

Financial Mathematics

MATH 5870/6870¹
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¹Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 13. Market-Making and Delta-Hedging

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§ 13.1 What do market-makers do?

§ 13.2 Market-maker risk

§ 13.3 Delta-Hedging

§ 13.4 The mathematics of Delta-hedging

§ 13.5 The Black-Scholes analysis

§ 13.6 Market-Making as insurance

§ 13.7 Problems

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§ 13.1 What do market-makers do?

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§ 13.7 Problems

First order (in \mathcal{S}) approximation
(with zero order in h):

$$C(\mathcal{S}_{t+h}, T - (t + h)) \approx C(\mathcal{S}_t, T - t) + \Delta(\mathcal{S}_t, T - t) \times (\mathcal{S}_{t+h} - \mathcal{S}_t)$$

Second order in \mathcal{S} approximation
(with zero order in h):

$$C(\mathcal{S}_{t+h}, T - (t + h)) \approx C(\mathcal{S}_t, T - t) + \Delta(\mathcal{S}_t, T - t) \times (\mathcal{S}_{t+h} - \mathcal{S}_t) \\ + \frac{1}{2} \times \Gamma(\mathcal{S}_t, T - t) \times (\mathcal{S}_{t+h} - \mathcal{S}_t)^2$$

Delta-Gamma approximation

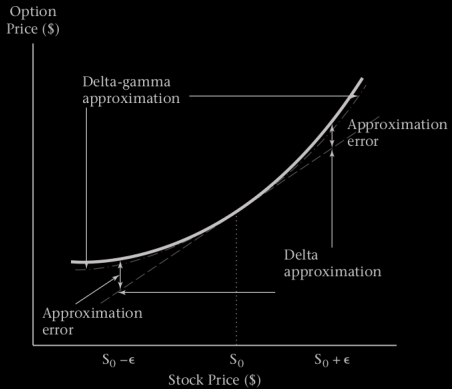
Explanations can be made
either using Taylor expansion or Δ_{average} .

Second order in \mathcal{S} approximation
(with first order in h):

$$\begin{aligned} C(\mathcal{S}_{t+h}, T - (t + h)) &\approx C(\mathcal{S}_t, T - t) \\ &+ \Delta(\mathcal{S}_t, T - t) \times (\mathcal{S}_{t+h} - \mathcal{S}_t) \\ &+ \frac{1}{2} \times \Gamma(\mathcal{S}_t, T - t) \times (\mathcal{S}_{t+h} - \mathcal{S}_t)^2 \\ &+ h \times \theta(\mathcal{S}_t, T - t) \end{aligned}$$

FIGURE 13.3

Delta- and delta-gamma approximations of option price. The true option price is represented by the bold line, and approximations by dashed lines.



Example 13.4-1 Given the first column of the following table, filling the details of the rest entries:

TABLE 13.4

Predicted option price over a period of 1 day, assuming stock price move of \$0.75, using equation (13.6). Assumes that $\sigma = 0.3$, $r = 0.08$, $T - t = 91$ days, and $\delta = 0$, and the initial stock price is \$40.

	Starting Price	$\epsilon \Delta$	$\frac{1}{2} \epsilon^2 \Gamma$	θh	Option Price 1 Day Later ($h = 1$ day)	
					Predicted	Actual
$S_{t+h} = \$40.75$	\$2.7804	0.4368	0.0183	-0.0173	\$3.2182	\$3.2176
$S_{t+h} = \$39.25$	\$2.7804	-0.4368	0.0183	-0.0173	\$2.3446	\$2.3452

Solution. Working with Mathematica code...



The value of the market-maker's investment:

$$\Delta_t S_t - C(S_t)$$

Market-maker's profit when the stock price changes
by ϵ over a time interval h

$$\underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} - \underbrace{rh[\Delta_t S_t - C(S_t)]}_{\text{interest charge}}$$

Now replace $\underbrace{C(S_{t+h}) - C(S_t)}_{\text{Changes in value of option}}$ by its second order approximation:

$$\begin{aligned} C(S_{t+h}) - C(S_t) &\approx \Delta_t \times (S_{t+h} - S_t) \\ &\quad + \frac{1}{2} \times \Gamma_t \times (S_{t+h} - S_t)^2 \\ &\quad + h \times \theta_t \end{aligned}$$

and $S_{t+h} - S_t$ by ϵ , we see that

Market-maker's profit

$$\begin{aligned} & \underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} - \underbrace{rh[\Delta_t S_t - C(S_t)]}_{\text{interest charge}} \\ & \parallel \\ & - \left(\frac{1}{2} \epsilon^2 \Gamma_t + \theta_t h + rh[\Delta_t S_t - C(S_t)] \right) \end{aligned}$$

We have seen that the market-maker approximately breaks even for a one-standard-deviation move in the stock:

$$\epsilon = \sigma S_t \sqrt{h} \quad \iff \quad \epsilon^2 = \sigma^2 S_t^2 h$$

Finally, we see that

$$\begin{aligned}
 & \text{Market-maker's profit} \\
 & \quad \parallel \\
 & \underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} - \underbrace{rh[\Delta_t S_t - C(S_t)]}_{\text{interest charge}} \\
 & \quad \parallel \\
 & - \left(\frac{1}{2} \sigma^2 S_t^2 \Gamma_t + \theta_t + r [\Delta_t S_t - C(S_t)] \right) h
 \end{aligned}$$