

# Financial Mathematics

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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

# Chapter 11. Binomial Option Pricing: Selected Topics

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§ 11.1 Understanding Early Exercise

§ 11.2 Understanding risk-neutral pricing

§ 11.3 The Binomial tree and lognormality

§ 11.4 Problems

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## Risk-Neutral Probability

Recall the binomial option pricing formula:

$$C = \Delta S + B = e^{-rh} \left[ p^* C_u + (1 - p^*) C_d \right]$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \sim \text{risk-neutral probability that the stock will go up}$$

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$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \iff p^* u S e^{\delta h} + (1 - p^*) d S e^{\delta h} = e^{rh} S$$

Two offers:

(a) \$1000 cash

(b) \$2000 or \$0 cash with probability  $1/2$  for each

Both offers have the same expected return,  
while (b) bears risk and (a) does not.

A risk-averse investor prefers (a).

A risk-neutral investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing. Hence, he/she prefers equally to (a) and (b).

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The option pricing formula can be said to price options  
as if investors are risk-neutral

Note that we are not assuming that investors are actually risk-neutral, and that risky assets are actually expected to earn the risk-free rate of return.

## Pricing an option using real probability

- ▶ Suppose that the continuously compounded expected return on the stock is  $\alpha$  and that the stock does not pay dividends.
- ▶ If  $p$  is the true probability of the stock going up,  $p$  must be consistent with  $u$ ,  $d$  and  $\alpha$

$$puS + (1 - p)dS = e^{\alpha h} S$$

- ▶ Solving for  $p$  gives us

$$p = \frac{e^{\alpha h} - d}{u - d}$$

- ▶ For  $p$  to be a probability, we have to have  $u \geq e^{\alpha h} \geq d$ .
- ▶ Using this  $p$ , the actual expected payoff to the option one period is

$$pC_u + (1 - p)C_d = \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d.$$

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At what rate do we discount this expected payoff?

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It is not correct to discount the option at the expected return on the stock,  $\alpha$ , because the option is equivalent to a leveraged investment in the stock and hence is riskier than the stock

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- ▶ Define the appropriate period discount rate for the option as  $e^{\alpha h}$ .
- ▶ Suppose option is replicated by holding a portfolio consisting of  $\Delta$  shares of stock and  $B$  bonds. The expected return on this portfolio is

$$R^P = \frac{\Delta S_u + B^P}{\Delta S + B^P} = \frac{B}{\Delta S + B^P}$$

- ▶ Hence, the discounted at this appropriate discount rate, the price for the option should be

$$C = e^{-\alpha h} \left[ \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d \right]$$

- ▶ By setting  $w = C$ , one obtains the simplest pricing procedure.
- ▶ This gives an alternative way to compute the option price instead of  $\Delta S + B$ .

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- ▶ Denote the appropriate per-period discount rate for the option as  $\gamma$
- ▶ Since an option is equivalent to holding a portfolio consisting of  $\Delta$  shares of stock and  $B$  bonds, the expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B} e^{\alpha h} + \frac{B}{S\Delta + B} e^{rh}$$

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One can use either

$$C = \Delta S + B$$

or

$$C = e^{-\gamma h} \left[ \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d \right]$$

to compute the option price

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- ▶ First equation is more efficient
- ▶ For the second one, in order to compute  $C$ , one needs to compute  $\Delta$  and  $B$  first and then obtains  $C$  via

$$e^{\alpha h} C = \frac{\Delta S \Delta}{\Delta S + B} e^{\alpha h} + \frac{B}{\Delta S + B} e^{\alpha h}$$

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Given the continuously compounded expected return of the stock  $\alpha$

1. Compute the probability that stock goes up

$$p = \frac{e^{\alpha h} - d}{u - d}$$

2. Compute the actual expected payoff (to be discounted)

$$X := pC_u + (1 - p)C_d$$

3. Using  $r$  and  $\delta$  to compute  $\Delta$  and  $B$ :

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u - d)} \quad \text{and} \quad B = e^{-rh} \frac{uC_d - dC_u}{u - d}.$$

4. Compute the discounted rate  $\gamma$ :

$$\gamma = \frac{1}{h} \log \left( \frac{S\Delta}{S\Delta + B} e^{\alpha h} + \frac{B}{S\Delta + B} e^{rh} \right)$$

5. Finally, the option price should be the discounted value:

$$Xe^{-\gamma h}.$$

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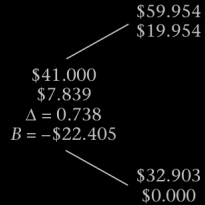
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# An one-period example

FIGURE 11.3

Binomial tree for pricing a European call option; assumes  $S = \$41.00$ ,  $K = \$40.00$ ,  $\sigma = 0.30$ ,  $r = 0.08$ ,  $T = 1.00$  years,  $\delta = 0.00$ , and  $h = 1.000$ . This is the same as Figure 10.3.



# A multi-period example

FIGURE 11.4

Binomial tree for pricing an American call option; assumes  $S = \$41.00$ ,  $K = \$40.00$ ,  $\sigma = 0.30$ ,  $r = 0.08$ ,  $T = 1.00$  years,  $\delta = 0.00$ , and  $h = 0.333$ . The continuously compounded true expected return on the stock,  $\alpha$ , is 15%. At each node the stock price, option price, and continuously compounded true discount rate for the option,  $\gamma$ , are given. Option price in ***bold italic*** signify that exercise is optimal at that node.

