Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 10. Binomial Option Pricing: Basic Concepts

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$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

- r: continuously compounded annual interest rate.
- δ: continuously dividend yield.
- σ: annual volatility.
- h: the length of a binomial period in years.

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Continuously Compounded Returns

$$r_{t,t+h} = \ln\left(S_{t_h}/S_t\right)$$

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$$

Go over 3 examples on p. 301

Volatility

The volatility of an asset is the standard deviation of continuously compounded returns.

- A year is dividend into n periods (say, $n = 12$) of length $h = 1/n$.
- Let σ^2 be the annual continuously compounded return.
- Assuming that the continuously compounded returns are independent and identically distributed
- We have

$$\sigma^2 = 12 \times \sigma^2_{\mathrm{monthly}}$$

and

$$\sigma_h = \sigma\sqrt{h} \quad \text{or} \quad \sigma = \frac{\sigma_h}{\sqrt{h}}.$$

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Constructing u and d

With no volatility

$$S_{t+h} = F_{t,t+h} = S_t e^{(r-\delta)h}$$

With volatility

$$uS_t = F_{t,t+h} e^{+\sigma\sqrt{h}}$$

$$dS_t = F_{t,t+h} e^{-\sigma\sqrt{h}}$$

$$\Downarrow$$

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$

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Estimating Historical Volatility

TABLE 10.1 Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 7/7/2010 to 9/8/2010.

	S&P 500		IBM	
Date	Price	$\ln(S_t/S_{t-1})$	Price	$\ln(S_t/S_{t-1})$
7/7/2010	1060.27		127	
7/14/2010	1095.17	0.03239	130.72	0.02887
7/21/2010	1069.59	−0.02363	125.27	−0.04259
7/28/2010	1106.13	0.03359	128.43	0.02491
8/4/2010	1127.24	0.01890	131.27	0.02187
8/11/2010	1089.47	−0.03408	129.83	−0.01103
8/18/2010	1094.16	0.00430	129.39	−0.00338
8/25/2010	1055.33	−0.03613	125.27	−0.03238
9/1/2010	1080.29	0.02338	125.77	0.00398
9/8/2010	1098.87	0.01705	126.08	0.00246
Standard deviation	0.02800		0.02486	
Standard deviation $\times \sqrt{52}$	0.20194		0.17926	

- Volatility computation should exclude dividend.
- But since dividends are small and infrequent; the standard deviation will be similar whether you exclude dividends or not when computing the standard deviation.

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One-period Example with a Forward Tree

Example 10.2-1 Consider a European call option on a stock, with a \$40 strike and 1 year to expiration. The stock does not pay dividends, and its current price is \$41. Suppose the volatility of the stock is 30%. The continuously compounded risk-free interest rate is 8%.

Use these inputs to calculate the followings:

1. the final stock prices uS and dS
2. the final option values C_u and C_d
3. Δ and B
4. the option price: $\Delta S + B$.

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Solution. In summary:

$$S = 41, K = 40, r = 0.08, \delta = 0, \sigma = 0.30, h = 1.$$

$uS = \$59.954$
$C_u = \$19.954$

$S = \$41.000$
Option price $= \$7.839$
$\Delta = 0.738$
$B = -\$22.405$

$dS = \$32.903$
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