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Adapted for

Emory University

Math 221

Linear Algebra

Sections **1** & **2**

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Course page

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Contents

2.4 Matrix Inverses

Three basic operations on matrices, addition, multiplication, and subtraction, are analogs for matrices of the same operations for numbers. In this section we introduce the matrix analog of numerical division.

To begin, consider how a numerical equation $ax = b$ is solved when a and b are known numbers. If $a = 0$, there is no solution (unless $b = 0$). But if $a \neq 0$, we can multiply both sides by the inverse $a^{-1} = \frac{1}{a}$ to obtain the solution $x = a^{-1}b$. Of course multiplying by a^{-1} is just dividing by a, and the property of a^{-1} that makes this work is that $a^{-1}a = 1$. Moreover, we saw in Section 2.2 that the role that 1 plays in arithmetic is played in matrix algebra by the identity matrix I. This suggests the following definition.

Definition 2.11 Matrix Inverses

If A is a square matrix, a matrix B is called an ***inverse*** *of A if and only if*

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an ***invertible matrix***.[8]

Example 2.4.1

Show that $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution. Compute AB and BA.

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $AB = I = BA$, so B is indeed an inverse of A.

Example 2.4.2

Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ has no inverse.

Solution. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote an arbitrary 2×2 matrix. Then

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a+3c & b+3d \end{bmatrix}$$

so AB has a row of zeros. Hence AB cannot equal I for any B.

[8] Only square matrices have inverses. Even though it is plausible that nonsquare matrices A and B could exist such that $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, we claim that this forces $n = m$. Indeed, if $m < n$ there exists a nonzero column $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ (by Theorem 1.3.1), so $\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B(\mathbf{0}) = \mathbf{0}$, a contradiction. Hence $m \geq n$. Similarly, the condition $AB = I_m$ implies that $n \geq m$. Hence $m = n$ so A is square.

The argument in Example 2.4.2 shows that no zero matrix has an inverse. But Example 2.4.2 also shows that, unlike arithmetic, *it is possible for a nonzero matrix to have no inverse.* However, if a matrix *does* have an inverse, it has only one.

Theorem 2.4.1

If B and C are both inverses of A, then $B = C$.

Proof. Since B and C are both inverses of A, we have $CA = I = AB$. Hence

$$B = IB = (CA)B = C(AB) = CI = C$$

□

If A is an invertible matrix, the (unique) inverse of A is denoted A^{-1}. Hence A^{-1} (when it exists) is a square matrix of the same size as A with the property that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

These equations characterize A^{-1} in the following sense:

> **Inverse Criterion:** *If somehow a matrix B can be found such that $AB = I$ and $BA = I$, then A is invertible and B is the inverse of A; in symbols, $B = A^{-1}$.*

This is a way to verify that the inverse of a matrix exists. Example 2.4.3 and Example 2.4.4 offer illustrations.

Example 2.4.3

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, show that $A^3 = I$ and so find A^{-1}.

Solution. We have $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, and so

$$A^3 = A^2A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $A^3 = I$, as asserted. This can be written as $A^2A = I = AA^2$, so it shows that A^2 is the inverse of A. That is, $A^{-1} = A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

The next example presents a useful formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when it exists. To state it, we define the **determinant** $\det A$ and the **adjugate** $\operatorname{adj} A$ of the matrix A as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \text{and} \quad \operatorname{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.4.4

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that A has an inverse if and only if $\det A \neq 0$, and in this case

$$A^{-1} = \tfrac{1}{\det A} \operatorname{adj} A$$

Solution. For convenience, write $e = \det A = ad - bc$ and $B = \operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then $AB = eI = BA$ as the reader can verify. So if $e \neq 0$, scalar multiplication by $\frac{1}{e}$ gives

$$A(\tfrac{1}{e}B) = I = (\tfrac{1}{e}B)A$$

Hence A is invertible and $A^{-1} = \frac{1}{e}B$. Thus it remains only to show that if A^{-1} exists, then $e \neq 0$.
We prove this by showing that assuming $e = 0$ leads to a contradiction. In fact, if $e = 0$, then $AB = eI = 0$, so left multiplication by A^{-1} gives $A^{-1}AB = A^{-1}0$; that is, $IB = 0$, so $B = 0$. But this implies that a, b, c, and d are *all* zero, so $A = 0$, contrary to the assumption that A^{-1} exists.

As an illustration, if $A = \begin{bmatrix} 2 & 4 \\ -3 & 8 \end{bmatrix}$ then $\det A = 2 \cdot 8 - 4 \cdot (-3) = 28 \neq 0$. Hence A is invertible and $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{28} \begin{bmatrix} 8 & -4 \\ 3 & 2 \end{bmatrix}$, as the reader is invited to verify.

