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LINEAR ALGEBRA with Applications

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Adapted for

Emory University

Math 221

Linear Algebra

Sections 1 & 2

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6.3 Linear Independence and Dimension

Definition 6.4 Linear Independence and Dependence

As in $\mathbb{R}^n$, a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ in a vector space V is called ***linearly independent*** *(or simply* ***independent****) if it satisfies the following condition:*

$$\text{If} \quad s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n = \mathbf{0}, \qquad \text{then} \quad s_1 = s_2 = \cdots = s_n = 0.$$

A set of vectors that is not linearly independent is said to be ***linearly dependent*** *(or simply* ***dependent****).*

The **trivial linear combination** of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$ is the one with every coefficient zero:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$$

This is obviously one way of expressing $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$, and they are linearly independent when it is the *only* way.

Example 6.3.1

Show that $\{1+x,\ 3x+x^2,\ 2+x-x^2\}$ is independent in $\mathbf{P}_2$.

Solution. Suppose a linear combination of these polynomials vanishes.

$$s_1(1+x) + s_2(3x+x^2) + s_3(2+x-x^2) = 0$$

Equating the coefficients of 1, x, and x^2 gives a set of linear equations.

$$\begin{aligned} s_1 + + 2s_3 &= 0 \\ s_1 + 3s_2 + s_3 &= 0 \\ s_2 - s_3 &= 0 \end{aligned}$$

The only solution is $s_1 = s_2 = s_3 = 0$.

Example 6.3.2

Show that $\{\sin x,\ \cos x\}$ is independent in the vector space $\mathbf{F}[0,\ 2\pi]$ of functions defined on the interval $[0,\ 2\pi]$.

Solution. Suppose that a linear combination of these functions vanishes.

$$s_1(\sin x) + s_2(\cos x) = 0$$

This must hold for *all* values of x in $[0,\ 2\pi]$ (by the definition of equality in $\mathbf{F}[0,\ 2\pi]$). Taking $x = 0$ yields $s_2 = 0$ (because $\sin 0 = 0$ and $\cos 0 = 1$). Similarly, $s_1 = 0$ follows from taking $x = \frac{\pi}{2}$ (because $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$).

Example 6.3.3

Suppose that $\{\mathbf{u},\ \mathbf{v}\}$ is an independent set in a vector space V. Show that $\{\mathbf{u}+2\mathbf{v},\ \mathbf{u}-3\mathbf{v}\}$ is also independent.

Solution. Suppose a linear combination of $\mathbf{u}+2\mathbf{v}$ and $\mathbf{u}-3\mathbf{v}$ vanishes:

$$s(\mathbf{u}+2\mathbf{v})+t(\mathbf{u}-3\mathbf{v})=\mathbf{0}$$

We must deduce that $s=t=0$. Collecting terms involving $\mathbf{u}$ and $\mathbf{v}$ gives

$$(s+t)\mathbf{u}+(2s-3t)\mathbf{v}=\mathbf{0}$$

Because $\{\mathbf{u},\ \mathbf{v}\}$ is independent, this yields linear equations $s+t=0$ and $2s-3t=0$. The only solution is $s=t=0$.

Example 6.3.4

Show that any set of polynomials of distinct degrees is independent.

Solution. Let $p_1,\ p_2,\ \dots,\ p_m$ be polynomials where $\deg(p_i)=d_i$. By relabelling if necessary, we may assume that $d_1>d_2>\cdots>d_m$. Suppose that a linear combination vanishes:

$$t_1p_1+t_2p_2+\cdots+t_mp_m=0$$

where each t_i is in $\mathbb{R}$. As $\deg(p_1)=d_1$, let ax^{d_1} be the term in p_1 of highest degree, where $a\neq 0$. Since $d_1>d_2>\cdots>d_m$, it follows that $t_1ax^{d_1}$ is the only term of degree d_1 in the linear combination $t_1p_1+t_2p_2+\cdots+t_mp_m=0$. This means that $t_1ax^{d_1}=0$, whence $t_1a=0$, hence $t_1=0$ (because $a\neq 0$). But then $t_2p_2+\cdots+t_mp_m=0$ so we can repeat the argument to show that $t_2=0$. Continuing, we obtain $t_i=0$ for each i, as desired.

