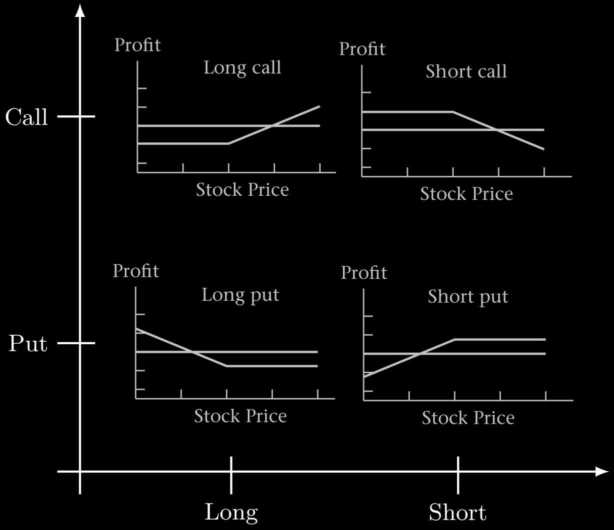
to

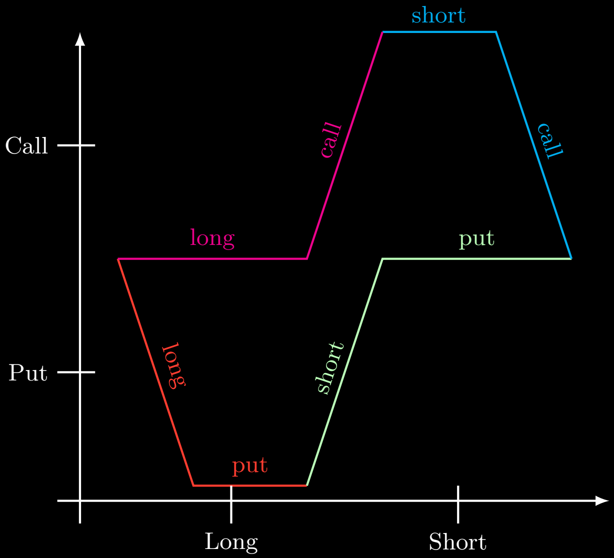
lower the cost of a position

by

reducing its payoff!

By combining two or more options, we find many well-known strategies.




short
call
call
long
put
Call
Put
long
short
put
Long
Short

An option spread is a position consisting of only calls or only puts, in which some options are purchased and some written.

- Bull and bear spreads
- Box spreads
- Ratio spreads
- Collars

Example for this section

Black-Scholes option prices

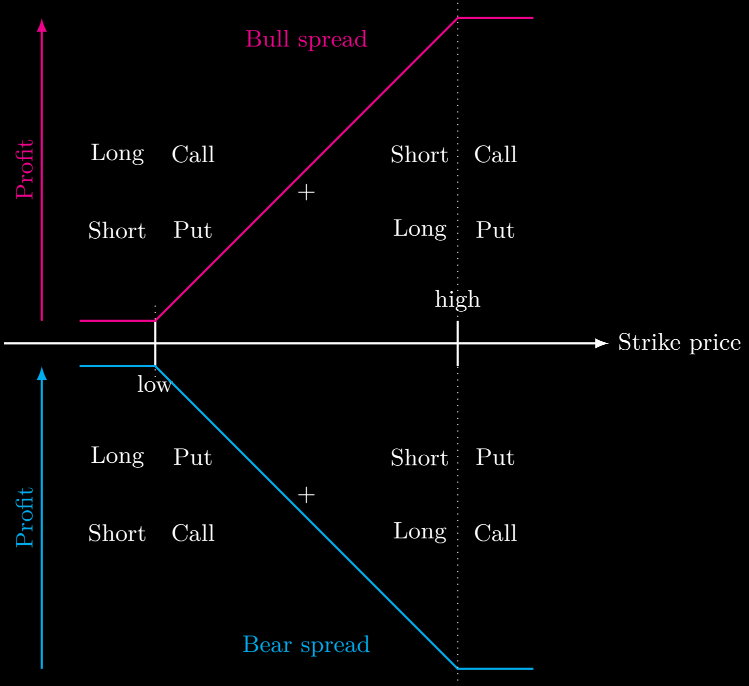
$$\begin{aligned} \text{Stock price} &= \$40 \\ \text{Volatility} &= 30\% \\ \text{Effective annual risk-free rate} &= 8.33\% \\ \text{Dividend yield} &= \$0 \\ \text{Expriation days} &= 91 \text{ days} \end{aligned}$$

