Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 18. The Lognormal Distribution

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§ 18.1 The normal distribution

§ 18.2 The lognormal distribution

§ 18.3 A lognormal model of stock prices

§ 18.4 Lognormal probability calculations

§ 18.5 Problems

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Theorem 18.4-1

$$\mathbb{P}\left(S_t < K\right) = N\left(-d_2\right) \qquad \text{with} \quad d_2 = \frac{\ln(S_0/K) + (\alpha - \delta - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}.$$

Or equivalently, $\mathbb{P}(S_t > K) = N(d_2)$.

Theorem 18.4-2 The $(1-p)\times 100\%$ prediction interval for S_t is $\left(S_t^L, S_t^U\right)$ with

$$S_t^L = S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}\,N^{-1}(p/2)}$$

$$S_t^U = S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}\,N^{-1}(1-p/2)}$$

Go over Examples 18.6 and 18.7 on P. 558-559.

Theorem 18.4-3 It holds that

$$\mathbb{E}\left(S_t | S_t < K\right) = S_0 e^{(\alpha-\delta)t} \frac{N\left(-d_1\right)}{N\left(-d_2\right)}$$

$$\mathbb{E}\left(S_t | S_t > K\right) = S_0 e^{(\alpha-\delta)t} \frac{N\left(+d_1\right)}{N\left(+d_2\right)}$$

where recall that

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Now we are ready to derive the Black-Scholes formula:

$$
\begin{aligned}
C(S, K, \sigma, r, t, \delta) &= e^{-rt}\mathbb{E}\left([S_t - K]\, 1_{\{S_t > K\}}\right) \\
&= e^{-rt}\mathbb{E}\left([S_t - K]\,|S_t > K\right)\mathbb{P}(S_t > K) \\
&= e^{-rt}\mathbb{E}\left(S_t|S_t > K\right)\mathbb{P}(S_t > K) + e^{-rt}\mathbb{E}\left(K|S_t > K\right)\mathbb{P}(S_t > K) \\
&= e^{-rt}\mathbb{E}\left(S_t|S_t > K\right)\mathbb{P}(S_t > K) + e^{-rt}K\mathbb{P}(S_t > K) \\
&\;\;\vdots
\end{aligned}
$$

Similar for put.