Example 6.3.5

Suppose that A is an $n\times n$ matrix such that $A^k=0$ but $A^{k-1}\neq 0$. Show that $B=\{I,\ A,\ A^2,\ \dots,\ A^{k-1}\}$ is independent in $\mathbf{M}_{nn}$.

Solution. Suppose $r_0I+r_1A+r_2A^2+\cdots+r_{k-1}A^{k-1}=0$. Multiply by A^{k-1}:

$$r_0A^{k-1}+r_1A^k+r_2A^{k+1}+\cdots+r_{k-1}A^{2k-2}=0$$

Since $A^k=0$, all the higher powers are zero, so this becomes $r_0A^{k-1}=0$. But $A^{k-1}\neq 0$, so $r_0=0$, and we have $r_1A^1+r_2A^2+\cdots+r_{k-1}A^{k-1}=0$. Now multiply by A^{k-2} to conclude that $r_1=0$. Continuing, we obtain $r_i=0$ for each i, so B is independent.

The next example collects several useful properties of independence for reference.

Example 6.3.6

Let V denote a vector space.

1. If $\mathbf{v} \neq \mathbf{0}$ in V, then $\{\mathbf{v}\}$ is an independent set.
2. No independent set of vectors in V can contain the zero vector.

Solution.

1. Let $t\mathbf{v} = \mathbf{0}$, t in $\mathbb{R}$. If $t \neq 0$, then $\mathbf{v} = 1\mathbf{v} = \frac{1}{t}(t\mathbf{v}) = \frac{1}{t}\mathbf{0} = \mathbf{0}$, contrary to assumption. So $t = 0$.
2. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent and (say) $\mathbf{v}_2 = \mathbf{0}$, then $0\mathbf{v}_1 + 1\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$ is a nontrivial linear combination that vanishes, contrary to the independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

A set of vectors is independent if $\mathbf{0}$ is a linear combination in a unique way. The following theorem shows that *every* linear combination of these vectors has uniquely determined coefficients, and so extends Theorem 5.2.1.

Theorem 6.3.1

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a linearly independent set of vectors in a vector space V. If a vector $\mathbf{v}$ has two (ostensibly different) representations

$$\begin{aligned} \mathbf{v} &= s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n \\ \mathbf{v} &= t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_n\mathbf{v}_n \end{aligned}$$

as linear combinations of these vectors, then $s_1 = t_1, s_2 = t_2, \dots, s_n = t_n$. In other words, every vector in V can be written in a unique way as a linear combination of the $\mathbf{v}_i$.

Proof. Subtracting the equations given in the theorem gives

$$(s_1 - t_1)\mathbf{v}_1 + (s_2 - t_2)\mathbf{v}_2 + \cdots + (s_n - t_n)\mathbf{v}_n = \mathbf{0}$$

The independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ gives $s_i - t_i = 0$ for each i, as required. □

The following theorem extends (and proves) Theorem 5.2.4, and is one of the most useful results in linear algebra.

Theorem 6.3.2: Fundamental Theorem

can be spanned by n vectors. If any set of m vectors in V is linearly independent, then $m \leq n$.

Proof. Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, and suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an independent set in V. Then $\mathbf{u}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where each a_i is in $\mathbb{R}$. As $\mathbf{u}_1 \neq \mathbf{0}$ (Example 6.3.6), not all of the

a_i are zero, say $a_1 \neq 0$ (after relabelling the $\mathbf{v}_i$). Then $V = \text{span}\{\mathbf{u}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ as the reader can verify. Hence, write $\mathbf{u}_2 = b_1\mathbf{u}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n$. Then some $c_i \neq 0$ because $\{\mathbf{u}_1, \mathbf{u}_2\}$ is independent; so, as before, $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$, again after possible relabelling of the $\mathbf{v}_i$. If $m > n$, this procedure continues until all the vectors $\mathbf{v}_i$ are replaced by the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. In particular, $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. But then $\mathbf{u}_{n+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ contrary to the independence of the $\mathbf{u}_i$. Hence, the assumption $m > n$ cannot be valid, so $m \leq n$ and the theorem is proved. □

If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, and if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an independent set in V, the above proof shows not only that $m \leq n$ but also that m of the (spanning) vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ can be replaced by the (independent) vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ and the resulting set will still span V. In this form the result is called the **Steinitz Exchange Lemma**.

