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LINEAR ALGEBRA with Applications

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Adapted for

Emory University

Math 221

Linear Algebra

Sections **1 & 2**

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Course page

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Contents

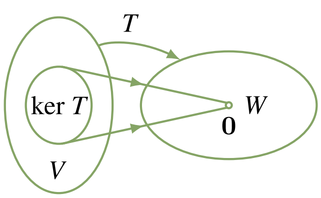
7.2 Kernel and Image of a Linear Transformation

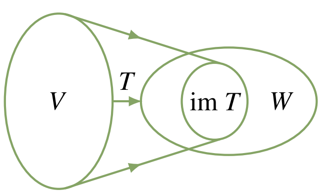
This section is devoted to two important subspaces associated with a linear transformation $T : V \to W$.

Definition 7.2 Kernel and Image of a Linear Transformation

*The **kernel** of T (denoted $\ker T$) and the **image** of T (denoted $\operatorname{im} T$ or $T(V)$) are defined by*

$$\begin{aligned} \ker T &= \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\} \\ \operatorname{im} T &= \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V) \end{aligned}$$





The kernel of T is often called the **nullspace** of T because it consists of all vectors $\mathbf{v}$ in V satisfying the *condition* that $T(\mathbf{v}) = \mathbf{0}$. The image of T is often called the **range** of T and consists of all vectors $\mathbf{w}$ in W of the *form* $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. These subspaces are depicted in the diagrams.

Example 7.2.1

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by the $m \times n$ matrix A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. Then

$$\begin{aligned} \ker T_A &= \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} = \operatorname{null} A \quad \text{and} \\ \operatorname{im} T_A &= \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \operatorname{im} A \end{aligned}$$

Hence the following theorem extends Example 5.1.2.

Theorem 7.2.1

Let $T : V \to W$ be a linear transformation.

1. $\ker T$ *is a subspace of* V.
2. $\operatorname{im} T$ *is a subspace of* W.

Proof. The fact that $T(\mathbf{0}) = \mathbf{0}$ shows that $\ker T$ and $\operatorname{im} T$ contain the zero vector of V and W respectively.

1. If $\mathbf{v}$ and $\mathbf{v}_1$ lie in $\ker T$, then $T(\mathbf{v}) = \mathbf{0} = T(\mathbf{v}_1)$, so

$$\begin{aligned} T(\mathbf{v} + \mathbf{v}_1) &= T(\mathbf{v}) + T(\mathbf{v}_1) = \mathbf{0} + \mathbf{0} = \mathbf{0} \\ T(r\mathbf{v}) &= rT(\mathbf{v}) = r\mathbf{0} = \mathbf{0} \quad \text{for all } r \text{ in } \mathbb{R} \end{aligned}$$

Hence $\mathbf{v}+\mathbf{v}_1$ and $r\mathbf{v}$ lie in $\ker T$ (they satisfy the required condition), so $\ker T$ is a subspace of V by the subspace test (Theorem 6.2.1).

2. If $\mathbf{w}$ and $\mathbf{w}_1$ lie in $\operatorname{im} T$, write $\mathbf{w} = T(\mathbf{v})$ and $\mathbf{w}_1 = T(\mathbf{v}_1)$ where $\mathbf{v},\ \mathbf{v}_1 \in V$. Then

$$\begin{aligned} \mathbf{w}+\mathbf{w}_1 &= T(\mathbf{v})+T(\mathbf{v}_1) = T(\mathbf{v}+\mathbf{v}_1) \\ r\mathbf{w} &= rT(\mathbf{v}) = T(r\mathbf{v}) \quad \text{for all } r \text{ in } \mathbb{R} \end{aligned}$$

Hence $\mathbf{w}+\mathbf{w}_1$ and $r\mathbf{w}$ both lie in $\operatorname{im} T$ (they have the required form), so $\operatorname{im} T$ is a subspace of W.

□

Given a linear transformation $T : V \to W$:

$\dim(\ker T)$ is called the **nullity** of T and denoted as $\operatorname{nullity}(T)$
$\dim(\operatorname{im} T)$ is called the **rank** of T and denoted as $\operatorname{rank}(T)$

The rank of a matrix A was defined earlier to be the dimension of $\operatorname{col} A$, the column space of A. The two usages of the word *rank* are consistent in the following sense. Recall the definition of T_A in Example 7.2.1.

