Financial Mathematics

MATH 5870/6870[1]
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Le Chen

lzc0090@auburn.edu

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Auburn University
Auburn AL

[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 20. Brownian Motion and Ito Lemma

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The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0. \tag{1}$$

$$\boxed{dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0}$$

- $S(t)$ is the stock price. $dS(t)$ is the instantaneous change in the stock price. S_0 is the initial asset value.
- α is the continuously compound expected return on the stock;
- σ is the volatility, i.e., the standard deviation of the instantaneous return;
- $Z(t)$ is the standard Brownian motion.
- $dZ(t)$ requires rigorous justification.

- Equation of this type is called stochastic differential equation.
- Solution to this specific equation is the geometric Brownian motion.

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Remark 20.1-1 We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(S(t)) \sim N\left(\ln(S_0) + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right) t,\ \sigma^2\ t\right), \quad \text{for all } t > 0.$$

Remark 20.1-2 Note that Remark 20.1-1 is valid for all $t > 0$. It works for the terminal time $t = T$. It can also help us solve path-dependent options.

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