Definition 6.5 Basis of a Vector Space

*As in $\mathbb{R}^n$, a set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of vectors in a vector space V is called a **basis** of V if it satisfies the following two conditions:*

1. $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ *is linearly independent*
2. $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$

Thus if a set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis, then *every* vector in V can be written as a linear combination of these vectors in a *unique* way (Theorem 6.3.1). But even more is true: Any two (finite) bases of V contain the same number of vectors.

Theorem 6.3.3: Invariance Theorem

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ be two bases of a vector space V. Then $n = m$.

Proof. Because $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is independent, it follows from Theorem 6.3.2 that $m \leq n$. Similarly $n \leq m$, so $n = m$, as asserted. □

Theorem 6.3.3 guarantees that no matter which basis of V is chosen it contains the same number of vectors as any other basis. Hence there is no ambiguity about the following definition.

Definition 6.6 Dimension of a Vector Space

*If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis of the nonzero vector space V, the number n of vectors in the basis is called the **dimension** of V, and we write*

$$\dim V = n$$

The zero vector space $\{\mathbf{0}\}$ is defined to have dimension 0:

$$\dim\{\mathbf{0}\} = 0$$

In our discussion to this point we have always assumed that a basis is nonempty and hence that the dimension of the space is at least 1. However, the zero space $\{\mathbf{0}\}$ has *no* basis (by Example 6.3.6) so our insistence that $\dim\{\mathbf{0}\} = 0$ amounts to saying that the *empty* set of vectors is a basis of $\{\mathbf{0}\}$. Thus the statement that "the dimension of a vector space is the number of vectors in any basis" holds even for the zero space.

We saw in Example 5.2.9 that $\dim(\mathbb{R}^n) = n$ and, if $\mathbf{e}_j$ denotes column j of I_n, that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis (called the standard basis). In Example 6.3.7 below, similar considerations apply to the space $\mathbf{M}_{mn}$ of all $m \times n$ matrices; the verifications are left to the reader.

Example 6.3.7

The space $\mathbf{M}_{mn}$ has dimension mn, and one basis consists of all $m \times n$ matrices with exactly one entry equal to 1 and all other entries equal to 0. We call this the **standard basis** of $\mathbf{M}_{mn}$.

Example 6.3.8

Show that $\dim \mathbf{P}_n = n+1$ and that $\{1, x, x^2, \ldots, x^n\}$ is a basis, called the **standard basis** of $\mathbf{P}_n$.

Solution. Each polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ in $\mathbf{P}_n$ is clearly a linear combination of $1, x, \ldots, x^n$, so $\mathbf{P}_n = \text{span}\{1, x, \ldots, x^n\}$. However, if a linear combination of these vectors vanishes, $a_0 1 + a_1 x + \cdots + a_n x^n = 0$, then $a_0 = a_1 = \cdots = a_n = 0$ because x is an indeterminate. So $\{1, x, \ldots, x^n\}$ is linearly independent and hence is a basis containing $n+1$ vectors. Thus, $\dim(\mathbf{P}_n) = n+1$.

Example 6.3.9

If $\mathbf{v} \neq \mathbf{0}$ is any nonzero vector in a vector space V, show that $\text{span}\{\mathbf{v}\} = \mathbb{R}\mathbf{v}$ has dimension 1.

Solution. $\{\mathbf{v}\}$ clearly spans $\mathbb{R}\mathbf{v}$, and it is linearly independent by Example 6.3.6. Hence $\{\mathbf{v}\}$ is a basis of $\mathbb{R}\mathbf{v}$, and so $\dim \mathbb{R}\mathbf{v} = 1$.

Example 6.3.10

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and consider the subspace

$$U = \{X \text{ in } \mathbf{M}_{22} \mid AX = XA\}$$

of $\mathbf{M}_{22}$. Show that $\dim U = 2$ and find a basis of U.

