Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 9. Parity and other option relationships

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European versus American options

$$C_{\mathrm{Amer}}(S, K, T) \geq C_{\mathrm{Eur}}(S, K, T)$$

$$P_{\mathrm{Amer}}(S, K, T) \geq P_{\mathrm{Eur}}(S, K, T)$$

Maximum and minimum option prices

$$S \geq C_{\mathrm{Amer}}(S, K, T) \geq C_{\mathrm{Eur}}(S, K, T) \geq \max\left(0, \mathrm{PV}_{0,T}(F_{0,T}) - \mathrm{PV}_{0,T}(K)\right)$$

$$K \geq P_{\mathrm{Amer}}(S, K, T) \geq P_{\mathrm{Eur}}(S, K, T) \geq \max\left(0, \mathrm{PV}(K) - \mathrm{PV}_{0,T}(F_{0,T})\right)$$

Early exercise for American options

Calls on stocks with no dividend

No early exercise!

$$
\begin{aligned}
C_{\mathrm{Ame}}(S_t, K, T-t) &\geq C_{\mathrm{Eur}}(S_t, K, T-t) \\
&= \underbrace{S_t - K}_{\text{Exercise value}} + \underbrace{P_{Eur}(S_t, K, T-t)}_{\text{Insurance against } S_T < K} + \underbrace{K\left(1 - e^{-r(T-t)}\right)}_{\text{Time value of money on } K} \\
&\geq S_t - K
\end{aligned}
$$

Instead of $C(S_t, K, T-t) \geq S_t - K$ one can prove a stronger version:

$$C(S_t, K, T-t) \geq S_t - Ke^{-r(T-t)}$$

Transaction	Time t	Expiration or Exercise, Time T	
		$S_T < K$	$S_T > K$
Buy call	$-C$	0	$S_T - K$
Short stock	S_t	$-S_T$	$-S_T$
Lend $Ke^{-r(T-t)}$	$-Ke^{-r(T-t)}$	K	K
Total	$S_t - Ke^{-r(T-t)} - C$	$K - S_T$	0

Early exercise for American options

Calls on stock with dividends

Interest beats dividends? $K - \mathrm{PV}_{t,T}(K) > \mathrm{PV}_{t,T}(\mathrm{Div})$	Early exercise?
✓	✗
✗	possibly

When dividends do make early exercise rational, one should exercise at the last moment before the ex-dividend date.

Early exercise for puts
(no dividend case)

In order to receive interest, one may exercise early
(think about the case when $S_T = 0$)

No early exercise	Early exercise
$PV_{t,T}(K)$	K

Early exercise for puts
(no dividend case)

No-exercise condition:

$$P\left(S_t, K, T-t\right) > K - S_t$$

$$\Updownarrow$$

$$C\left(S_t, K, T-t\right) > K - \mathrm{PV}_{t,T}(K)$$

$$P(S_t, K, T-t) = C(S_t, K, T-t) - S_t + \mathrm{PV}_{t,T}(K)$$

	calls	puts
Receive	stock	cash
Motivation for early exercise	sufficient dividends	sufficient interest

One can view interest as the dividend on cash.

Dividends are the sole reason to early-exercise an option.

Time to expiration – the K fixed

The longer the more expensive

- American call/put options
- European call option on stock with no dividend

$$C_{Eur} = C_{Ame}$$

The longer, might be cheaper

- European call option on stock with dividend
- European put option

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Time to expiration – $K_t = ke^{rt}$

Theorem 9.3-1 When $K_t = e^{rt}K$, i.e., the strike grows at the interest rate, the premiums on European calls and puts on a non-dividend-paying stock increases with time to maturity.

Proof. We only prove the case for puts and leave the calls as exercise. Let $T > t$. In order to show that

$$P_{\text{Euro}}(S_T, K_T, T) > P_{\text{Euro}}(S_t, K_t, t),$$

it suffices to find an arbitrage when

$$P_{\text{Euro}}(S_T, K_T, T) \leq P_{\text{Euro}}(S_t, K_t, t).$$

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Proof (continued).

		Payoff at Time T			
		$S_T < K_T$		$S_T > K_T$	
		Payoff at Time t			
Transaction	**Time 0**	$S_t < K_t$	$S_t > K_t$	$S_t < K_t$	$S_t > K_t$
Sell $P(t)$	$P(t)$	$S_T - K_T$	0	$S_T - K_T$	0
Buy $P(T)$	$-P(T)$	$K_T - S_T$	$K_T - S_T$	0	0
Total	$P(t) - P(T)$	0	$K_T - S_T$	$S_T - K_T$	0

For example, at time t, if $S_t < K_t$, one has to buy the stock. The payoff at time t is

$$S_t - K_t = S_t - Ke^{rt}$$

One keeps this stock to time T, the stock price becomes S_T, and the future value of Ke^{rt} that one spent at time t becomes $Ke^{rt+r(T-t)} = Ke^{rT}$. Hence, the payoff of this strategy at time T is

$$S_T - Ke^{rT} = S_T - K_T.$$

□

Different strike prices

$$K_1 \le K_2 \le K_3$$

Relation	Ideas in proof, arbitrage in
$C(K_1) \ge C(K_2)$	a call bull spread
$P(K_1) \le P(K_2)$	a put bear spread
$C(K_1) - C(K_2) \le K_2 - K_1$	a call bear spread
$P(K_2) - P(K_1) \le K_2 - K_1$	a put bull spread
$\dfrac{C(K_1) - C(K_2)}{K_2 - K_1} \ge \dfrac{C(K_2) - C(K_3)}{K_3 - K_2}$	an asymmetric butterfly spread
$\dfrac{P(K_2) - P(K_1)}{K_2 - K_1} \le \dfrac{P(K_3) - P(K_2)}{K_3 - K_2}$	an asymmetric butterfly spread

Convexity revisited

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

$$\Updownarrow$$

$$C(K_2) \geq \lambda C(K_1) + (1 - \lambda) C(K_3).$$

with

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1}$$

Example 9.3-1 Suppose that

Strike	50	55
Call Premium	18	12

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution. Check Example 9.4 on p. 283. □

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Example 9.3-2 Suppose that

Strike	50	59	65
Call premium	14	8.9	5

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Solution. Check Example 9.5 on p. 284. □

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Example 9.3-3 Suppose that

Strike	50	55	70
Put premium	4	8	16

1. What no-arbitrage property is violated?
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Solution. Check Example 9.6 on p. 284. □

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