Example 7.2.2

Given an $m \times n$ matrix A, show that $\operatorname{im} T_A = \operatorname{col} A$, so $\operatorname{rank} T_A = \operatorname{rank} A$.

Solution. Write $A = \left[\begin{array}{ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{array}\right]$ in terms of its columns. Then

$$\operatorname{im} T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \{x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n \mid x_i \text{ in } \mathbb{R}\}$$

using Definition 2.5. Hence $\operatorname{im} T_A$ is the column space of A; the rest follows.

Often, a useful way to study a subspace of a vector space is to exhibit it as the kernel or image of a linear transformation. Here is an example.

Example 7.2.3

Define a transformation $P : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ by $P(A) = A - A^T$ for all A in $\mathbf{M}_{nn}$. Show that P is linear and that:

a. $\ker P$ consists of all symmetric matrices.

b. $\operatorname{im} P$ consists of all skew-symmetric matrices.

Solution. The verification that P is linear is left to the reader. To prove part (a), note that a matrix A lies in $\ker P$ just when $0 = P(A) = A - A^T$, and this occurs if and only if $A = A^T$—that is, A is symmetric. Turning to part (b), the space $\operatorname{im} P$ consists of all matrices $P(A)$, A in $\mathbf{M}_{nn}$. Every such matrix is skew-symmetric because

$$P(A)^T = (A - A^T)^T = A^T - A = -P(A)$$

On the other hand, if S is skew-symmetric (that is, $S^T = -S$), then S lies in $\operatorname{im} P$. In fact,

$$P\left[\tfrac{1}{2}S\right] = \tfrac{1}{2}S - \left[\tfrac{1}{2}S\right]^T = \tfrac{1}{2}(S - S^T) = \tfrac{1}{2}(S + S) = S$$

One-to-One and Onto Transformations

Definition 7.3 One-to-one and Onto Linear Transformations

Let $T : V \to W$ be a linear transformation.

1. *T is said to be* **onto** *if* $\operatorname{im} T = W$.
2. *T is said to be* **one-to-one** *if $T(\mathbf{v}) = T(\mathbf{v}_1)$ implies $\mathbf{v} = \mathbf{v}_1$.*

A vector $\mathbf{w}$ in W is said to be **hit** by T if $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. Then T is onto if every vector in W is hit at least once, and T is one-to-one if no element of W gets hit twice. Clearly the onto transformations T are those for which $\operatorname{im} T = W$ is as large a subspace of W as possible. By contrast, Theorem 7.2.2 shows that the one-to-one transformations T are the ones with $\ker T$ as *small* a subspace of V as possible.

Theorem 7.2.2

If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Proof. If T is one-to-one, let $\mathbf{v}$ be any vector in $\ker T$. Then $T(\mathbf{v}) = \mathbf{0}$, so $T(\mathbf{v}) = T(\mathbf{0})$. Hence $\mathbf{v} = \mathbf{0}$ because T is one-to-one. Hence $\ker T = \{\mathbf{0}\}$.

Conversely, assume that $\ker T = \{\mathbf{0}\}$ and let $T(\mathbf{v}) = T(\mathbf{v}_1)$ with $\mathbf{v}$ and $\mathbf{v}_1$ in V. Then $T(\mathbf{v} - \mathbf{v}_1) = T(\mathbf{v}) - T(\mathbf{v}_1) = \mathbf{0}$, so $\mathbf{v} - \mathbf{v}_1$ lies in $\ker T = \{\mathbf{0}\}$. This means that $\mathbf{v} - \mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v} = \mathbf{v}_1$, proving that T is one-to-one. □

Example 7.2.4

The identity transformation $1_V : V \to V$ is both one-to-one and onto for any vector space V.

Example 7.2.5

Consider the linear transformations

$$\begin{aligned} S &: \mathbb{R}^3 \to \mathbb{R}^2 \quad \text{given by } S(x,\ y,\ z) = (x + y,\ x - y) \\ T &: \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{given by } T(x,\ y) = (x + y,\ x - y,\ x) \end{aligned}$$

Show that T is one-to-one but not onto, whereas S is onto but not one-to-one.