Solution. It was shown in Example 6.2.3 that U is a subspace for any choice of the matrix A. In the present case, if $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is in U, the condition $AX = XA$ gives $z = 0$ and $x = y + w$. Hence each matrix X in U can be written

$$X = \begin{bmatrix} y+w & y \\ 0 & w \end{bmatrix} = y \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + w \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $U = \text{span}\, B$ where $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Moreover, the set B is linearly independent (verify this), so it is a basis of U and $\dim U = 2$.

Example 6.3.11

Show that the set V of all symmetric 2×2 matrices is a vector space, and find the dimension of V.

Solution. A matrix A is symmetric if $A^T = A$. If A and B lie in V, then

$$(A+B)^T = A^T + B^T = A + B \quad \text{and} \quad (kA)^T = kA^T = kA$$

using Theorem 2.1.2. Hence $A+B$ and kA are also symmetric. As the 2×2 zero matrix is also in V, this shows that V is a vector space (being a subspace of $\mathbf{M}_{22}$). Now a matrix A is symmetric when entries directly across the main diagonal are equal, so each 2×2 symmetric matrix has the form

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence the set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans V, and the reader can verify that B is linearly independent. Thus B is a basis of V, so $\dim V = 3$.

It is frequently convenient to alter a basis by multiplying each basis vector by a nonzero scalar. The next example shows that this always produces another basis. The proof is left as Exercise 6.3.22.

Example 6.3.12

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be nonzero vectors in a vector space V. Given nonzero scalars $a_1, a_2, \dots, a_n$, write $D = \{a_1\mathbf{v}_1, a_2\mathbf{v}_2, \dots, a_n\mathbf{v}_n\}$. If B is independent or spans V, the same is true of D. In particular, if B is a basis of V, so also is D.

Exercises for 6.3

Exercise 6.3.1 Show that each of the following sets of vectors is independent.

a. $\{1+x, 1-x, x+x^2\}$ in $\mathbf{P}_2$

b. $\{x^2, x+1, 1-x-x^2\}$ in $\mathbf{P}_2$

c. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathbf{M}_{22}$

d. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathbf{M}_{22}$

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b. If $ax^2 + b(x+1) + c(1-x-x^2) = 0$, then $a+c = 0$, $b-c = 0$, $b+c = 0$, so $a = b = c = 0$.

d. If $a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $a+c+d=0$, $a+b+d=0$, $a+b+c=0$, and $b+c+d=0$, so $a=b=c=d=0$.

Exercise 6.3.2 Which of the following subsets of V are independent?

a. $V=\mathbf{P}_2$; $\{x^2+1,\ x+1,\ x\}$

b. $V=\mathbf{P}_2$; $\{x^2-x+3,\ 2x^2+x+5,\ x^2+5x+1\}$

c. $V=\mathbf{M}_{22}$; $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

d. $V=\mathbf{M}_{22}$;
$\left\{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right\}$

e. $V=\mathbf{F}[1,\ 2]$; $\left\{\frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3}\right\}$

f. $V=\mathbf{F}[0,\ 1]$; $\left\{\frac{1}{x^2+x-6},\ \frac{1}{x^2-5x+6},\ \frac{1}{x^2-9}\right\}$

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b. $3(x^2-x+3)-2(2x^2+x+5)+(x^2+5x+1)=0$

d. $2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

f. $\frac{5}{x^2+x-6}+\frac{1}{x^2-5x+6}-\frac{6}{x^2-9}=0$

Exercise 6.3.3 Which of the following are independent in $\mathbf{F}[0,\ 2\pi]$?

a. $\{\sin^2 x,\ \cos^2 x\}$

b. $\{1,\ \sin^2 x,\ \cos^2 x\}$

c. $\{x,\ \sin^2 x,\ \cos^2 x\}$

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b. Dependent: $1-\sin^2 x-\cos^2 x=0$

Exercise 6.3.4 Find all values of a such that the following are independent in $\mathbb{R}^3$.

a. $\{(1,\ -1,\ 0),\ (a,\ 1,\ 0),\ (0,\ 2,\ 3)\}$

b. $\{(2,\ a,\ 1),\ (1,\ 0,\ 1),\ (0,\ 1,\ 3)\}$

---

b. $x \neq -\frac{1}{3}$

Exercise 6.3.5 Show that the following are bases of the space V indicated.