The determinant and adjugate will be defined in Chapter 3 for any square matrix, and the conclusions in Example 2.4.4 will be proved in full generality.

Inverses and Linear Systems

Matrix inverses can be used to solve certain systems of linear equations. Recall that a *system* of linear equations can be written as a *single* matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A and $\mathbf{b}$ are known and $\mathbf{x}$ is to be determined. If A is invertible, we multiply each side of the equation on the left by A^{-1} to get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This gives the solution to the system of equations (the reader should verify that $\mathbf{x} = A^{-1}\mathbf{b}$ really does satisfy $A\mathbf{x} = \mathbf{b}$). Furthermore, the argument shows that if $\mathbf{x}$ is *any* solution, then necessarily $\mathbf{x} = A^{-1}\mathbf{b}$, so the solution is unique. Of course the technique works only when the coefficient matrix A has an inverse. This proves Theorem 2.4.2.

Theorem 2.4.2

Suppose a system of n equations in n variables is written in matrix form as

$$A\mathbf{x} = \mathbf{b}$$

If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example 2.4.5

Use Example 2.4.4 to solve the system $\begin{cases} 5x_1 - 3x_2 = -4 \\ 7x_1 + 4x_2 = 8 \end{cases}$.

Solution. In matrix form this is $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 5 & -3 \\ 7 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$. Then $\det A = 5 \cdot 4 - (-3) \cdot 7 = 41$, so A is invertible and $A^{-1} = \frac{1}{41}\begin{bmatrix} 4 & 3 \\ -7 & 5 \end{bmatrix}$ by Example 2.4.4. Thus Theorem 2.4.2 gives

$$\mathbf{x} = A^{-1}\mathbf{b} = \tfrac{1}{41}\begin{bmatrix} 4 & 3 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 8 \end{bmatrix} = \tfrac{1}{41}\begin{bmatrix} 8 \\ 68 \end{bmatrix}$$

so the solution is $x_1 = \frac{8}{41}$ and $x_2 = \frac{68}{41}$.

An Inversion Method

If a matrix A is $n \times n$ and invertible, it is desirable to have an efficient technique for finding the inverse. The following procedure will be justified in Section 2.5.

Matrix Inversion Algorithm

If A is an invertible (square) matrix, there exists a sequence of elementary row operations that carry A to the identity matrix I of the same size, written $A \to I$. This same series of row operations carries I to A^{-1}; that is, $I \to A^{-1}$. The algorithm can be summarized as follows:

$$\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

where the row operations on A and I are carried out simultaneously.

Example 2.4.6

Use the inversion algorithm to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution. Apply elementary row operations to the double matrix

$$\left[\begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A & I \end{array}\right]$$

so as to carry A to I. First interchange rows 1 and 2.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next subtract 2 times row 1 from row 2, and subtract row 1 from row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Continue to reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]$$

Hence $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$, as is readily verified.

Given any $n \times n$ matrix A, Theorem 1.2.1 shows that A can be carried by elementary row operations to a matrix R in reduced row-echelon form. If $R = I$, the matrix A is invertible (this will be proved in the next section), so the algorithm produces A^{-1}. If $R \neq I$, then R has a row of zeros (it is square), so no system of linear equations $A\mathbf{x} = \mathbf{b}$ can have a unique solution. But then A is not invertible by Theorem 2.4.2. Hence, the algorithm is effective in the sense conveyed in Theorem 2.4.3.

Theorem 2.4.3

If A is an $n \times n$ matrix, either A can be reduced to I by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

Properties of Inverses

The following properties of an invertible matrix are used everywhere.

Example 2.4.7: Cancellation Laws

Let A be an invertible matrix. Show that:

1. If $AB = AC$, then $B = C$.
2. If $BA = CA$, then $B = C$.

Solution. Given the equation $AB = AC$, left multiply both sides by A^{-1} to obtain $A^{-1}AB = A^{-1}AC$. Thus $IB = IC$, that is $B = C$. This proves (1) and the proof of (2) is left to the reader.

Properties (1) and (2) in Example 2.4.7 are described by saying that an invertible matrix can be "left cancelled" and "right cancelled", respectively. Note however that "mixed" cancellation does not hold in general: If A is invertible and $AB = CA$, then B and C may *not* be equal, even if both are 2×2. Here is a specific example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Sometimes the inverse of a matrix is given by a formula. Example 2.4.4 is one illustration; Example 2.4.8 and Example 2.4.9 provide two more. The idea is the *Inverse Criterion*: If a matrix B can be found such that $AB = I = BA$, then A is invertible and $A^{-1} = B$.

Example 2.4.8

If A is an invertible matrix, show that the transpose A^T is also invertible. Show further that the inverse of A^T is just the transpose of A^{-1}; in symbols, $(A^T)^{-1} = (A^{-1})^T$.