Solution. The verification that they are linear is omitted. T is one-to-one because

$$\ker T = \{(x,\ y) \mid x+y = x-y = x = 0\} = \{(0,\ 0)\}$$

However, it is not onto. For example $(0,\ 0,\ 1)$ does not lie in $\operatorname{im} T$ because if $(0,\ 0,\ 1) = (x+y,\ x-y,\ x)$ for some x and y, then $x+y = 0 = x-y$ and $x = 1$, an impossibility. Turning to S, it is not one-to-one by Theorem 7.2.2 because $(0,\ 0,\ 1)$ lies in $\ker S$. But every element $(s,\ t)$ in $\mathbb{R}^2$ lies in $\operatorname{im} S$ because $(s,\ t) = (x+y,\ x-y) = S(x,\ y,\ z)$ for some x, y, and z (in fact, $x = \frac{1}{2}(s+t)$, $y = \frac{1}{2}(s-t)$, and $z = 0$). Hence S is onto.

Example 7.2.6

Let U be an invertible $m \times m$ matrix and define

$$T : \mathbf{M}_{mn} \to \mathbf{M}_{mn} \quad \text{by} \quad T(X) = UX \text{ for all } X \text{ in } \mathbf{M}_{mn}$$

Show that T is a linear transformation that is both one-to-one and onto.

Solution. The verification that T is linear is left to the reader. To see that T is one-to-one, let $T(X) = 0$. Then $UX = 0$, so left-multiplication by U^{-1} gives $X = 0$. Hence $\ker T = \{\mathbf{0}\}$, so T is one-to-one. Finally, if Y is any member of $\mathbf{M}_{mn}$, then $U^{-1}Y$ lies in $\mathbf{M}_{mn}$ too, and $T(U^{-1}Y) = U(U^{-1}Y) = Y$. This shows that T is onto.

The linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ all have the form T_A for some $m \times n$ matrix A (Theorem 2.6.2). The next theorem gives conditions under which they are onto or one-to-one. Note the connection with Theorem 5.4.3 and Theorem 5.4.4.

Theorem 7.2.3

Let A be an $m \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$.

1. *T_A is onto if and only if* $\operatorname{rank} A = m$.
2. *T_A is one-to-one if and only if* $\operatorname{rank} A = n$.

Proof.

1. We have that $\operatorname{im} T_A$ is the column space of A (see Example 7.2.2), so T_A is onto if and only if the column space of A is $\mathbb{R}^m$. Because the rank of A is the dimension of the column space, this holds if and only if $\operatorname{rank} A = m$.
2. $\ker T_A = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, so (using Theorem 7.2.2) T_A is one-to-one if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. This is equivalent to $\operatorname{rank} A = n$ by Theorem 5.4.3. □

The Dimension Theorem

Let A denote an $m \times n$ matrix of rank r and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ denote the corresponding matrix transformation given by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. It follows from Example 7.2.1 and Example 7.2.2 that $\text{im } T_A = \text{col } A$, so $\dim(\text{im } T_A) = \dim(\text{col } A) = r$. On the other hand Theorem 5.4.2 shows that $\dim(\ker T_A) = \dim(\text{null } A) = n - r$. Combining these we see that

$$\dim(\text{im } T_A) + \dim(\ker T_A) = n \quad \text{for every } m \times n \text{ matrix } A$$

The main result of this section is a deep generalization of this observation.

Theorem 7.2.4: Dimension Theorem

Let $T : V \to W$ be any linear transformation and assume that $\ker T$ and $\text{im } T$ are both finite dimensional. Then V is also finite dimensional and

$$\dim V = \dim(\ker T) + \dim(\text{im } T)$$

In other words, $\dim V = \text{nullity}(T) + \text{rank}(T)$.

Proof. Every vector in $\text{im } T = T(V)$ has the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V. Hence let $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_r)\}$ be a basis of $\text{im } T$, where the $\mathbf{e}_i$ lie in V. Let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k\}$ be any basis of $\ker T$. Then $\dim(\text{im } T) = r$ and $\dim(\ker T) = k$, so it suffices to show that $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{f}_1, \dots, \mathbf{f}_k\}$ is a basis of V.