a. $\{(1,\ 1,\ 0),\ (1,\ 0,\ 1),\ (0,\ 1,\ 1)\}$; $V=\mathbb{R}^3$

b. $\{(-1,\ 1,\ 1),\ (1,\ -1,\ 1),\ (1,\ 1,\ -1)\}$; $V=\mathbb{R}^3$

c. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$; $V=\mathbf{M}_{22}$

d. $\{1+x,\ x+x^2,\ x^2+x^3,\ x^3\}$; $V=\mathbf{P}_3$

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b. If $r(-1,\ 1,\ 1)+s(1,\ -1,\ 1)+t(1,\ 1,\ -1)=(0,\ 0,\ 0)$, then $-r+s+t=0$, $r-s+t=0$, and $r-s-t=0$, and this implies that $r=s=t=0$. This proves independence. To prove that they span $\mathbb{R}^3$, observe that $(0,\ 0,\ 1)=\frac{1}{2}[(-1,\ 1,\ 1)+(1,\ -1,\ 1)]$ so $(0,\ 0,\ 1)$ lies in span $\{(-1,\ 1,\ 1),\ (1,\ -1,\ 1),\ (1,\ 1,\ -1)\}$. The proof is similar for $(0,\ 1,\ 0)$ and $(1,\ 0,\ 0)$.

d. If $r(1+x)+s(x+x^2)+t(x^2+x^3)+ux^3=0$, then $r=0$, $r+s=0$, $s+t=0$, and $t+u=0$, so $r=s=t=u=0$. This proves independence. To show that they span $\mathbf{P}_3$, observe that $x^2=(x^2+x^3)-x^3$, $x=(x+x^2)-x^2$, and $1=(1+x)-x$, so $\{1,\ x,\ x^2,\ x^3\}\subseteq$ span $\{1+x,\ x+x^2,\ x^2+x^3,\ x^3\}$.

Exercise 6.3.6 Exhibit a basis and calculate the dimension of each of the following subspaces of $\mathbf{P}_2$.

a. $\{a(1+x)+b(x+x^2)\mid a \text{ and } b \text{ in } \mathbb{R}\}$

b. $\{a+b(x+x^2)\mid a \text{ and } b \text{ in } \mathbb{R}\}$

c. $\{p(x)\mid p(1)=0\}$

d. $\{p(x)\mid p(x)=p(-x)\}$

---

b. $\{1,\ x+x^2\}$; dimension $=2$

d. $\{1,\ x^2\}$; dimension $=2$

Exercise 6.3.7 Exhibit a basis and calculate the dimension of each of the following subspaces of $\mathbf{M}_{22}$.

a. $\{A \mid A^T = -A\}$

b. $\left\{A \,\middle|\, A\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} A\right\}$

c. $\left\{A \,\middle|\, A\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

d. $\left\{A \,\middle|\, A\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} A\right\}$

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b. $\left\{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$; dimension $=2$

d. $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$; dimension $=2$

Exercise 6.3.8 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and define $U = \{X \mid X \in \mathbf{M}_{22} \text{ and } AX = X\}$.

a. Find a basis of U containing A.

b. Find a basis of U not containing A.

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b. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

Exercise 6.3.9 Show that the set $\mathbb{C}$ of all complex numbers is a vector space with the usual operations, and find its dimension.

Exercise 6.3.10

a. Let V denote the set of all 2×2 matrices with equal column sums. Show that V is a subspace of $\mathbf{M}_{22}$, and compute $\dim V$.

b. Repeat part (a) for 3×3 matrices.

c. Repeat part (a) for $n \times n$ matrices.

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b. $\dim V = 7$

Exercise 6.3.11

a. Let $V = \{(x^2+x+1)p(x) \mid p(x) \text{ in } \mathbf{P}_2\}$. Show that V is a subspace of $\mathbf{P}_4$ and find $\dim V$. [*Hint*: If $f(x)g(x) = 0$ in $\mathbf{P}$, then $f(x) = 0$ or $g(x) = 0$.]

b. Repeat with $V = \{(x^2-x)p(x) \mid p(x) \text{ in } \mathbf{P}_3\}$, a subset of $\mathbf{P}_5$.

c. Generalize.

