

# Financial Mathematics

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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

## Chapter 12. The Black-Scholes Formula

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§ 12.1 Introduction to the Black-Scholes formula

§ 12.2 Applying the formula to other assets

§ 12.3 Option Greeks

§ 12.4 Problems

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§ 12.4 Problems

The **Black-Scholes formula** is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.

TABLE 12.1

Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price  $S = \$41$ , the strike price  $K = \$40$ , volatility  $\sigma = 0.30$ , risk-free rate  $r = 0.08$ , time to expiration  $T = 1$ , and dividend yield  $\delta = 0$ .

Number of Steps (n)	Binomial Call Price (\$)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
$\infty$	6.961

Check Python code Figure12-1.py

- ▶ Consider an European call (or put) option written on a stock
- ▶ Assume that the stock pays dividend at the continuous rate  $\delta$

Call options	Put options
$C(S, K, \sigma, r, T, \delta)$	$P(S, K, \sigma, r, T, \delta)$
$Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$	$Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Put-call Parity

$$P = C + Ke^{-rT} - Se^{-\delta T}$$

$$d_1 - d_2 = \sigma\sqrt{T}$$

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$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{x^2}{2}} dx$$

Example 12.1-1 Verify that the Black-Scholes formula for call and put

$$C := C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

$$P := P(S, K, \sigma, r, T, \delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

with

$$d_i = \frac{\ln(S/K) + (r - \delta - (-1)^i \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad i = 1, 2$$

satisfies the call-put parity:  $C - P = Se^{-\delta T} - Ke^{-rT}$ .

Solution.

□

Example 12.1-2 Plot the functions

$$S \rightarrow C(S, K, \sigma, r, T - t, \delta) = Se^{-\delta(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2)$$

$$S \rightarrow P(S, K, \sigma, r, T - t, \delta) = Ke^{-r(T-t)} N(-d_2) - Se^{-\delta(T-t)} N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

with  $\sigma, r, \delta, K$  fixed for various values of  $T - t = 2, 1.5, 1, 0.5, 0$ .

Solution. Try code

CallPut\_vs\_T-t.nb



**Example 12.1-3** Let  $S = \$41$ ,  $K = \$40$ ,  $\sigma = 0.3$ ,  $r = 8\%$ ,  $T = 0.25$  (3 months), and  $\delta = 0$ . Compute the Black-Scholes call and put prices. Compare what you obtained with the results obtained from the binomial tree.

Check code  
Example12-1.py

# When is the Black-Scholes formula valid?

## Assumptions about stock return distribution

- ▶ Continuously compounded returns on the stock are normally distributed and independent over time (no “jumps”)
- ▶ The volatility of continuously compounded returns is known and constant
- ▶ Future dividends are known, either as dollar amount or as a fixed dividend yield

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## Assumptions about the economic environment

- ▶ The risk-free rate is known and constant
- ▶ There are no transaction costs or taxes
- ▶ It is possible to short-sell endlessly and to borrow at the risk-free rate

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