Financial Mathematics

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Chapter 5. Financial Forwards and Futures

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Four different payment and receipt timing combinations

1. Outright purchase: ordinary transaction
2. Fully leveraged purchase: investor borrows the full amount
3. Prepaid forward contract: pay today, receive the share later
4. Forward contract: agree on price now, pay/receive later

	Day 0	Day T	Payment
Outright purchase	pay+receive	—	S_0
Fully leveraged purchase	receive	pay	$S_0 e^{rT}$
Prepaid forward contract	pay	receive	?
Forward contract	—	pay+receive	$? \times e^{rT}$

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Three ways to determine the payment for the prepaid forward contracts (no dividend case)

- ▶ Pricing the prepaid forward by analogy
- ▶ Pricing the prepaid forward by discounted present value
- ▶ Pricing the prepaid forward by arbitrage

In the absence of dividends, whether you receive physical possession today or at time T is irrelevant: In either case you own the stock, and at time T it will be exactly as if you had owned the stock the whole time. Hence,

$$F_{0,T}^{p} = S_0$$

Pricing the prepaid forward by discounted present value

Let α be the expected return on the stock.

Let $\mathbb{E}_0(S_T)$ be the expected stock price at time T.

Hence,

$$F_{0,T}^p = \underbrace{\mathbb{E}_0(S_T)}_{=S_0 \times e^{\alpha T}} \times e^{-\alpha T} = S_0$$

Pricing the prepaid forward by arbitrage

Arbitrage = Free money

The price of a derivative should be such that

no arbitrage is possible.

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1. If $F_{0,T}^p > S_0$: find the arbitrage.

2. If $F_{0,T}^p < S_0$: find the arbitrage.

   Hence, $F_{0,T}^p = S_0$.

Pricing prepaid forwards with dividends
– Discrete dividends

Suppose a stock is expected to make dividend payments of D_{t_i} at time t_i, $i = 1, \cdots, n$. Then

$$F_{0,T}^P = S_0 - \sum_{i=1}^{n} \mathrm{PV}_{0,t_i}\left(D_{t_i}\right),$$

where $\mathrm{PV}_{0,t}(\cdot)$ is the present value at time zero of a time t_i payment.

Example 5.2-1 Suppose XYZ stock costs \$100 today and is expected to pay a \$1.25 quarterly dividend, with the first coming 3 months from today and the last just prior to the delivery of the stock. Suppose the annual continuously compounded risk-free rate is 10%. The quarterly continuously compounded rate is therefore 2.5%. Find a 1-year prepaid forward contract for the stock would cost.

Solution.

$$F_{0,1}^T = \$100 - \sum_{i=1}^{4} \$1.25 \times e^{-0.025i} = \$93.30.$$

□

Pricing prepaid forwards with dividends – Continuous dividends

Let δ be the compounded dividend yield. Then

$$F_{0,T}^P = S_0 e^{-\delta T}$$

Example 5.2-2 Suppose that the index is \$125 and the annualized daily compounded dividend yield is 3%. Find the prepaid forward price at one year.

Solution.

$$F_{0,1}^p = \$125e^{-0.03 \times 1} = \$121.306.$$

□

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Forward price is the future value of the prepaid forward price:

$$F_{0,T} = \text{FV}\left(F_{0,T}^p\right)$$

Example 5.3-1 Continuous dividends

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

- ▶ r: risk-free rate.
- ▶ δ: the dividend yield.

$$\text{Forward premium} = \frac{F_{0,T}}{S_0}$$

$$\text{Annualized forward premium} = \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right)$$

Does the forward price predict the future spot price?

Buying a stock

Compensation for	Earn	Buying a stock
time value of the money	interest	✓
the risk of the stock	risk premium	✓

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Entering a forward contract

Compensation for	Earn	Entering a forward contract
time value of the money	interest	✗
the risk of the stock	risk premium	✓

The forward price is the expected future spot price, discounted at the risk premium.

$$F_{0,T} = e^{rT} \times \underbrace{F^p_{0,T}}_{=\mathbb{E}_0(S_T)e^{-\alpha T}} = \mathbb{E}_0(S_T)e^{-(\alpha-r)T}$$

Creating a synthetic forward contract

Assuming that the dividends are continuous and paid at the rate δ.