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b. $\{x^2 - x,\ \ x(x^2 - x),\ \ x^2(x^2 - x),\ \ x^3(x^2 - x)\}$; $\dim V = 4$

Exercise 6.3.12 In each case, either prove the assertion or give an example showing that it is false.

a. Every set of four nonzero polynomials in $\mathbf{P}_3$ is a basis.

b. $\mathbf{P}_2$ has a basis of polynomials $f(x)$ such that $f(0) = 0$.

c. $\mathbf{P}_2$ has a basis of polynomials $f(x)$ such that $f(0) = 1$.

d. Every basis of $\mathbf{M}_{22}$ contains a noninvertible matrix.

e. No independent subset of $\mathbf{M}_{22}$ contains a matrix A with $A^2 = 0$.

f. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent then, $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$ for some a, b, c.

g. $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent if $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$ for some a, b, c.

h. If $\{\mathbf{u},\ \mathbf{v}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{u}+\mathbf{v}\}$.

i. If $\{\mathbf{u},\ \mathbf{v}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{v},\ \mathbf{u}+\mathbf{v}\}$.

j. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{v}\}$.

k. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, so is $\{\mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}$.

l. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, so is $\{\mathbf{u}+\mathbf{v}+\mathbf{w}\}$.

m. If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ then $\{\mathbf{u}, \mathbf{v}\}$ is dependent if and only if one is a scalar multiple of the other.

n. If $\dim V = n$, then no set of more than n vectors can be independent.

o. If $\dim V = n$, then no set of fewer than n vectors can span V.

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b. No. Any linear combination f of such polynomials has $f(0) = 0$.

d. No. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$; consists of invertible matrices.

f. Yes. $0\mathbf{u}+0\mathbf{v}+0\mathbf{w} = \mathbf{0}$ for every set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

h. Yes. $s\mathbf{u}+t(\mathbf{u}+\mathbf{v}) = \mathbf{0}$ gives $(s+t)\mathbf{u}+t\mathbf{v} = \mathbf{0}$, whence $s+t = 0 = t$.

j. Yes. If $r\mathbf{u}+s\mathbf{v} = \mathbf{0}$, then $r\mathbf{u}+s\mathbf{v}+0\mathbf{w} = \mathbf{0}$, so $r = 0 = s$.

l. Yes. $\mathbf{u}+\mathbf{v}+\mathbf{w} \neq \mathbf{0}$ because $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent.

n. Yes. If I is independent, then $|I| \leq n$ by the fundamental theorem because any basis spans V.

Exercise 6.3.13 Let $A \neq 0$ and $B \neq 0$ be $n \times n$ matrices, and assume that A is symmetric and B is skew-symmetric (that is, $B^T = -B$). Show that $\{A, B\}$ is independent.

Exercise 6.3.14 Show that every set of vectors containing a dependent set is again dependent.

Exercise 6.3.15 Show that every nonempty subset of an independent set of vectors is again independent.

If a linear combination of the subset vanishes, it is a linear combination of the vectors in the larger set (coefficients outside the subset are zero) so it is trivial.

Exercise 6.3.16 Let f and g be functions on $[a, b]$, and assume that $f(a) = 1 = g(b)$ and $f(b) = 0 = g(a)$. Show that $\{f, g\}$ is independent in $\mathbf{F}[a, b]$.

Exercise 6.3.17 Let $\{A_1, A_2, \dots, A_k\}$ be independent in $\mathbf{M}_{mn}$, and suppose that U and V are invertible matrices of size $m \times m$ and $n \times n$, respectively. Show that $\{UA_1V, UA_2V, \dots, UA_kV\}$ is independent.

Exercise 6.3.18 Show that $\{\mathbf{v}, \mathbf{w}\}$ is independent if and only if neither $\mathbf{v}$ nor $\mathbf{w}$ is a scalar multiple of the other.

Exercise 6.3.19 Assume that $\{\mathbf{u}, \mathbf{v}\}$ is independent in a vector space V. Write $\mathbf{u}' = a\mathbf{u}+b\mathbf{v}$ and $\mathbf{v}' = c\mathbf{u}+d\mathbf{v}$, where a, b, c, and d are numbers. Show that $\{\mathbf{u}', \mathbf{v}'\}$ is independent if and only if the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is invertible. [*Hint*: Theorem 2.4.5.]