Solution. A^{-1} exists (by assumption). Its transpose $(A^{-1})^T$ is the candidate proposed for the inverse of A^T. Using the inverse criterion, we test it as follows:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Hence $(A^{-1})^T$ is indeed the inverse of A^T; that is, $(A^T)^{-1} = (A^{-1})^T$.

Example 2.4.9

If A and B are invertible $n \times n$ matrices, show that their product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Solution. We are given a candidate for the inverse of AB, namely $B^{-1}A^{-1}$. We test it as follows:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Hence $B^{-1}A^{-1}$ is the inverse of AB; in symbols, $(AB)^{-1} = B^{-1}A^{-1}$.

We now collect several basic properties of matrix inverses for reference.

Theorem 2.4.4

All the following matrices are square matrices of the same size.

1. *I is invertible and $I^{-1} = I$.*
2. *If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.*
3. *If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *If A_1, A_2, ..., A_k are all invertible, so is their product $A_1A_2\cdots A_k$, and*
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$
5. *If A is invertible, so is A^k for any $k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.*
6. *If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.*
7. *If A is invertible, so is its transpose A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Proof.

1. This is an immediate consequence of the fact that $I^2 = I$.
2. The equations $AA^{-1} = I = A^{-1}A$ show that A is the inverse of A^{-1}; in symbols, $(A^{-1})^{-1} = A$.
3. This is Example 2.4.9.
4. Use induction on k. If $k = 1$, there is nothing to prove, and if $k = 2$, the result is property 3. If $k > 2$, assume inductively that $(A_1A_2\cdots A_{k-1})^{-1} = A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$. We apply this fact together with property 3 as follows:
$$\begin{aligned}[A_1A_2\cdots A_{k-1}A_k]^{-1} &= [(A_1A_2\cdots A_{k-1})A_k]^{-1} \\ &= A_k^{-1}(A_1A_2\cdots A_{k-1})^{-1} \\ &= A_k^{-1}\left(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}\right)\end{aligned}$$

So the proof by induction is complete.

5. This is property 4 with $A_1 = A_2 = \cdots = A_k = A$.

6. This is left as Exercise 2.4.29.

7. This is Example 2.4.8. □

The reversal of the order of the inverses in properties 3 and 4 of Theorem 2.4.4 is a consequence of the fact that matrix multiplication is not commutative. Another manifestation of this comes when matrix equations are dealt with. If a matrix equation $B = C$ is given, it can be *left-multiplied* by a matrix A to yield $AB = AC$. Similarly, *right-multiplication* gives $BA = CA$. However, we cannot mix the two: If $B = C$, it need *not* be the case that $AB = CA$ even if A is invertible, for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = C$.

Part 7 of Theorem 2.4.4 together with the fact that $(A^T)^T = A$ gives

Corollary 2.4.1

A square matrix A is invertible if and only if A^T is invertible.

Example 2.4.10

Find A if $(A^T - 2I)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

Solution. By Theorem 2.4.4(2) and Example 2.4.4, we have

$$(A^T - 2I) = \left[(A^T - 2I)^{-1}\right]^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

Hence $A^T = 2I + \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, so $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ by Theorem 2.4.4(7).

The following important theorem collects a number of conditions all equivalent[9] to invertibility. It will be referred to frequently below.

Theorem 2.4.5: Inverse Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*

2. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*

[9] If p and q are statements, we say that p **implies** q (written $p \Rightarrow q$) if q is true whenever p is true. The statements are called **equivalent** if both $p \Rightarrow q$ and $q \Rightarrow p$ (written $p \Leftrightarrow q$, spoken "p if and only if q"). See Appendix **??**.

3. *A can be carried to the identity matrix I_n by elementary row operations.*
4. *The system $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x}$ for every choice of column $\mathbf{b}$.*
5. *There exists an $n \times n$ matrix C such that $AC = I_n$.*

Proof. We show that each of these conditions implies the next, and that (5) implies (1).

(1) $\Rightarrow$ (2). If A^{-1} exists, then $A\mathbf{x} = \mathbf{0}$ gives $\mathbf{x} = I_n\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

(2) $\Rightarrow$ (3). Assume that (2) is true. Certainly $A \to R$ by row operations where R is a reduced, row-echelon matrix. It suffices to show that $R = I_n$. Suppose that this is not the case. Then R has a row of zeros (being square). Now consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{0}$. Then $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right] \to \left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ is the reduced form, and $\left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ also has a row of zeros. Since R is square there must be at least one nonleading variable, and hence at least one parameter. Hence the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, contrary to (2). So $R = I_n$ after all.

(3) $\Rightarrow$ (4). Consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{b}$. Using (3), let $A \to I_n$ by a sequence of row operations. Then these same operations carry $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \to \left[\begin{array}{c|c} I_n & \mathbf{c} \end{array}\right]$ for some column $\mathbf{c}$. Hence the system $A\mathbf{x} = \mathbf{b}$ has a solution (in fact unique) by gaussian elimination. This proves (4).

