# Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

# Chapter 20. Brownian Motion and Ito Lemma

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**Definition 20.2-1** A real-valued stochastic process $Z(t)$ is called a Brownian motion or Wiener process if

1. It starts at 0:

$$Z(0) = 0.$$

2. For $0 \leq s < t$, the increment $Z(t) - Z(s)$ is normally distributed with mean zero and variance $t - s$:

$$Z(t) - Z(s) \sim N(0, t - s).$$

3. Its increments are independent: if

$$0 \leq t_0 \leq t_1 \leq \cdots \leq t_k,$$

then

$$\mathbb{P}\left(Z(t_i) - Z(t_{i-1}) \in H_i,\ 1 \leq i \leq k\right) = \prod_{i=1}^{k} \mathbb{P}\left(Z(t_i) - Z(t_{i-1}) \in H_i\right).$$

**Remark 20.2-1** One can always construct a continuous version of the Brownian motion; from now on, we always assume that Brownian motion is a continuous process.

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**Remark 20.2-1** One can always construct a continuous version of the Brownian motion; from now on, we always assume that Brownian motion is a continuous process.

Theorem 20.2-1 (Some properties of Brownian motion)

1. $Z(t)$ is nowhere differentiable.

   (Hence, $dZ(t)$ requires some special treatment.)

2. $Z(t)$ satisfies the scaling property:
$$\widetilde{Z}(t) := \frac{1}{\sqrt{c}} Z(ct) \text{ is also a B.M. for all } c > 0.$$

3. $Z(t)$ is a martingale, namely,
$$\mathbb{E}\left(Z(t+s) | Z(t)\right) = Z(t).$$

4. For any $t > 0$, $Z(t) \sim N(0, t)$ and
$$\mathbb{E}(Z(t)Z(s)) = \min(t, s) \quad \text{for all } t, s \geq 0.$$

5. $Z(t)$ is translation invariant, namely,
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Proof. Part (1) goes beyond this course. All the rest could be proved using our current knowledge.

□

# Arithmetic Brownian motion

**Definition 20.2-2** Let $Z(t)$ be a B.M. Then the process $X(t)$ given by

$$dX(t) = \alpha dt + \sigma dZ(t)$$

is called an arithmetic Brownian motion. Equivalently, $X(t)$ can be written in the following integral representation:

$$X(t) = X(0) + \int_0^t \alpha ds + \int_0^t \sigma dZ(s).$$

### Remark 20.2-2

1. $X(t)$ is normally distributed:

$$X(t) = \sigma t + \sigma Z(t) \sim N\left(\sigma t, \sigma^2 t\right).$$

2. $X(t)$ takes both positive and negative values almost surely.
3. $\alpha t$ is a drift term.

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# The Ornstein-Uhlenbeck process

**Definition 20.2-3** Let $Z(t)$ be a B.M. Then the process $X(t)$ given by

$$dX(t) = \lambda\left(\alpha - X(t)\right)dt + \sigma dZ(t)$$

is called the Ornstein-Uhlenbeck process.

Remark 20.2-3 Equivalently, $X(t)$ can be written in the following integral representation:

$$X(t) = X(0) + \lambda \int_0^t (\alpha - X(s))\, ds + \int_0^t \sigma dZ(s),$$

which is an integral equation (unknown $X$ appears on both sides).

Remark 20.2-4 We have introduced mean reversion in the drift term.