Strike	Call	Put
35	6.13	0.44
40	2.78	1.99
45	0.97	5.08

Bull and bear spreads

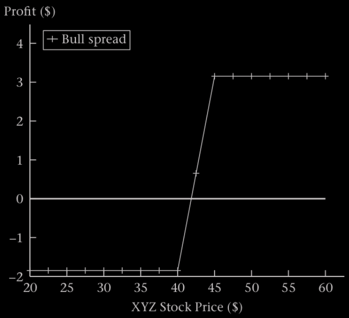
A position in which you buy a call and sell an otherwise identical call with a higher strike price is an example of a bull spread. Bull spreads can also be constructed using puts.

The opposite of a bull spread is a bear spread.


Bull spread
Profit
Long Call
Short Put
+
Short Call
Long Put
high
Strike price
low
Long Put
Short Call
+
Short Put
Long Call
Profit
Bear spread

Example 3.3-1 Draw profit diagram for a 40-45 bull spread, namely, buying a 40-strike call and selling a 45-strike call.

Solution.



We only need to determine the two levels.

Solution(Continued).

(a) Suppose that the index price is \$ 30 at the expiration:

$$(\$2.78 - \$0.97) \times (1 + 0.0833)^{1/4} = \$1.85.$$

(b) Suppose that the index price is \$50 at the expiration:

$$(\$50 - \$40) - (\$40 - \$45) - \$1.85 = \$3.15.$$

□

Box spreads

A **box spread** is accomplished by using options to create a synthetic long forward at one price and a synthetic short forward at a different price.

This strategy guarantees a cash flow in the future.

Hence, it is an option spread that is purely a means of borrowing or lending money. It is costly but has no stock price risk.

Example 3.3-2 Suppose we simultaneously enter into the following two transactions:

1. Buy a 40-strike call and sell a 40-strike put.
2. Sell a 45-strike call and buy a 45-strike put.

Explain why there is no free lunch. Draw the profit diagram.

Solution. The profit is

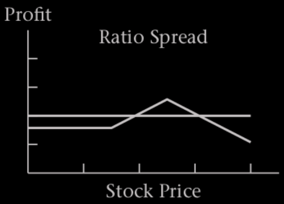
$$5 + \underbrace{(1.99 - 2.78) \times (1.0833)^{1/4}}_{\text{Synthetic long forward}} + \underbrace{(0.97 - 5.08) \times (1.0833)^{1/4}}_{\text{Synthetic short forward}} = \$0.0099851.$$

□

Buy a $40-strike call	Sell a $40-strike put	Synthetic long forward
Sell a $45-strike call	Buy a $45-strike put	Synthetic short forward
↑ Bull spread	↑ Bear spread	

Ratio spreads

A **ratio spread** is constructed by buying m options at one strike and selling n options at a different strike, with all options having the same type (call or put), same time to maturity, and same underlying asset.



Example 3.3-3 (Problem 3.15) Compute profit diagrams for the following ratio spreads:

a Buy 950-strike call, sell two 1050-strike calls.

b Buy two 950-strike calls, sell three 1050-strike calls.

c Consider buying n 950-strike calls and selling m 1050-strike calls so that the premium of the position is zero. Considering your analysis in (a) and (b), what can you say about n/m? What exact ratio gives you a zero premium?

Strike	Call	Put
$950	$120.405	$51.777
1000	93.809	74.201
1020	84.470	84.470
1050	71.802	101.214
1107	51.873	137.167

Solution. ... □

Collars

A **collar** is the purchase of a put option and the sale of a call option with a higher strike price, with both options having the same underlying asset and having the same expiration date.

If the position is reversed, i.e., sale of a put and purchase of a call, the collar is written.

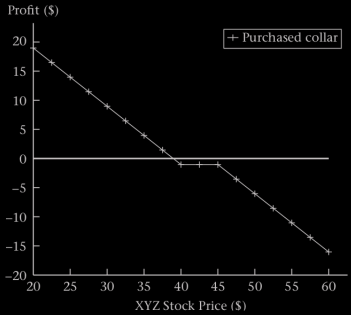
The **collar width** is the difference between the call and put strikes.