(4) $\Rightarrow$ (5). Write $I_n = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right]$ where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the columns of I_n. For each $j = 1, 2, \ldots, n$, the system $A\mathbf{x} = \mathbf{e}_j$ has a solution $\mathbf{c}_j$ by (4), so $A\mathbf{c}_j = \mathbf{e}_j$. Now let $C = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right]$ be the $n \times n$ matrix with these matrices $\mathbf{c}_j$ as its columns. Then Definition 2.9 gives (5):

$$AC = A\left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{c}_1 & A\mathbf{c}_2 & \cdots & A\mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = I_n$$

(5) $\Rightarrow$ (1). Assume that (5) is true so that $AC = I_n$ for some matrix C. Then $C\mathbf{x} = 0$ implies $\mathbf{x} = \mathbf{0}$ (because $\mathbf{x} = I_n\mathbf{x} = AC\mathbf{x} = A\mathbf{0} = \mathbf{0}$). Thus condition (2) holds for the matrix C rather than A. Hence the argument above that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) (with A replaced by C) shows that a matrix C' exists such that $CC' = I_n$. But then

$$A = AI_n = A(CC') = (AC)C' = I_nC' = C'$$

Thus $CA = CC' = I_n$ which, together with $AC = I_n$, shows that C is the inverse of A. This proves (1). □

The proof of (5) $\Rightarrow$ (1) in Theorem 2.4.5 shows that if $AC = I$ for square matrices, then necessarily $CA = I$, and hence that C and A are inverses of each other. We record this important fact for reference.

Corollary 2.4.1

If A and C are square matrices such that $AC = I$, then also $CA = I$. In particular, both A and C are invertible, $C = A^{-1}$, and $A = C^{-1}$.

Here is a quick way to remember Corollary 2.4.1. If A is a square matrix, then

1. If $AC = I$ then $C = A^{-1}$.

2. If $CA = I$ then $C = A^{-1}$.

Observe that Corollary 2.4.1 is false if A and C are not square matrices. For example, we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = I_2 \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq I_3$$

In fact, it is verified in the footnote on page 91 that if $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, then $m = n$ and A and B are (square) inverses of each other.

An $n \times n$ matrix A has $\textsf{rank}\, n$ if and only if (3) of Theorem 2.4.5 holds. Hence

Corollary 2.4.2

An $n \times n$ matrix A is invertible if and only if $\textsf{rank}\, A = n$.

Here is a useful fact about inverses of block matrices.

Example 2.4.11

Let $P = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ and $Q = \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix}$ be block matrices where A is $m \times m$ and B is $n \times n$ (possibly $m \neq n$).

a. Show that P is invertible if and only if A and B are both invertible. In this case, show that
$$P^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

b. Show that Q is invertible if and only if A and B are both invertible. In this case, show that
$$Q^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}YA^{-1} & B^{-1} \end{bmatrix}$$

Solution. We do (a.) and leave (b.) for the reader.

a. If A^{-1} and B^{-1} both exist, write $R = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$. Using block multiplication, one verifies that $PR = I_{m+n} = RP$, so P is invertible, and $P^{-1} = R$.
Conversely, suppose that P is invertible, and write $P^{-1} = \begin{bmatrix} C & V \\ W & D \end{bmatrix}$ in block form, where C is $m \times m$ and D is $n \times n$.
Then the equation $PP^{-1} = I_{n+m}$ becomes

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \begin{bmatrix} C & V \\ W & D \end{bmatrix} = \begin{bmatrix} AC + XW & AV + XD \\ BW & BD \end{bmatrix} = I_{m+n} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$$

using block notation. Equating corresponding blocks, we find

$$AC + XW = I_m, \quad BW = 0, \quad \text{and } BD = I_n$$

Hence B is invertible because $BD = I_n$ (by Corollary 2.4.1), then $W = 0$ because $BW = 0$, and finally, $AC = I_m$ (so A is invertible, again by Corollary 2.4.1).

Inverses of Matrix Transformations

Let $T = T_A : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by the $n \times n$ matrix A. Since A is square, it may very well be invertible, and this leads to the question:

What does it mean geometrically for T that A is invertible?

