Financial Mathematics

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Le Chen

lzc0090@auburn.edu

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Auburn University
Auburn AL

[1]Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 9. Parity and other option relationships

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§ 9.1 Put-call parity

§ 9.2 Generalized parity and exchange options

§ 9.3 Comparing options with respect to style, maturity, and strike

§ 9.4 Problems

How does one value the right to back away from a commitment?

TABLE 9.1	IBM option prices, dollars per share, May 6, 2011. The closing price of IBM on that day was $168.89.				

		Calls		Puts	
Strike	Expiration	Bid ($)	Ask ($)	Bid ($)	Ask ($)
160	June	10.05	10.15	1.16	1.20
165	June	6.15	6.25	2.26	2.31
170	June	3.20	3.30	4.25	4.35
175	June	1.38	1.43	7.40	7.55
160	October	14.10	14.20	5.70	5.80
165	October	10.85	11.00	7.45	7.60
170	October	8.10	8.20	9.70	9.85
175	October	5.80	5.90	12.40	12.55

Source: Chicago Board Options Exchange.

- ▶ What determines the difference between put and call prices at a given strike?
- ▶ How would the premiums change if these options were European rather than American?
- ▶ It appears that, for a given strike, the October options are more expensive than the June options. Is this necessarily true?
- ▶ Do call premiums always decrease as the strike price increases? Do put premiums always increase as the strike price increases?
- ▶ Both call and put premiums change by less than the change in the strike price. Does this always happen?

European options

$$C(K, T) - P(K, T) = \text{PV}_{0,T}\left(F_{0,T} - K\right)$$
$$= e^{-rT}\left(F_{0,T} - K\right)$$

Buying a call and selling a put
with the strike both equal to the forward price (i.e., $K = F_{0,T}$)
creates a synthetic forward contract
and hence must have a zero price.

Parity generally fails for American options!

Parity for stocks

$$C(K, T) = P(K, T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT}K$$

Example 9.1-1   Suppose that the price of a non-dividend-paying stock is \$40, the continuously compounded interest rate is 8%, and options have 3 months to expiration. If a 40-strike European call sells for \$2.78, find the price for a 40-strike European put sells.

Solution.   Let the price for put be y. Then

$$\$2.78 = y + \$40 - \$40e^{-0.08 \times 0.25}$$

Hence,

$$y = \$1.99.$$

□

Why is a call more expensive than a put?

When $S_0 = K$ and $\text{Div} = 0$, then

$$C(K, T) - P(K, T) = K\left(1 - e^{-rT}\right)$$

The difference of a call and put is
the time value of money.

Example 9.1-2 Make the same assumptions as in Example 9.1-1, except suppose that the stock pays a $5 dividend just before expiration. If the price of the European call is $0.74, what would be the price of the European put?

Solution. Let the price for put be y. Then

$$\$0.74 = y + \left(\$40 - \$5e^{-0.08 \times 0.25}\right) - \$40e^{-0.08 \times 0.25}$$

Hence,

$$y = \$4.85.$$

□

Synthetic securities

$$C(K, T) = P(K, T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT}K$$

▶ Synthetic stock

$$S_0 = C(K, T) - P(K, T) + \mathrm{PV}_{0,T}(\mathrm{Div}) + e^{-rT}K$$

$$C(K, T) = P(K, T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT}K$$

▶ Synthetic Treasury bill (T-bill)

$$\underbrace{S_0 - C(K, T) + P(K, T)}_{\text{a conversion}} = \mathrm{PV}_{0,T}(\mathrm{Div}) + e^{-rT}K$$

Motivation:

A hedged position that has no risk but requires investment.

T-bills are taxed differently than stocks.

Synthetic securities

$$C(K, T) = P(K, T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT} K$$

▶ Synthetic options

$$C(K, T) = P(K, T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT} K$$

$$P(K, T) = C(K, T) - (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) + e^{-rT} K$$

§ 9.1 Put-call parity

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Generalize the parity to apply to the case where the strike asset is not necessarily cash but could be any other asset.

We will skip this section and leave it for motivated students.

§ 9.1 Put-call parity

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$$C_{\mathrm{Amer}}(S, K, T) \geq C_{\mathrm{Eur}}(S, K, T)$$

$$P_{\mathrm{Amer}}(S, K, T) \geq P_{\mathrm{Eur}}(S, K, T)$$

$$S \geq C_{\mathrm{Amer}}(S, K, T) \geq C_{\mathrm{Eur}}(S, K, T) \geq \max\left(0, \mathrm{PV}_{0,T}(F_{0,T}) - \mathrm{PV}_{0,T}(K)\right)$$

$$K \geq P_{\mathrm{Amer}}(S, K, T) \geq P_{\mathrm{Eur}}(S, K, T) \geq \max\left(0, \mathrm{PV}(K) - \mathrm{PV}_{0,T}(F_{0,T})\right)$$

Early exercise for American options

Calls on stocks with no dividend

No early exercise!