1. B *spans* V. If $\mathbf{v}$ lies in V, then $T(\mathbf{v})$ lies in $\text{im } V$, so

$$T(\mathbf{v}) = t_1T(\mathbf{e}_1) + t_2T(\mathbf{e}_2) + \cdots + t_rT(\mathbf{e}_r) \quad t_i \text{ in } \mathbb{R}$$

This implies that $\mathbf{v} - t_1\mathbf{e}_1 - t_2\mathbf{e}_2 - \cdots - t_r\mathbf{e}_r$ lies in $\ker T$ and so is a linear combination of $\mathbf{f}_1, \dots, \mathbf{f}_k$. Hence $\mathbf{v}$ is a linear combination of the vectors in B.

2. B is *linearly independent*. Suppose that t_i and s_j in $\mathbb{R}$ satisfy

$$t_1\mathbf{e}_1 + \cdots + t_r\mathbf{e}_r + s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0} \tag{7.1}$$

Applying T gives $t_1T(\mathbf{e}_1) + \cdots + t_rT(\mathbf{e}_r) = \mathbf{0}$ (because $T(\mathbf{f}_i) = \mathbf{0}$ for each i). Hence the independence of $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ yields $t_1 = \cdots = t_r = 0$. But then (7.1) becomes

$$s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0}$$

so $s_1 = \cdots = s_k = 0$ by the independence of $\{\mathbf{f}_1, \dots, \mathbf{f}_k\}$. This proves that B is linearly independent.

□

Note that the vector space V is not assumed to be finite dimensional in Theorem 7.2.4. In fact, verifying that $\ker T$ and $\text{im } T$ are both finite dimensional is often an important way to *prove* that V is finite dimensional.

Note further that $r+k=n$ in the proof so, after relabelling, we end up with a basis

$$B = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_r,\ \mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$$

of V with the property that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$ and $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T$. In fact, if V is known in advance to be finite dimensional, then *any* basis $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ of $\ker T$ can be extended to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ of V by Theorem 6.4.1. Moreover, it turns out that, no matter how this is done, the vectors $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ will be a basis of $\operatorname{im} T$. This result is useful, and we record it for reference. The proof is much like that of Theorem 7.2.4 and is left as Exercise 7.2.26.

Theorem 7.2.5

Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$. Then $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T$, and hence $r = \operatorname{rank} T$.

The dimension theorem is one of the most useful results in all of linear algebra. It shows that if either $\dim(\ker T)$ or $\dim(\operatorname{im} T)$ can be found, then the other is automatically known. In many cases it is easier to compute one than the other, so the theorem is a real asset. The rest of this section is devoted to illustrations of this fact. The next example uses the dimension theorem to give a different proof of the first part of Theorem 5.4.2.

Example 7.2.7

Let A be an $m \times n$ matrix of $\operatorname{rank} r$. Show that the space $\operatorname{null} A$ of all solutions of the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n variables has dimension $n - r$.

Solution. The space in question is just $\ker T_A$, where $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. But $\dim(\operatorname{im} T_A) = \operatorname{rank} T_A = \operatorname{rank} A = r$ by Example 7.2.2, so $\dim(\ker T_A) = n - r$ by the dimension theorem.

Example 7.2.8

If $T : V \to W$ is a linear transformation where V is finite dimensional, then

$$\dim(\ker T) \leq \dim V \quad \text{and} \quad \dim(\operatorname{im} T) \leq \dim V$$

Indeed, $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$ by Theorem 7.2.4. Of course, the first inequality also follows because $\ker T$ is a subspace of V.

Example 7.2.9

Let $D : \mathbf{P}_n \to \mathbf{P}_{n-1}$ be the differentiation map defined by $D[p(x)] = p'(x)$. Compute $\ker D$ and hence conclude that D is onto.

Solution. Because $p'(x) = 0$ means $p(x)$ is constant, we have $\dim(\ker D) = 1$. Since $\dim \mathbf{P}_n = n+1$, the dimension theorem gives

$$\dim(\operatorname{im} D) = (n+1) - \dim(\ker D) = n = \dim(\mathbf{P}_{n-1})$$

This implies that $\operatorname{im} D = \mathbf{P}_{n-1}$, so D is onto.

Of course it is not difficult to verify directly that each polynomial $q(x)$ in $\mathbf{P}_{n-1}$ is the derivative of some polynomial in $\mathbf{P}_n$ (simply integrate $q(x)$!), so the dimension theorem is not needed in this case. However, in some situations it is difficult to see directly that a linear transformation is onto, and the method used in Example 7.2.9 may be by far the easiest way to prove it. Here is another illustration.