To answer this, let $T' = T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$ denote the transformation induced by A^{-1}. Then

$$\begin{aligned} T'[T(\mathbf{x})] &= A^{-1}[A\mathbf{x}] = I\mathbf{x} = \mathbf{x} \\ T[T'(\mathbf{x})] &= A\left[A^{-1}\mathbf{x}\right] = I\mathbf{x} = \mathbf{x} \end{aligned} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2.8}$$

The first of these equations asserts that, if T carries $\mathbf{x}$ to a vector $T(\mathbf{x})$, then T' carries $T(\mathbf{x})$ right back to $\mathbf{x}$; that is T' "reverses" the action of T. Similarly T "reverses" the action of T'. Conditions (2.8) can be stated compactly in terms of composition:

$$T' \circ T = 1_{\mathbb{R}^n} \quad \text{and} \quad T \circ T' = 1_{\mathbb{R}^n} \tag{2.9}$$

When these conditions hold, we say that the matrix transformation T' is an **inverse** of T, and we have shown that if the matrix A of T is invertible, then T has an inverse (induced by A^{-1}).

The converse is also true: If T has an inverse, then its matrix A must be invertible. Indeed, suppose $S : \mathbb{R}^n \to \mathbb{R}^n$ is any inverse of T, so that $S \circ T = 1_{\mathbb{R}_n}$ and $T \circ S = 1_{\mathbb{R}_n}$. It can be shown that S is also a matrix transformation. If B is the matrix of S, we have

$$BA\mathbf{x} = S[T(\mathbf{x})] = (S \circ T)(\mathbf{x}) = 1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} = I_n\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

It follows by Theorem 2.2.6 that $BA = I_n$, and a similar argument shows that $AB = I_n$. Hence A is invertible with $A^{-1} = B$. Furthermore, the inverse transformation S has matrix A^{-1}, so $S = T'$ using the earlier notation. This proves the following important theorem.

Theorem 2.4.6

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by an $n \times n$ matrix A. Then

A is invertible if and only if T has an inverse.

In this case, T has exactly one inverse (which we denote as T^{-1}), and $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the

transformation induced by the matrix A^{-1}. In other words

$$(T_A)^{-1} = T_{A^{-1}}$$

The geometrical relationship between T and T^{-1} is embodied in equations (2.8) above:

$$T^{-1}[T(\mathbf{x})] = \mathbf{x} \quad \text{and} \quad T\left[T^{-1}(\mathbf{x})\right] = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

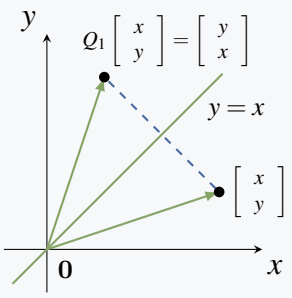
These equations are called the **fundamental identities** relating T and T^{-1}. Loosely speaking, they assert that each of T and T^{-1} "reverses" or "undoes" the action of the other.

This geometric view of the inverse of a linear transformation provides a new way to find the inverse of a matrix A. More precisely, if A is an invertible matrix, we proceed as follows:

1. *Let T be the linear transformation induced by A.*
2. *Obtain the linear transformation T^{-1} which "reverses" the action of T.*
3. *Then A^{-1} is the matrix of T^{-1}.*

Here is an example.

Example 2.4.12



Find the inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by viewing it as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

Solution. If $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ the vector $A\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ is the result of reflecting $\mathbf{x}$ in the line $y = x$ (see the diagram). Hence, if $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection in the line $y = x$, then A is the matrix of Q_1. Now observe that Q_1 reverses itself because reflecting a vector $\mathbf{x}$ twice results in $\mathbf{x}$. Consequently $Q_1^{-1} = Q_1$. Since A^{-1} is the matrix of Q_1^{-1} and A is the matrix of Q, it follows that $A^{-1} = A$. Of course this conclusion is clear by simply observing directly that $A^2 = I$, but the geometric method can often work where these other methods may be less straightforward.

Exercises for 2.4

Exercise 2.4.1 In each case, show that the matrices are inverses of each other.

a. $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 1 & -4 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 1 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$

Exercise 2.4.2 Find the inverse of each of the following matrices.

a) $\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{bmatrix}$

e) $\begin{bmatrix} 3 & 5 & 0 \\ 3 & 7 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 5 & -1 \end{bmatrix}$

g) $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 3 & 2 \\ 4 & 1 & 4 \end{bmatrix}$ h) $\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

i) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ j) $\begin{bmatrix} -1 & 4 & 5 & 2 \\ 0 & 0 & 0 & -1 \\ 1 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

k) $\begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & 3 & 6 \\ 1 & -1 & 5 & 2 \\ 1 & -1 & 5 & 1 \end{bmatrix}$ l) $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

f. $\frac{1}{10}\begin{bmatrix} 1 & 4 & -1 \\ -2 & 2 & 2 \\ -9 & 14 & -1 \end{bmatrix}$

h. $\frac{1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & 2 & 5 \\ -3 & 2 & -1 \end{bmatrix}$

j. $\begin{bmatrix} 0 & 0 & 1 & -2 \\ -1 & -2 & -1 & -3 \\ 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

l. $\begin{bmatrix} 1 & -2 & 6 & -30 & 210 \\ 0 & 1 & -3 & 15 & -105 \\ 0 & 0 & 1 & -5 & 35 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 2.4.3 In each case, solve the systems of equations by finding the inverse of the coefficient matrix.