Recall that

Payoff of a long forward position at expiration

$$||$$
$$S_T - F_{0,T}$$
$$||$$
$$S_T - S_0 e^{(r-\delta)T}$$

Forward = Stock − Zero-coupon bond

| | Cash Flows | |
Transaction	Time 0	Time T (expiration)
Buy $e^{-\delta T}$ units of the index	$-S_0 e^{-\delta T}$	$+ S_T$
Borrow $S_0 e^{-\delta T}$	$+S_0 e^{-\delta T}$	$- S_0 e^{(r-\delta)T}$
Total	0	$S_T - S_0 e^{(r-\delta)T}$

Stock = Forward + Zero-coupon bond

Transaction	Cash Flows	
	Time 0	Time T (expiration)
Long one forward	0	$S_T - F_{0,T}$
Lend $S_0 e^{-\delta T}$	$-S_0 e^{-\delta T}$	$+S_0 e^{(r-\delta)T}$
Total	$-S_0 e^{-\delta T}$	S_T

Zero-coupon bond = Stock − Forward

Transaction	Cash Flows	
	Time 0	**Time T (expiration)**
Buy $e^{-\delta T}$ units of the index	$-S_0 e^{-\delta T}$	$+S_T$
Short one forward	0	$F_{0,T} - S_T$
Total	$-S_0 e^{-\delta T}$	$F_{0,T}$

Cash-and-carry is a transaction in which one buys the underlying asset and short the offsetting forward contract.

A cash-and-carry has no risk because
You have an obligation to deliver the asset
that you have already owned.

Transaction	Cash Flows	
	Time 0	**Time T (expiration)**
Buy tailed position in stock, paying $S_0 e^{-\delta T}$	$-S_0 e^{-\delta T}$	$+S_T$
Borrow $S_0 e^{-\delta T}$	$+S_0 e^{-\delta T}$	$-S_0 e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0 e^{(r-\delta)T}$

Cash-and-carry

| | Cash Flows | |
Transaction	Time 0	Time T (expiration)
Buy tailed position in stock, paying $S_0 e^{-\delta T}$	$-S_0 e^{-\delta T}$	$+S_T$
Borrow $S_0 e^{-\delta T}$	$+S_0 e^{-\delta T}$	$-S_0 e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0 e^{(r-\delta)T}$

Arbitrage when $F_{0,T} > S_0 e^{(r-\delta)T}$

Reverse cash-and-carry

| | Cash Flows | |
Transaction	Time 0	Time T (expiration)
Short tailed position in stock, receiving $S_0 e^{-\delta T}$	$+S_0 e^{-\delta T}$	$-S_T$
Lend $S_0 e^{-\delta T}$	$-S_0 e^{-\delta T}$	$+S_0 e^{(r-\delta)T}$
Long forward	0	$S_T - F_{0,T}$
Total	0	$S_0 e^{(r-\delta)T} - F_{0,T}$

Arbitrage when $F_{0,T} < S_0 e^{(r-\delta)T}$

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§ 5.1 Alternative ways to buy a stock

§ 5.2 Prepaid forward contracts on stock

§ 5.3 Forward contracts on stock

§ 5.4 Futures contracts

§ 5.5 Problems

Chapter 5. Financial Forwards and Futures

Definition 5.4-1 **Future contracts** are essentially exchange-traded forward contracts.

Typical features of futures contracts include:

- ▶ Standardized, with specified delivery dates, locations, procedures
- ▶ A clearinghouse

    Matches buy and sell orders

    Keeps track of members' obligations and payments

    After matching the trades, becomes counterparty

Differences from forward contracts

- Settled daily through the mark-to-market process
- Highly liquid: easier to offset an existing position
- Highly standardized structure
- Low credit risk
- There are typically daily price limits.

We will not go further in this section. Interested students can read the textbook.

Chapter 5. Financial Forwards and Futures

Problems: 5.2, 5.3, 5.4, 5.5, 5.8, 5.10.

Due Date: TBA