Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 13. Market-Making and Delta-Hedging

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First order (in S) approximation
(with zero order in h):

$$C\left(S_{t+h}, T-(t+h)\right) \approx C(S_t, T-t) + \Delta(S_t, T-t) \times \left(S_{t+h} - S_t\right)$$

Second order in S approximation
(with zero order in h):

$$C\left(S_{t+h}, T-(t+h)\right) \approx C(S_t, T-t) + \Delta(S_t, T-t) \times \left(S_{t+h} - S_t\right) + \frac{1}{2} \times \Gamma(S_t, T-t) \times \left(S_{t+h} - S_t\right)^2$$

Delta-Gamma approximation

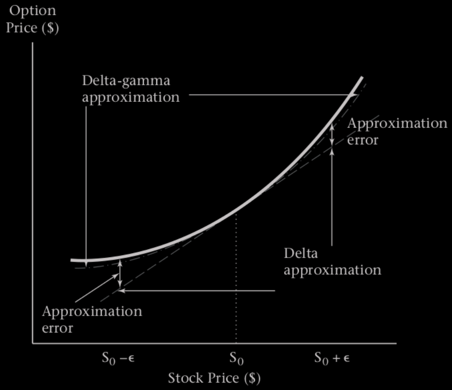
Explanations can be made
either using Taylor expansion or Δ_{average}.

Second order in S approximation
(with first order in h):

$$
\begin{aligned}
C\left(S_{t+h}, T-(t+h)\right) \approx \quad & C(S_t, T-t) \\
& + \Delta(S_t, T-t) \times (S_{t+h} - S_t) \\
& + \frac{1}{2} \times \Gamma(S_t, T-t) \times (S_{t+h} - S_t)^2 \\
& + h \times \theta(S_t, T-t)
\end{aligned}
$$

FIGURE 13.3

Delta- and delta-gamma approximations of option price. The true option price is represented by the bold line, and approximations by dashed lines.



Example 13.4-1 Given the first column of the following table, filling the details of the rest entries:

TABLE 13.4 Predicted option price over a period of 1 day, assuming stock price move of \$0.75, using equation (13.6). Assumes that $\sigma = 0.3$, $r = 0.08$, $T - t = 91$ days, and $\delta = 0$, and the initial stock price is \$40.

					Option Price 1 Day Later ($h = 1$ day)	
	Starting Price	$\epsilon\Delta$	$\frac{1}{2}\epsilon^2\Gamma$	θh	Predicted	Actual
$S_{t+h} = \$40.75$	\$2.7804	0.4368	0.0183	−0.0173	\$3.2182	\$3.2176
$S_{t+h} = \$39.25$	\$2.7804	−0.4368	0.0183	−0.0173	\$2.3446	\$2.3452

Solution. Working with Mathematica code... □

The value of the market-maker's investment:

$$\Delta_t S_t - C(S_t)$$

Market-marker's profit when the stock price changes by ϵ over a time interval h

$$\underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} - \underbrace{rh\,[\Delta_t S_t - C(S_t)]}_{\text{interest charge}}$$

Now replace $\underbrace{C(S_{t+h}) - C(S_t)}_{\text{Changes in value of option}}$ by its second order approximation:

$$\begin{aligned} C\left(S_{t+h}\right) - C(S_t) \approx \quad & \Delta_t \times (S_{t+h} - S_t) \\ & + \frac{1}{2} \times \Gamma_t \times (S_{t+h} - S_t)^2 \\ & + h \times \theta_t \end{aligned}$$

and $S_{t+h} - S_t$ by ϵ, we see that

$$\text{Market-maker's profit}$$

$$||$$

$$\underbrace{\Delta_t(S_{t+h} - S_t)}_{\text{Changes in value of stock}} \quad - \quad \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} \quad - rh \underbrace{[\Delta_t S_t - C(S_t)]}_{\text{interest charge}}$$

$$||$$

$$-\left(\frac{1}{2}\epsilon^2\Gamma_t + \theta_t h + rh\,[\Delta_t S_t - C(S_t)]\right)$$

We have seen that the market-maker approximately breaks even for a one-standard-deviation move in the stock:

$$\epsilon = \sigma S_t \sqrt{h} \qquad \Longleftrightarrow \qquad \epsilon^2 = \sigma^2 S_t^2 h$$

Finally, we see that

$$\begin{gathered}\text{Market-maker's profit} \\ \| \\ \underbrace{\Delta_t (S_{t+h} - S_t)}_{\text{Changes in value of stock}} - \underbrace{[C(S_{t+h}) - C(S_t)]}_{\text{Changes in value of option}} - rh \underbrace{[\Delta_t S_t - C(S_t)]}_{\text{interest charge}} \\ \| \\ -\left(\frac{1}{2}\sigma^2 S_t^2 \Gamma_t + \theta_t + r\,[\Delta_t S_t - C(S_t)]\right) h\end{gathered}$$