a) $\begin{aligned} 3x - y &= 5 \\ 2x + 2y &= 1 \end{aligned}$

b) $\begin{aligned} 2x - 3y &= 0 \\ x - 4y &= 1 \end{aligned}$

c) $\begin{aligned} x + y + 2z &= 5 \\ x + y + z &= 0 \\ x + 2y + 4z &= -2 \end{aligned}$

d) $\begin{aligned} x + 4y + 2z &= 1 \\ 2x + 3y + 3z &= -1 \\ 4x + y + 4z &= 0 \end{aligned}$

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b. $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 \\ -2 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 9 & -14 & 6 \\ 4 & -4 & 1 \\ -10 & 15 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 23 \\ 8 \\ -25 \end{bmatrix}$

Exercise 2.4.4 Given $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$:

a. Solve the system of equations $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

b. Find a matrix B such that $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

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b. $\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

c. Find a matrix C such that $CA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$.

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b. $B = A^{-1}AB = \begin{bmatrix} 4 & -2 & 1 \\ 7 & -2 & 4 \\ -1 & 2 & -1 \end{bmatrix}$

Exercise 2.4.5 Find A when

a) $(3A)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ b) $(2A)^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1}$

c) $(I+3A)^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

d) $(I-2A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

e) $\left(A \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

f) $\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} A\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

g) $\left(A^T - 2I\right)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

h) $\left(A^{-1} - 2I\right)^T = -2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

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b. $\frac{1}{10}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

d. $\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

f. $\frac{1}{2}\begin{bmatrix} 2 & 0 \\ -6 & 1 \end{bmatrix}$

h. $-\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 2.4.6 Find A when:

a) $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ b) $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

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b. $A = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}$

Exercise 2.4.7 Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, express the variables x_1, x_2, and x_3 in terms of z_1, z_2, and z_3.

Exercise 2.4.8

a. In the system $\begin{matrix} 3x+4y=7 \\ 4x+5y=1 \end{matrix}$, substitute the new variables x' and y' given by $\begin{matrix} x = -5x'+4y' \\ y = \ 4x'-3y' \end{matrix}$. Then find x and y.

b. Explain part (a) by writing the equations as $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x' \\ y' \end{bmatrix}$. What is the relationship between A and B?

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b. A and B are inverses.

Exercise 2.4.9 In each case either prove the assertion or give an example showing that it is false.

a. If $A \neq 0$ is a square matrix, then A is invertible.

b. If A and B are both invertible, then $A+B$ is invertible.

c. If A and B are both invertible, then $(A^{-1}B)^T$ is invertible.

d. If $A^4 = 3I$, then A is invertible.

e. If $A^2 = A$ and $A \neq 0$, then A is invertible.

f. If $AB = B$ for some $B \neq 0$, then A is invertible.

g. If A is invertible and skew symmetric ($A^T = -A$), the same is true of A^{-1}.

h. If A^2 is invertible, then A is invertible.

i. If $AB = I$, then A and B commute.

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b. False. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

d. True. $A^{-1} = \frac{1}{3}A^3$

f. False. $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

h. True. If $(A^2)B = I$, then $A(AB) = I$; use Theorem 2.4.5.

Exercise 2.4.10

a. If A, B, and C are square matrices and $AB = I$, $I = CA$, show that A is invertible and $B = C = A^{-1}$.

b. If $C^{-1} = A$, find the inverse of C^T in terms of A.

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b. $(C^T)^{-1} = (C^{-1})^T = A^T$ because $C^{-1} = (A^{-1})^{-1} = A$.

Exercise 2.4.11 Suppose $CA = I_m$, where C is $m \times n$ and A is $n \times m$. Consider the system $A\mathbf{x} = \mathbf{b}$ of n equations in m variables.

a. Show that this system has a unique solution CB if it is consistent.

b. If $C = \begin{bmatrix} 0 & -5 & 1 \\ 3 & 0 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \\ 6 & -10 \end{bmatrix}$, find $\mathbf{x}$ (if it exists) when (i) $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$; and (ii) $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ 22 \end{bmatrix}$.

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b. (i) Inconsistent. (ii) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Exercise 2.4.12 Verify that $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ satisfies $A^2 - 3A + 2I = 0$, and use this fact to show that $A^{-1} = \frac{1}{2}(3I - A)$.

Exercise 2.4.13 Let $Q = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$. Compute QQ^T and so find Q^{-1} if $Q \neq 0$.

Exercise 2.4.14 Let $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that each of U, $-U$, and $-I_2$ is its own inverse and that the product of any two of these is the third.

Exercise 2.4.15 Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{bmatrix}$. Find the inverses by computing (a) A^6; (b) B^4; and (c) C^3.