Example 7.2.10

Given a in $\mathbb{R}$, the evaluation map $E_a : \mathbf{P}_n \to \mathbb{R}$ is given by $E_a[p(x)] = p(a)$. Show that E_a is linear and onto, and hence conclude that $\{(x-a), (x-a)^2, \dots, (x-a)^n\}$ is a basis of $\ker E_a$, the subspace of all polynomials $p(x)$ for which $p(a) = 0$.

Solution. E_a is linear by Example 7.1.3; the verification that it is onto is left to the reader. Hence $\dim(\operatorname{im} E_a) = \dim(\mathbb{R}) = 1$, so $\dim(\ker E_a) = (n+1) - 1 = n$ by the dimension theorem. Now each of the n polynomials $(x-a), (x-a)^2, \dots, (x-a)^n$ clearly lies in $\ker E_a$, and they are linearly independent (they have distinct degrees). Hence they are a basis because $\dim(\ker E_a) = n$.

We conclude by applying the dimension theorem to the rank of a matrix.

Example 7.2.11

If A is any $m \times n$ matrix, show that $\operatorname{rank} A = \operatorname{rank} A^TA = \operatorname{rank} AA^T$.

Solution. It suffices to show that $\operatorname{rank} A = \operatorname{rank} A^TA$ (the rest follows by replacing A with A^T). Write $B = A^TA$, and consider the associated matrix transformations

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{and} \quad T_B : \mathbb{R}^n \to \mathbb{R}^n$$

The dimension theorem and Example 7.2.2 give

$$\begin{aligned}\operatorname{rank} A &= \operatorname{rank} T_A = \dim(\operatorname{im} T_A) = n - \dim(\ker T_A)\\ \operatorname{rank} B &= \operatorname{rank} T_B = \dim(\operatorname{im} T_B) = n - \dim(\ker T_B)\end{aligned}$$

so it suffices to show that $\ker T_A = \ker T_B$. Now $A\mathbf{x} = \mathbf{0}$ implies that $B\mathbf{x} = A^TA\mathbf{x} = \mathbf{0}$, so $\ker T_A$ is contained in $\ker T_B$. On the other hand, if $B\mathbf{x} = \mathbf{0}$, then $A^TA\mathbf{x} = \mathbf{0}$, so

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0$$

This implies that $A\mathbf{x} = \mathbf{0}$, so $\ker T_B$ is contained in $\ker T_A$.

Exercises for 7.2

Exercise 7.2.1 For each matrix A, find a basis for the kernel and image of T_A, and find the rank and nullity of T_A.

a) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & -4 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \\ 0 & 3 & -6 \end{bmatrix}$

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b. $\left\{ \begin{bmatrix} -3 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$; 2, 2

d. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\}$; 2, 1

Exercise 7.2.2 In each case, (i) find a basis of $\ker T$, and (ii) find a basis of $\operatorname{im} T$. You may assume that T is linear.

a. $T : \mathbf{P}_2 \to \mathbb{R}^2$; $T(a + bx + cx^2) = (a, b)$

b. $T : \mathbf{P}_2 \to \mathbb{R}^2$; $T(p(x)) = (p(0), p(1))$

c. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x + y, x + y, 0)$

d. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x, x, y, y)$

e. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix}$

f. $T : \mathbf{M}_{22} \to \mathbb{R}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$

g. $T : \mathbf{P}_n \to \mathbb{R}$; $T(r_0 + r_1 x + \cdots + r_n x^n) = r_n$

h. $T : \mathbb{R}^n \to \mathbb{R}$; $T(r_1, r_2, \ldots, r_n) = r_1 + r_2 + \cdots + r_n$

i. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = XA - AX$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

j. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = XA$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

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b. $\{x^2 - x\}$; $\{(1, 0), (0, 1)\}$

d. $\{(0, 0, 1)\}$; $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$

f. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$; $\{1\}$

h. $\{(1, 0, 0, \ldots, 0, -1), (0, 1, 0, \ldots, 0, -1), \ldots, (0, 0, 0, \ldots, 1, -1)\}$; $\{1\}$

j. $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$;
$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

Exercise 7.2.3 Let $P : V \to \mathbb{R}$ and $Q : V \to \mathbb{R}$ be linear transformations, where V is a vector space. Define $T : V \to \mathbb{R}^2$ by $T(\mathbf{v}) = (P(\mathbf{v}), Q(\mathbf{v}))$.

a. Show that T is a linear transformation.

b. Show that $\ker T = \ker P \cap \ker Q$, the set of vectors in both $\ker P$ and $\ker Q$.