Example 3.3-4 Draw the profit diagram for a purchased collar:

selling a 45-strike call + buying a 40-strike put.

Solution. One can easily draw the profit graph. We only need to determine the level when the curve is flat. Hence, suppose the price is \$43. Then the profit is

$$(0.97 - 1.99) \times (1.083)^{1/4} = -\$1.0405.$$



□

It is possible to find strike prices for the put and call such that the two premiums exactly offset one another. This position is called a **zero-cost collar**.

Example 3.3-5 Consider XYZ:

Strike	Call	Put
35	6.13	0.44
40	2.78	1.99
41.72	1.99	—
45	0.97	5.08

where we need to use Black-Scholes formula to find out the strike price, which is 41.72, when the put premium is $1.99. This gives a zero-cost collar.

Chapter 3. Insurance, Collars, and Other Strategies

Chapter 3. Insurance, Collars, and Other Strategies

Directional positions

- Bull spread
- Bear spread
- Collars
- Box spreads

Nondirectional positions

- Straddles
- Strangle
- Butterfly spread

Investors who do not care whether the stock goes up or down, but only how much it moves.

Investors are speculating on

volatility

Example for this section

Black-Scholes option prices

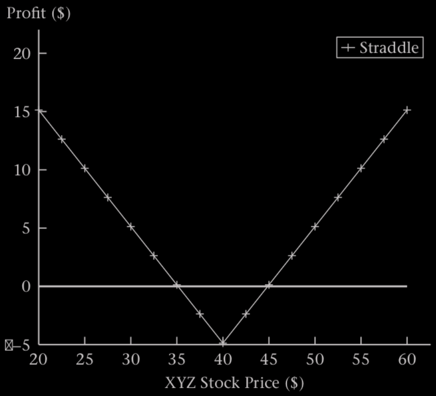
$$
\begin{aligned}
\text{Stock price} &= \$40 \\
\text{Volatility} &= 30\% \\
\text{Effective annual risk-free rate} &= 8.33\% \\
\text{Dividend yield} &= \$0 \\
\text{Expriation days} &= 91 \text{ days}
\end{aligned}
$$

Strike	Call	Put
35	6.13	0.44
40	2.78	1.99
45	0.97	5.08

Straddles

Straddle is the strategy of buying a call and a put with the same strike price and time to expiration.

A straddle is a bet that volatility will be high relative to the market's assessment

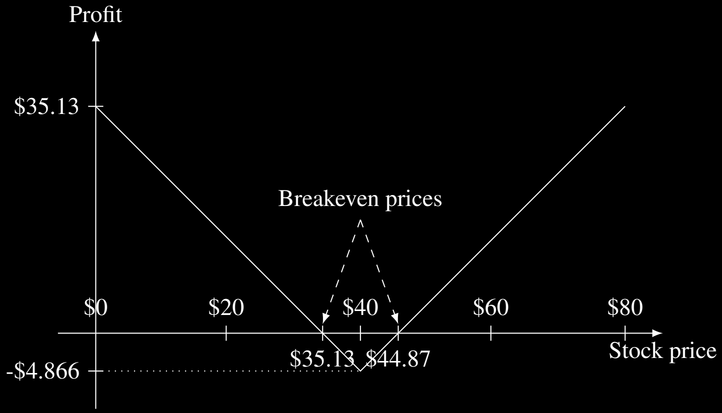


Example 3.4-1 Draw the profit graph for a \$40=strike straddle.

Solution. We only need to determine the tip of the graph:

$$-(2.78 + 1.99) \times (1 + 0.083)^{1/4} = -\$4.8660.$$

Hence,



□

Strangle

Strangle is the strategy of buying an out-of-the-money call and put with the same time to expiration.

A strangle can be used to reduce the high premium cost, associated with a straddle.

	Buying call at a strike price	Buying put at a strike price
Straddle	Same	Same
Strangle	High	Low

Example 3.4-2 Draw profit diagram for 40-strike straddle and strangle composed of

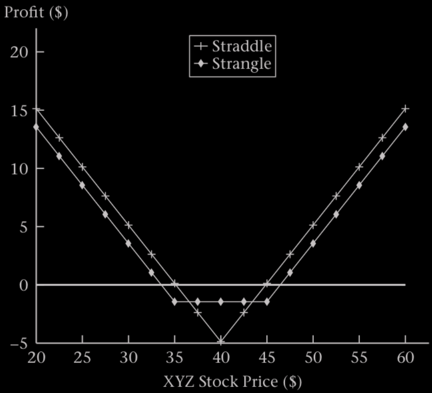
35-strike put + 45-strike call.

