Financial Mathematics

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 12. The Black-Scholes Formula

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The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.

TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

Check Python code Figure12-1.py

- Consider an European call (or put) option written on a stock
- Assume that the stock pays dividend at the continuous rate δ

Call options	Put options
$C(S, K, \sigma, r, T, \delta)$	$P(S, K, \sigma, r, T, \delta)$
$\vert\vert$	$\vert\vert$
$Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$	$Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Put-call Parity

$$P = C + Ke^{-rT} - Se^{-\delta T}$$

$$d_1 - d_2 = \sigma\sqrt{T}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx$$

Example 12.1-1 Verify that the Black-Scholes formula for call and put

$$C := C(S, K, \sigma, r, T, \delta) = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$
$$P := P(S, K, \sigma, r, T, \delta) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

with

$$d_i = \frac{\ln(S/K) + (r - \delta - (-1)^i \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad i = 1, 2$$

satisfies the call-put parity: $C - P = Se^{-\delta T} - Ke^{-rT}$.

Solution.

□

Example 12.1-2 Plot the functions

$$S \to C(S, K, \sigma, r, T-t, \delta) = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$S \to P(S, K, \sigma, r, T-t, \delta) = Ke^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

with σ, r, δ, K fixed for various values of $T - t = 2, 1.5, 1, 0.5, 0$.

Solution. Try code

CallPut_vs_T-t.nb

□

Example 12.1-3 Let $S = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 8\%$, $T = 0.25$ (3 months), and $\delta = 0$. Compute the Black-Scholes call and put prices. Compare what you obtained with the results obtained from the binomial tree.

Check code
Example12-1.py

When is the Black-Scholes formula valid?

Assumptions about stock return distribution

- Continuously compounded returns on the stock are normally distributed and independent over time (no “jumps”)
- The volatility of continuously compounded returns is known and constant
- Future dividends are known, either as dollar amount or as a fixed dividend yield

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Assumptions about the economic environment

- The risk-free rate is known and constant
- There are no transaction costs or taxes
- It is possible to short-sell costlessly and to borrow at the risk-free rate