$$C_{\mathrm{Ame}}(S_t, K, T-t) \geq C_{\mathrm{Eur}}(S_t, K, T-t)$$

$$= \underbrace{S_t - K}_{\text{Exercise value}} + \underbrace{P_{Eur}(S_t, K, T-t)}_{\text{Insurance against } S_T < K} + \underbrace{K\left(1 - e^{-r(T-t)}\right)}_{\text{Time value of money on } K}$$

$$\geq S_t - K$$

Instead of $C(S_t, K, T-t) \geq S_t - K$ one can prove a stronger version:

$$C(S_t, K, T-t) \geq S_t - Ke^{-r(T-t)}$$

		Expiration or Exercise, Time T	
Transaction	**Time t**	$\mathbf{S_T < K}$	$\mathbf{S_T > K}$
Buy call	$-C$	0	$S_T - K$
Short stock	S_t	$-S_T$	$-S_T$
Lend $Ke^{-r(T-t)}$	$-Ke^{-r(T-t)}$	K	K
Total	$S_t - Ke^{-r(T-t)} - C$	$K - S_T$	0

Early exercise for American options

Calls on stock with dividends

Interest beats dividends? $K - \mathrm{PV}_{t,T}(K) > \mathrm{PV}_{t,T}(\mathrm{Div})$	Early exercise?
✓	✗
✗	possibly

When dividends do make early exercise rational, one should exercise at the last moment before the ex-dividend date.

In order to receive interest, one may exercise early
(think about the case when $S_T = 0$)

No early exercise	Early exercise
$PV_{t,T}(K)$	K

Early exercise for puts
(no dividend case)

No-exercise condition:

$$P\left(S_t, K, T - t\right) > K - S_t$$

$$\Updownarrow$$

$$C\left(S_t, K, T - t\right) > K - \mathrm{PV}_{t,T}(K)$$

$$P(S_t, K, T - t) = C(S_t, K, T - t) - S_t + \mathrm{PV}_{t,T}(K)$$

	calls	puts
Receive	stock	cash
Motivation for early exercise	sufficient dividends	sufficient interest

One can view interest as the dividend on cash.

Dividends are the sole reason to early-exercise an option.

The longer the more expensive

► American call/put options

► European call option on stock with no dividend

$$C_{Eur} = C_{Ame}$$

The longer, might be cheaper

► European call option on stock with dividend

► European put option

Time to expiration
$- K_t = ke^{rt}$

Theorem 9.3-1 When $K_t = e^{rt}K$, i.e., the strike grows at the interest rate, the premiums on European calls and puts on a non-dividend-paying stock increases with time to maturity.

Proof. We only prove the case for puts and leave the calls as exercise.
Let $T > t$. In order to show that

$$P_{\text{Euro}}(S_T, K_T, T) > P_{\text{Euro}}(S_t, K_t, t),$$

it suffices to find an arbitrage when

$$P_{\text{Euro}}(S_T, K_T, T) \leq P_{\text{Euro}}(S_t, K_t, t).$$

		Payoff at Time T			
		$S_T < K_T$		$S_T > K_T$	
		Payoff at Time t			
Transaction	Time 0	$S_t < K_t$	$S_t > K_t$	$S_t < K_t$	$S_t > K_t$
Sell $P(t)$	$P(t)$	$S_T - K_T$	0	$S_T - K_T$	0
Buy $P(T)$	$-P(T)$	$K_T - S_T$	$K_T - S_T$	0	0
Total	$P(t) - P(T)$	0	$K_T - S_T$	$S_T - K_T$	0

For example, at time t, if $S_t < K_t$, one has to buy the stock. The payoff at time t is

$$S_t - K_t = S_t - Ke^{rt}$$

One keeps this stock to time T, the stock price becomes S_T, and the future value of Ke^{rt} that one spent at time t becomes $Ke^{rt+r(T-t)} = Ke^{rT}$. Hence, the payoff of this strategy at time T is

$$S_T - Ke^{rT} = S_T - K_T.$$

$\square$

Different strike prices

$$K_1 \leq K_2 \leq K_3$$

Relation	Ideas in proof, arbitrage in
$C(K_1) \geq C(K_2)$	a call bull spread
$P(K_1) \leq P(K_2)$	a put bear spread
$C(K_1) - C(K_2) \leq K_2 - K_1$	a call bear spread
$P(K_2) - P(K_1) \leq K_2 - K_1$	a put bull spread
$\dfrac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \dfrac{C(K_2) - C(K_3)}{K_3 - K_2}$	an asymmetric butterfly spread
$\dfrac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \dfrac{P(K_3) - P(K_2)}{K_3 - K_2}$	an asymmetric butterfly spread

Convexity revisited

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

$$\Updownarrow$$

$$C(K_2) \geq \lambda C(K_1) + (1 - \lambda) C(K_3).$$

with

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1}$$

Example 9.3-1 Suppose that

$$
\begin{array}{lcc}
\text{Strike} & 50 & 55 \\
\text{Call Premium} & 18 & 12
\end{array}
$$

1. What no-arbitrage property is violated?

2. What spread position would you use to effect arbitrage?

3. Demonstrate that the spread position is an arbitrage.

Solution. Check Example 9.4 on p. 283. □

Example 9.3-2 Suppose that

Strike	50	59	65
Call premium	14	8.9	5

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution.   Check Example 9.5 on p. 284.   □

Example 9.3-3 Suppose that

$$
\begin{array}{lccc}
\text{Strike} & 50 & 55 & 70 \\
\text{Put premium} & 4 & 8 & 16
\end{array}
$$

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution. Check Example 9.6 on p. 284.   □

Problems: 9.1, 9.2, 9.3, 9.4, 9.8, 9.9, 9.10, 9.11, 9.15.

Due Date: TBA