Solution. We know the shape of the graph and need only to determine the level of the flat part. Hence, suppose the stock price is \$40. Then the profit is

$$-(0.44 + 0.97) \times (1 + 0.083)^{1/4} = -\$1.4384.$$

The breakeven prices are

$$45 + 1.4384 = \$46.4384 \quad \text{and} \quad 35 - 1.4384 = \$33.562.$$


Profit ($)
Straddle
Strangle
20
15
10
5
0
−5
20
25
30
35
40
45
50
55
60
XYZ Stock Price ($)

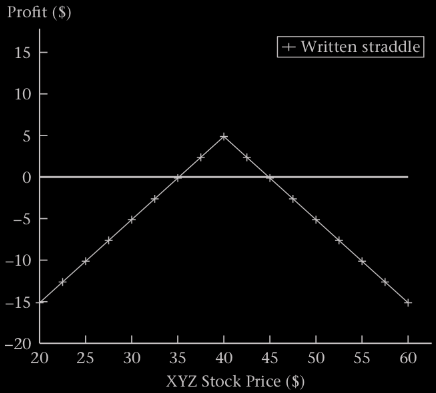
Written straddles

Written straddle is the strategy of selling a call and put with the same strike price and time to maturity.

Unlike a purchased straddle, a written straddle is a bet that

volatility will be low

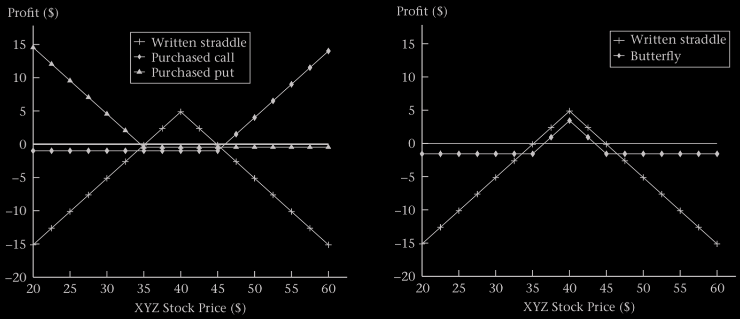
relative to the market's assessment.



Butterfly spreads

$$\textbf{Butterfly spreads} = \text{Insured wrien straddle}$$
$$= \text{Written straddle} + \text{purchased straggle}$$

A butterfly spread insures against large losses on a straddle.



Example 3.4-3 Draw the profit graph for the butterfly spread:

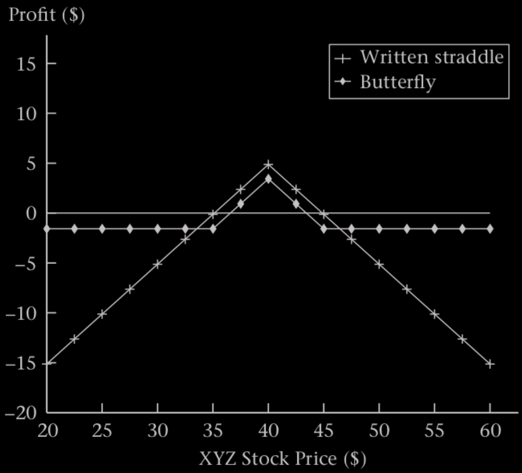
Written \$40 straddle + purchased 35-45 straggle.

Solution. First notice that this spread corresponds:

Strike	Call	Put
35	6.13	0.44 (long)
40	2.78 (short)	1.99 (short)
45	0.97 (long)	5.08

We know the general shape of the profit graph and need only to determine the level when the graph is flat. For this, suppose that the stock price is \$ $x < 30$. In this case, only both puts are in the money and the profit is

$$(2.78 + 1.99 - 0.44 - 0.97) \times (1 + 0.083)^{1/4} + (35 - x) + (x - 40) = -\$1.5724.$$



Profit ($)
Written straddle
Butterfly
15
10
5
0
–5
–10
–15
–20
20
25
30
35
40
45
50
55
60
XYZ Stock Price ($)

Chapter 3. Insurance, Collars, and Other Strategies

Problems: 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.11, 3.13, 3.14, 3.15, 3.17, 3.18.

Due Date: TBA