Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 11. Binomial Option Pricing: Selected Topics

The usefulness of the binomial pricing model hinges on

the binomial tree providing

a reasonable representation of

the stock price distribution

The binomial tree approximates a lognormal distribution

Random Walk

- Let Y_i be a sequence of i.i.d. random variables, each following

$$Y_i = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

- Random walk Z_n is defined to be

$$Z_n = \sum_{i=1}^{n} Y_i.$$

- From the walk Z_n, one can also retrieve the values of Y_n

$$Y_n = Z_n - Z_{n-1}$$

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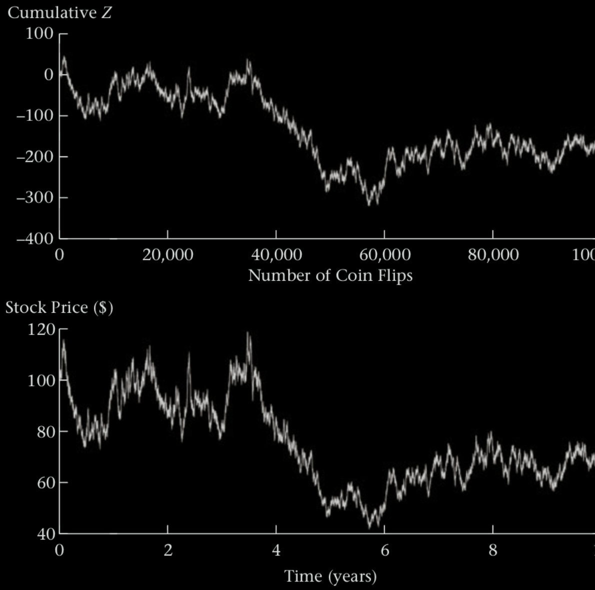
FIGURE 11.5

In the top panel is an illustration of a random walk, where the counter, Z, increases by 1 when a fair coin flip comes up heads, and decreases by 1 with tails. In the bottom panel is a particular path through a 10,000-step binomial tree, where the up and down moves are the same as in the top panel. Assumes $S_0 = \$100$, $r = 6\%$, $\sigma = 30\%$, $T = 10$ years, and $h = 0.0001$.



Modelling Stock prices as a random walk

The idea that asset prices should follow a random walk was articulated in Samuelson (1965)

In efficient markets, an asset price should reflect all available information. In response to new information the price is equally likely to move up or down, as with the coin flip.

The price after a period of time is the initial price plus the cumulative up and down movements due to informational surprises

Modelling Stock prices as a random walk – Issues and Binomial Model

If by chance we get enough cumulative down movements, the stock price will become negative

The stock, on average, should have a positive return. The random walk model taken literally does not permit this

The magnitude of the move ($1) should depend upon how quickly the coin flips occur and the level of the stock price

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The binomial model is a variant of the random walk model that solves all of these problems at once:

$$S_{t+h} = S_t e^{(r-\delta)h \pm \sigma h},$$

which says, instead of the prices jumping like a random walk, the compound rate follows a random walk.

Lognormality of the binomial model

The binomial tree approximates a lognormal distribution, which is commonly used to model stock prices

The lognormal distribution is the probability distribution that arises from the assumption that continuously compounded returns on the stock are normally distributed

With the lognormal distribution, the stock price is positive, and the distribution is skewed to the right, that is, there is a chance that extremely high stock prices will occur

FIGURE 11.6

Construction of a binomial tree depicting stock price paths, along with risk-neutral probabilities of reaching the various terminal prices.

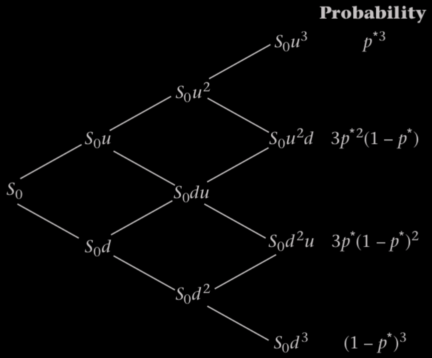


FIGURE 11.7

Comparison of lognormal distribution with three-period binomial approximation.

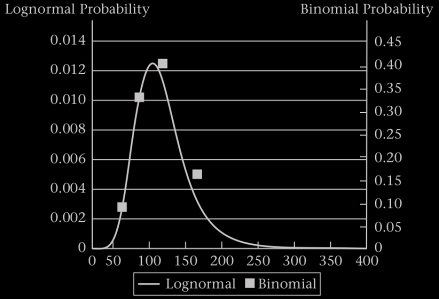
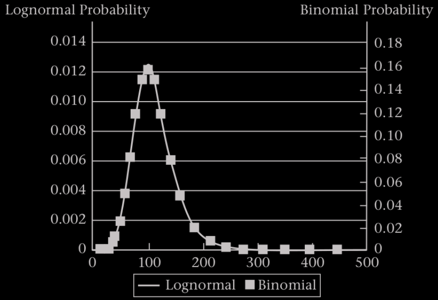


FIGURE 11.8

Comparison of lognormal distribution with 25-period binomial approximation.



Alternative Binomial trees

Binomial Tree	Cox-Ross-Rubinstein binomial tree	Lognormal tree
$u = e^{(r-\delta)h+\sigma\sqrt{h}}$	$u = e^{+\sigma\sqrt{h}}$	$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}}$
$d = e^{(r-\delta)h-\sigma\sqrt{h}}$	$d = e^{-\sigma\sqrt{h}}$	$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}}$

- Even though the values of u and d are deferent, the ratio, which measures volatility, remains the same:

$$u/d = e^{2\sigma\sqrt{h}}.$$

- Once u and d are determined, the rest computations for option price remain the same.
- All three methods of constructing a binomial tree yield different option prices for finite n, but they approach the same price as $n \to \infty$.

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Is the Binomial model realistic?

The binomial model is a form of the random walk model, adapted to modeling stock prices. The lognormal random walk model in this section assumes among other things, that

Volatility is constant

“Large” stock price movements do not occur

Returns are independent over time

All of these assumptions appear to be violated in the data