

# Financial Mathematics

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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

## Chapter 12. The Black-Scholes Formula

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§ 12.1 Introduction to the Black-Scholes formula

§ 12.2 Applying the formula to other assets

§ 12.3 Option Greeks

§ 12.4 Problems

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## What happens to the option price when one and only one input changes?

- ▶ Delta ( $\Delta$ ): change in option price when stock price increases by \$1
- ▶ Gamma ( $\Gamma$ ): change in delta when option price increases by \$1
- ▶ Vega: change in option price when volatility increases by 1%
- ▶ Theta ( $\theta$ ): change in option price when time to maturity decreases by 1 day
- ▶ Rho ( $\rho$ ): change in option price when interest rate increases by 1%
- ▶ Psi ( $\psi$ ): change in the option premium due to a change in the dividend yield

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- ▶ The Greek measure of a portfolio is weighted average of Greeks of individual portfolio components

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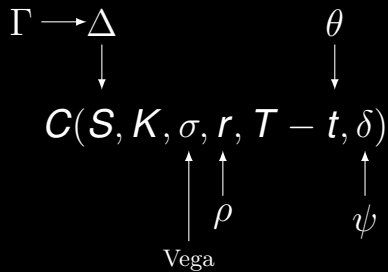
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## Delta

Delta ( $\Delta$ ): change in option price when stock price increases by \$1.

$$\Delta = \begin{cases} \frac{\partial C(S, K, \sigma, T - t, \delta)}{\partial S} = +e^{-\delta(T-t)} N(+d_1) & \text{Call} \\ \frac{\partial P(S, K, \sigma, T - t, \delta)}{\partial S} = -e^{-\delta(T-t)} N(-d_1) & \text{Put} \end{cases}$$

Example 12.3-1 Demonstrate that

$$\Delta = \begin{cases} \frac{\partial C(S, K, \sigma, T - t, \delta)}{\partial S} = +e^{-\delta(T-t)} N(+d_1) & \text{Call} \\ \frac{\partial P(S, K, \sigma, T - t, \delta)}{\partial S} = -e^{-\delta(T-t)} N(-d_1) & \text{Put.} \end{cases}$$

Solution. We only show the call part. By the chain rule:

$$\begin{aligned} \frac{\partial C}{\partial S} &= e^{-\delta(T-t)} N(d_1) \\ &+ Se^{-\delta(T-t)} N'(d_1) \frac{\partial d_1}{\partial S} - Ke^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S}. \end{aligned}$$

Because  $d_2 = d_1 - \sigma\sqrt{T-t}$ , we see that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}.$$

It suffices to prove that

$$Se^{\delta(T-t)} N'(d_1) = Ke^{-r(T-t)} N'(d_2).$$

**Solution.** (Continued) Notice that

$$N'(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}.$$

The above relation is equivalent to

$$\frac{Se^{(r-\delta)(T-t)}}{K} = \exp\left(\frac{d_1^2 - d_2^2}{2}\right). \quad (\star)$$

Now, from the definitions of  $d_1$  and  $d_2$ , we see that

$$\begin{aligned} d_1^2 - d_2^2 &= d_1^2 - \left(d_1 - \sigma\sqrt{T-t}\right)^2 \\ &= 2d_1\sigma\sqrt{T-t} - \sigma^2(T-t) \\ &= 2(\ln(S/K) + (r-\delta)(T-t)) \\ &= 2\ln\left(\frac{Se^{(r-\delta)(T-t)}}{K}\right). \end{aligned}$$