Because $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, $s\mathbf{u}'+t\mathbf{v}' = \mathbf{0}$ is equivalent to $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Now apply Theorem 2.4.5.

Exercise 6.3.20 If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent and $\mathbf{w}$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, show that:

a. $\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent.

b. $\{\mathbf{v}_1+\mathbf{w}, \mathbf{v}_2+\mathbf{w}, \dots, \mathbf{v}_k+\mathbf{w}\}$ is independent.

Exercise 6.3.21 If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent, show that $\{\mathbf{v}_1, \mathbf{v}_1+\mathbf{v}_2, \dots, \mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_k\}$ is also independent.

Exercise 6.3.22 Prove Example 6.3.12.

Exercise 6.3.23 Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ be independent. Which of the following are dependent?

a. $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{w}-\mathbf{u}\}$

b. $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$

c. $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{w}-\mathbf{z}, \mathbf{z}-\mathbf{u}\}$

d. $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{z}, \mathbf{z}+\mathbf{u}\}$

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b. Independent.

d. Dependent. For example, $(\mathbf{u}+\mathbf{v})-(\mathbf{v}+\mathbf{w})+(\mathbf{w}+\mathbf{z})-(\mathbf{z}+\mathbf{u})=\mathbf{0}$.

Exercise 6.3.24 Let U and W be subspaces of V with bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ respectively. If U and W have only the zero vector in common, show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2\}$ is independent.

Exercise 6.3.25 Let $\{p, q\}$ be independent polynomials. Show that $\{p, q, pq\}$ is independent if and only if $\deg p \geq 1$ and $\deg q \geq 1$.

Exercise 6.3.26 If z is a complex number, show that $\{z, z^2\}$ is independent if and only if z is not real.

If z is not real and $az+bz^2=0$, then $a+bz=0 (z\neq 0)$. Hence if $b\neq 0$, then $z=-ab^{-1}$ is real. So $b=0$, and so $a=0$. Conversely, if z is real, say $z=a$, then $(-a)z+1z^2=0$, contrary to the independence of $\{z, z^2\}$.

Exercise 6.3.27 Let $B=\{A_1, A_2, \dots, A_n\}\subseteq \mathbf{M}_{mn}$, and write $B'=\{A_1^T, A_2^T, \dots, A_n^T\}\subseteq \mathbf{M}_{nm}$. Show that:

a. B is independent if and only if B' is independent.

b. B spans $\mathbf{M}_{mn}$ if and only if B' spans $\mathbf{M}_{nm}$.

Exercise 6.3.28 If $V=\mathbf{F}[a, b]$ as in Example 6.1.7, show that the set of constant functions is a subspace of dimension 1 (f is **constant** if there is a number c such that $f(x)=c$ for all x).

Exercise 6.3.29

a. If U is an invertible $n\times n$ matrix and $\{A_1, A_2, \dots, A_{mn}\}$ is a basis of $\mathbf{M}_{mn}$, show that $\{A_1U, A_2U, \dots, A_{mn}U\}$ is also a basis.

b. Show that part (a) fails if U is not invertible. [*Hint*: Theorem 2.4.5.]

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b. If $U\mathbf{x}=\mathbf{0}$, $\mathbf{x}\neq\mathbf{0}$ in $\mathbb{R}^n$, then $R\mathbf{x}=\mathbf{0}$ where $R\neq 0$ is row 1 of U. If $B\in\mathbf{M}_{mn}$ has each row equal to R, then $B\mathbf{x}\neq\mathbf{0}$. But if $B=\sum r_iA_iU$, then $B\mathbf{x}=\sum r_iA_iU\mathbf{x}=\mathbf{0}$. So $\{A_iU\}$ cannot span $\mathbf{M}_{mn}$.

Exercise 6.3.30 Show that $\{(a, b), (a_1, b_1)\}$ is a basis of $\mathbb{R}^2$ if and only if $\{a+bx, a_1+b_1x\}$ is a basis of $\mathbf{P}_1$.