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b. $B^4 = I$, so $B^{-1} = B^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Exercise 2.4.16 Find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ c & 1 & c \\ 3 & c & 2 \end{bmatrix}$ in terms of c.

$\begin{bmatrix} c^2 - 2 & -c & 1 \\ -c & 1 & 0 \\ 3 - c^2 & c & -1 \end{bmatrix}$

Exercise 2.4.17 If $c \neq 0$, find the inverse of $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 2 & c \end{bmatrix}$ in terms of c.

Exercise 2.4.18 Show that A has no inverse when:

a. A has a row of zeros.

b. A has a column of zeros.

c. each row of A sums to 0. [*Hint*: Theorem 2.4.5(2).]

d. each column of A sums to 0. [*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

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b. If column j of A is zero, $A\mathbf{y} = \mathbf{0}$ where $\mathbf{y}$ is column j of the identity matrix. Use Theorem 2.4.5.

d. If each column of A sums to 0, $XA = 0$ where X is the row of 1s. Hence $A^T X^T = 0$ so A has no inverse by Theorem 2.4.5 ($X^T \neq 0$).

Exercise 2.4.19 Let A denote a square matrix.

a. Let $YA = 0$ for some matrix $Y \neq 0$. Show that A has no inverse. [*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

b. Use part (a) to show that (i) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$; and (ii) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ have no inverse. [*Hint*: For part (ii) compare row 3 with the difference between row 1 and row 2.]

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b. (ii) $(-1,\ 1,\ 1)A = 0$

Exercise 2.4.20 If A is invertible, show that

a) $A^2 \neq 0$. b) $A^k \neq 0$ for all $k = 1,\ 2,\ \ldots$.

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b. Each power A^k is invertible by Theorem 2.4.4 (because A is invertible). Hence A^k cannot be 0.

Exercise 2.4.21 Suppose $AB = 0$, where A and B are square matrices. Show that:

a. If one of A and B has an inverse, the other is zero.

b. It is impossible for both A and B to have inverses.

c. $(BA)^2 = 0$.

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b. By (a), if one has an inverse the other is zero and so has no inverse.

Exercise 2.4.22 Find the inverse of the x-expansion in Example 2.2.16 and describe it geometrically.

If $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, $a > 1$, then $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}$ is an x-compression because $\frac{1}{a} < 1$.

Exercise 2.4.23 Find the inverse of the shear transformation in Example 2.2.17 and describe it geometrically.

Exercise 2.4.24 In each case assume that A is a square matrix that satisfies the given condition. Show that A is invertible and find a formula for A^{-1} in terms of A.

a. $A^3 - 3A + 2I = 0$.

b. $A^4 + 2A^3 - A - 4I = 0$.

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b. $A^{-1} = \frac{1}{4}(A^3 + 2A^2 - 1)$

Exercise 2.4.25 Let A and B denote $n \times n$ matrices.

a. If A and AB are invertible, show that B is invertible using only (2) and (3) of Theorem 2.4.4.

b. If AB is invertible, show that both A and B are invertible using Theorem 2.4.5.

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b. If $B\mathbf{x} = \mathbf{0}$, then $(AB)\mathbf{x} = (A)B\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ because AB is invertible. Hence B is invertible by Theorem 2.4.5. But then $A = (AB)B^{-1}$ is invertible by Theorem 2.4.4.

Exercise 2.4.26 In each case find the inverse of the matrix A using Example 2.4.11.

a) $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ b) $A = \begin{bmatrix} 3 & 1 & 0 \\ 5 & 2 & 0 \\ 1 & 3 & -1 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 1 & 4 \end{bmatrix}$

d) $A = \begin{bmatrix} 2 & 1 & 5 & 2 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

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b. $\left[\begin{array}{rr|r} 2 & -1 & 0 \\ -5 & 3 & 0 \\ \hline -13 & 8 & -1 \end{array}\right]$

d. $\left[\begin{array}{rr|rr} 1 & -1 & -14 & 8 \\ -1 & 2 & 16 & -9 \\ \hline 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Exercise 2.4.27 If A and B are invertible symmetric matrices such that $AB = BA$, show that A^{-1}, AB, AB^{-1}, and $A^{-1}B^{-1}$ are also invertible and symmetric.

Exercise 2.4.28 Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix.

a. If $A^2 = 0$, verify that $(I - A)^{-1} = I + A$.

b. If $A^3 = 0$, verify that $(I - A)^{-1} = I + A + A^2$.

c. Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

d. If $A^n = 0$, find the formula for $(I - A)^{-1}$.

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d. If $A^n = 0$, $(I - A)^{-1} = I + A + \cdots + A^{n-1}$.