Plugging the above expression back to  $(\star)$  proves the case. □

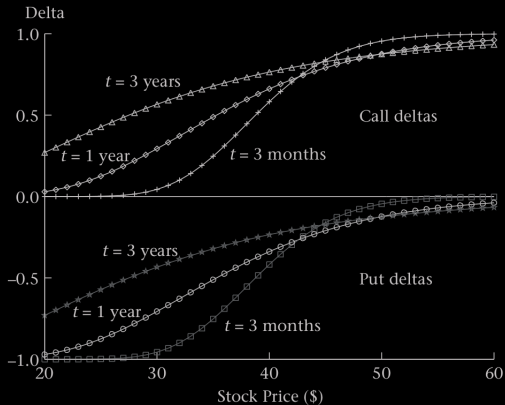
In the above proof, we have showed the following relation, which will be useful in the computations of other Greeks:

$$Se^{-\delta(T-t)}N'(d_1) = Ke^{-r(T-t)}N'(d_2)$$



**FIGURE 12.1**

Call (top graph) and put (bottom graph) deltas for 40-strike options with different times to expiration. Assumes  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .



## Gamma and Vega

**Gamma ( $\Gamma$ ):** change in delta when option price increases by \$1

$$\Gamma = \frac{\partial^2 C(S, K, \sigma, r, T - t, \delta)}{\partial S^2} = \frac{\partial^2 P(S, K, \sigma, r, T - t, \delta)}{\partial S^2} = \frac{e^{-\delta(T-t)} N'(d_1)}{S\sigma\sqrt{T-t}}$$

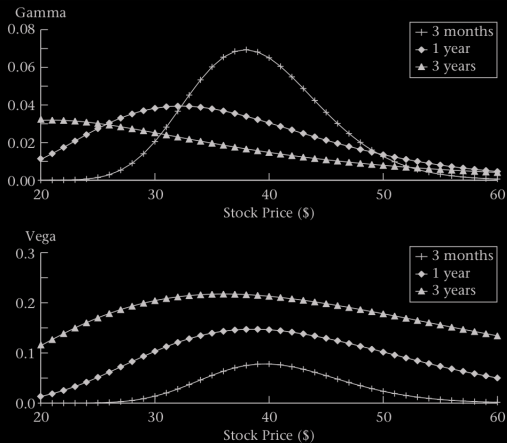
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**Vega:** change in option price when volatility increases by 1%

$$\text{Vega} = \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial \sigma} = \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial \sigma} = Se^{-\delta(T-t)} N'(d_1) \sqrt{T-t}$$

**FIGURE 12.2**

Gamma (top panel) and vega (bottom panel) for 40-strike options with different times to expiration. Assumes  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ . Vega is the sensitivity of the option price to a 1 percentage point change in volatility. Otherwise identical calls and puts have the same gamma and vega.



# Theta

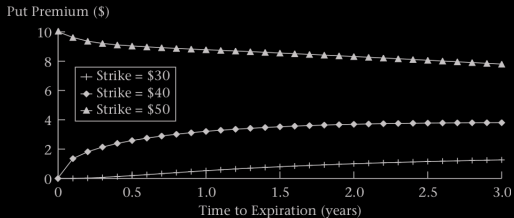
Theta ( $\theta$ ): change in option price when time to maturity decreases by 1 day

$$\begin{aligned}\text{Call } \theta &= \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial t} \\ &= \delta S e^{-\delta(T-t)} N(d_1) - r K e^{-r(T-t)} N(d_2) - \frac{K e^{r(T-t)} N'(d_2) \sigma}{2\sqrt{T-t}}\end{aligned}$$

$$\begin{aligned}\text{Put } \theta &= \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial t} \\ &= \text{Call } \theta + r K e^{-r(T-t)} + \delta S e^{-\delta(T-t)}\end{aligned}$$

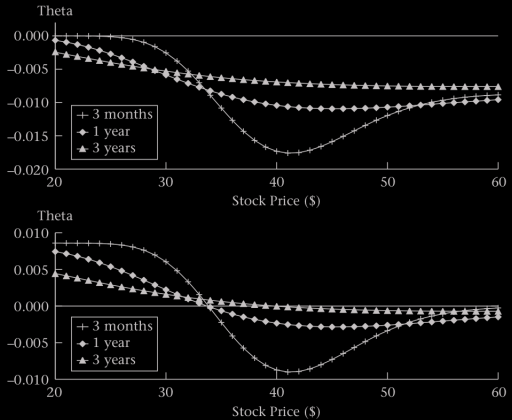
**FIGURE 12.3**

Call (top panel) and put (bottom panel) prices for options with different strikes at different times to expiration. Assumes  $S = \$40$ ,  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .



**FIGURE 12.4**

Theta for calls (top panel) and puts (bottom panel) with different expirations at different stock prices. Assumes  $K = \$40$ ,  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .



## Rho and Psi

**Rho ( $\rho$ ):** change in option price when interest rate increases by 1%

$$\text{Call } \rho = \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial r} = +(T - t)Ke^{-r(T-t)}N(+d_2)$$

$$\text{Put } \rho = \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial r} = -(T - t)Ke^{-r(T-t)}N(-d_2)$$

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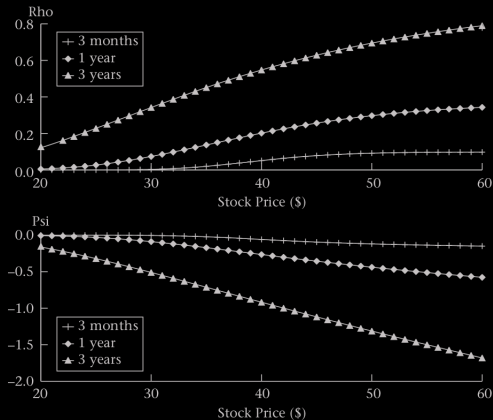
**Psi ( $\psi$ ):** change in the option premium due to a change in the dividend yield

$$\text{Call } \psi = \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial \delta} = -(T - t)Ke^{-\delta(T-t)}N(+d_1)$$

$$\text{Put } \psi = \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial \delta} = +(T - t)Ke^{-\delta(T-t)}N(-d_1)$$

**FIGURE 12.5**

Rho (top panel) and psi (bottom panel) at different stock prices for call options with different maturities. Assumes  $K = \$40$ ,  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .





## Do these Greeks satisfy some relation?

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Theorem 12.3-1 Let  $V(t, S)$  denote the option price for either European call or put. Recall that

$$V_t = \theta, \quad V_S = \Delta, \quad \text{and} \quad V_{SS} = \Gamma.$$

Then, these three Greeks have to satisfy the Black-Scholes equation:

$$\boxed{V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \delta)SV_S - rV = 0} \quad 0 \leq t \leq T, \quad (\text{BS})$$

with the boundary conditions:

Condition	call	put
$V(T, S)$	$\max(S - K, 0)$	$\max(K - S, 0)$
$V(t, S)$	0	$Ke^{-r(T-t)}$
$\lim_{S \rightarrow \infty} V(t, S)$	$S$	0

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**Proof.** We will only verify (BS). This can be easily done by the symbolic computations via Mathematica. Check

Greeks-BS-Equation.nb



## Questions:

- (1) How to derive this Black-Scholes equation?
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TABLE 12.2

Greeks for a bull spread where  $S = \$40$ ,  $\sigma = 0.3$ ,  $r = 0.08$ , and  $T = 91$  days, with a purchased 40-strike call and a written 45-strike call. The column titled “combined” is the difference between column 1 and column 2.

	40-Strike Call	45-Strike Call	Combined
$\omega_i$	1	-1	—
Price	2.7804	0.9710	1.8094
Delta	0.5824	0.2815	0.3009
Gamma	0.0652	0.0563	0.0088
Vega	0.0780	0.0674	0.0106
Theta	-0.0173	-0.0134	-0.0040
Rho	0.0511	0.0257	0.0255

Delta ( $\Delta$ ): change in option price when stock price increases by \$1

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Option Elasticity ( $\Omega$ ): If stock price  $S$  changes by 1%, what is the percentage change in the value of the option  $C$ :

$$\Omega = \frac{\text{Percentage change in option price}}{\text{Percentage change in stock price}} = \frac{\frac{\epsilon \Delta}{C}}{\frac{\epsilon}{S}} = \frac{S \Delta}{C}.$$