Exercise 6.3.31 Find the dimension of the subspace $\text{span}\{1, \sin^2\theta, \cos 2\theta\}$ of $\mathbf{F}[0, 2\pi]$.

Exercise 6.3.32 Show that $\mathbf{F}[0, 1]$ is not finite dimensional.

Exercise 6.3.33 If U and W are subspaces of V, define their intersection $U\cap W$ as follows: $U\cap W=\{\mathbf{v}\mid \mathbf{v}\text{ is in both }U\text{ and }W\}$

a. Show that $U\cap W$ is a subspace contained in U and W.

b. Show that $U\cap W=\{\mathbf{0}\}$ if and only if $\{\mathbf{u}, \mathbf{w}\}$ is independent for any nonzero vectors $\mathbf{u}$ in U and $\mathbf{w}$ in W.

c. If B and D are bases of U and W, and if $U\cap W=\{\mathbf{0}\}$, show that $B\cup D=\{\mathbf{v}\mid \mathbf{v}\text{ is in }B\text{ or }D\}$ is independent.

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b. If $U\cap W=0$ and $r\mathbf{u}+s\mathbf{w}=\mathbf{0}$, then $r\mathbf{u}=-s\mathbf{w}$ is in $U\cap W$, so $r\mathbf{u}=\mathbf{0}=s\mathbf{w}$. Hence $r=0=s$ because $\mathbf{u}\neq\mathbf{0}\neq\mathbf{w}$. Conversely, if $\mathbf{v}\neq\mathbf{0}$ lies in $U\cap W$, then $1\mathbf{v}+(-1)\mathbf{v}=\mathbf{0}$, contrary to hypothesis.

Exercise 6.3.34 If U and W are vector spaces, let $V=\{(\mathbf{u}, \mathbf{w})\mid \mathbf{u}\text{ in }U\text{ and }\mathbf{w}\text{ in }W\}$.

a. Show that V is a vector space if $(\mathbf{u}, \mathbf{w})+(\mathbf{u}_1, \mathbf{w}_1)=(\mathbf{u}+\mathbf{u}_1, \mathbf{w}+\mathbf{w}_1)$ and $a(\mathbf{u}, \mathbf{w})=(a\mathbf{u}, a\mathbf{w})$.

b. If $\dim U=m$ and $\dim W=n$, show that $\dim V=m+n$.

c. If $V_1, \dots, V_m$ are vector spaces, let

$$\begin{aligned} V &= V_1\times\cdots\times V_m \\ &= \{(\mathbf{v}_1, \dots, \mathbf{v}_m)\mid \mathbf{v}_i\in V_i\text{ for each }i\} \end{aligned}$$

denote the space of n-tuples from the V_i with componentwise operations (see Exercise 6.1.17). If $\dim V_i=n_i$ for each i, show that $\dim V=n_1+\cdots+n_m$.

Exercise 6.3.35 Let $\mathbf{D}_n$ denote the set of all functions f from the set $\{1, 2, \dots, n\}$ to $\mathbb{R}$.

a. Show that $\mathbf{D}_n$ is a vector space with pointwise addition and scalar multiplication.

b. Show that $\{S_1, S_2, \dots, S_n\}$ is a basis of $\mathbf{D}_n$ where, for each $k = 1, 2, \dots, n$, the function S_k is defined by $S_k(k) = 1$, whereas $S_k(j) = 0$ if $j \neq k$.

Exercise 6.3.36 A polynomial $p(x)$ is called **even** if $p(-x) = p(x)$ and **odd** if $p(-x) = -p(x)$. Let E_n and O_n denote the sets of even and odd polynomials in $\mathbf{P}_n$.

a. Show that E_n is a subspace of $\mathbf{P}_n$ and find $\dim E_n$.

b. Show that O_n is a subspace of $\mathbf{P}_n$ and find $\dim O_n$.

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b. $\dim O_n = \frac{n}{2}$ if n is even and $\dim O_n = \frac{n+1}{2}$ if n is odd.

Exercise 6.3.37 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be independent in a vector space V, and let A be an $n \times n$ matrix. Define $\mathbf{u}_1, \dots, \mathbf{u}_n$ by

$$\begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = A \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$$

(See Exercise 6.1.18.) Show that $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is independent if and only if A is invertible.