Exercise 2.4.29 Prove property 6 of Theorem 2.4.4: If A is invertible and $a \neq 0$, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$

Exercise 2.4.30 Let A, B, and C denote $n \times n$ matrices. Using only Theorem 2.4.4, show that:

a. If A, C, and ABC are all invertible, B is invertible.

b. If AB and BA are both invertible, A and B are both invertible.

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b. $A[B(AB)^{-1}] = I = [(BA)^{-1}B]A$, so A is invertible by Exercise 2.4.10.

Exercise 2.4.31 Let A and B denote invertible $n \times n$ matrices.

a. If $A^{-1} = B^{-1}$, does it mean that $A = B$? Explain.

b. Show that $A = B$ if and only if $A^{-1}B = I$.

Exercise 2.4.32 Let A, B, and C be $n \times n$ matrices, with A and B invertible. Show that

a. If A commutes with C, then A^{-1} commutes with C.

b. If A commutes with B, then A^{-1} commutes with B^{-1}.

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a. Have $AC = CA$. Left-multiply by A^{-1} to get $C = A^{-1}CA$. Then right-multiply by A^{-1} to get $CA^{-1} = A^{-1}C$.

Exercise 2.4.33 Let A and B be square matrices of the same size.

a. Show that $(AB)^2 = A^2B^2$ if $AB = BA$.

b. If A and B are invertible and $(AB)^2 = A^2B^2$, show that $AB = BA$.

c. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(AB)^2 = A^2B^2$ but $AB \neq BA$.

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b. Given $ABAB = AABB$. Left multiply by A^{-1}, then right multiply by B^{-1}.

Exercise 2.4.34 Let A and B be $n \times n$ matrices for which AB is invertible. Show that A and B are both invertible.

If $B\mathbf{x} = \mathbf{0}$ where $\mathbf{x}$ is $n \times 1$, then $AB\mathbf{x} = \mathbf{0}$ so $\mathbf{x} = \mathbf{0}$ as AB is invertible. Hence B is invertible by Theorem 2.4.5, so $A = (AB)B^{-1}$ is invertible.

Exercise 2.4.35 Consider $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & -7 & 13 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & -3 \\ -2 & 5 & 17 \end{bmatrix}$.

a. Show that A is not invertible by finding a nonzero 1×3 matrix Y such that $YA = 0$. [*Hint*: Row 3 of A equals 2(row 2) -3(row 1).]

b. Show that B is not invertible. [*Hint*: Column 3 = 3(column 2) $-$ column 1.]

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b. $B \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = 0$ so B is not invertible by Theorem 2.4.5.

Exercise 2.4.36 Show that a square matrix A is invertible if and only if it can be left-cancelled: $AB = AC$ implies $B = C$.

Exercise 2.4.37 If $U^2 = I$, show that $I + U$ is not invertible unless $U = I$.

Exercise 2.4.38

a. If J is the 4×4 matrix with every entry 1, show that $I - \frac{1}{2}J$ is self-inverse and symmetric.

b. If X is $n \times m$ and satisfies $X^TX = I_m$, show that $I_n - 2XX^T$ is self-inverse and symmetric.

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b. Write $U = I_n - 2XX^T$. Then $U^T = I_n^T - 2X^{TT}X^T = U$, and $U^2 = I_n^2 - (2XX^T)I_n - I_n(2XX^T) + 4(XX^T)(XX^T) = I_n - 4XX^T + 4XX^T = I_n$.

Exercise 2.4.39 An $n \times n$ matrix P is called an idempotent if $P^2 = P$. Show that:

a. I is the only invertible idempotent.

b. P is an idempotent if and only if $I - 2P$ is self-inverse.

c. U is self-inverse if and only if $U = I - 2P$ for some idempotent P.

d. $I - aP$ is invertible for any $a \neq 1$, and that $(I - aP)^{-1} = I + \left(\frac{a}{1-a}\right)P$.

---

b. $(I - 2P)^2 = I - 4P + 4P^2$, and this equals I if and only if $P^2 = P$.

Exercise 2.4.40 If $A^2 = kA$, where $k \neq 0$, show that A is invertible if and only if $A = kI$.

Exercise 2.4.41 Let A and B denote $n \times n$ invertible matrices.

a. Show that $A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1}$.

b. If $A + B$ is also invertible, show that $A^{-1} + B^{-1}$ is invertible and find a formula for $(A^{-1} + B^{-1})^{-1}$.

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b. $(A^{-1} + B^{-1})^{-1} = B(A + B)^{-1}A$

Exercise 2.4.42 Let A and B be $n \times n$ matrices, and let I be the $n \times n$ identity matrix.

a. Verify that $A(I + BA) = (I + AB)A$ and that $(I + BA)B = B(I + AB)$.

b. If $I + AB$ is invertible, verify that $I + BA$ is also invertible and that $(I + BA)^{-1} = I - B(I + AB)^{-1}A$.