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b. $T(\mathbf{v}) = \mathbf{0} = (0, 0)$ if and only if $P(\mathbf{v}) = 0$ and $Q(\mathbf{v}) = 0$; that is, if and only if $\mathbf{v}$ is in $\ker P \cap \ker Q$.

Exercise 7.2.4 In each case, find a basis $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ of V such that $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ is a basis of $\ker T$, and verify Theorem 7.2.5.

a. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x - y + 2z, x + y - z, 2x + z, 2y - 3z)$

b. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x + y + z, 2x - y + 3z, z - 3y, 3x + 4z)$

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b. $\ker T = \operatorname{span}\{(-4,\ 1,\ 3)\}$; $B = \{(1,\ 0,\ 0),\ (0,\ 1,\ 0),\ (-4,\ 1,\ 3)\}$, $\operatorname{im} T = \operatorname{span}\{(1,\ 2,\ 0,\ 3),\ (1,\ -1,\ -3,\ 0)\}$

Exercise 7.2.5 Show that every matrix X in $\mathbf{M}_{nn}$ has the form $X = A^T - 2A$ for some matrix A in $\mathbf{M}_{nn}$. [*Hint*: The dimension theorem.]

Exercise 7.2.6 In each case either prove the statement or give an example in which it is false. Throughout, let $T : V \to W$ be a linear transformation where V and W are finite dimensional.

a. If $V = W$, then $\ker T \subseteq \operatorname{im} T$.

b. If $\dim V = 5$, $\dim W = 3$, and $\dim(\ker T) = 2$, then T is onto.

c. If $\dim V = 5$ and $\dim W = 4$, then $\ker T \neq \{\mathbf{0}\}$.

d. If $\ker T = V$, then $W = \{\mathbf{0}\}$.

e. If $W = \{\mathbf{0}\}$, then $\ker T = V$.

f. If $W = V$, and $\operatorname{im} T \subseteq \ker T$, then $T = 0$.

g. If $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ is a basis of V and $T(\mathbf{e}_1) = \mathbf{0} = T(\mathbf{e}_2)$, then $\dim(\operatorname{im} T) \leq 1$.

h. If $\dim(\ker T) \leq \dim W$, then $\dim W \geq \frac{1}{2}\dim V$.

i. If T is one-to-one, then $\dim V \leq \dim W$.

j. If $\dim V \leq \dim W$, then T is one-to-one.

k. If T is onto, then $\dim V \geq \dim W$.

l. If $\dim V \geq \dim W$, then T is onto.

m. If $\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_k)\}$ is independent, then $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_k\}$ is independent.

n. If $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_k\}$ spans V, then $\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_k)\}$ spans W.

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b. Yes. $\dim(\operatorname{im} T) = 5 - \dim(\ker T) = 3$, so $\operatorname{im} T = W$ as $\dim W = 3$.

d. No. $T = 0 : \mathbb{R}^2 \to \mathbb{R}^2$

f. No. $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,\ y) = (y,\ 0)$. Then $\ker T = \operatorname{im} T$

h. Yes. $\dim V = \dim(\ker T) + \dim(\operatorname{im} T) \leq \dim W + \dim W = 2\dim W$

j. No. Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x,\ y) = (y,\ 0)$.

l. No. Same example as **(j)**.

n. No. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,\ y) = (x,\ 0)$. If $\mathbf{v}_1 = (1,\ 0)$ and $\mathbf{v}_2 = (0,\ 1)$, then $\mathbb{R}^2 = \operatorname{span}\{\mathbf{v}_1,\ \mathbf{v}_2\}$ but $\mathbb{R}^2 \neq \operatorname{span}\{T(\mathbf{v}_1),\ T(\mathbf{v}_2)\}$.

Exercise 7.2.7 Show that linear independence is preserved by one-to-one transformations and that spanning sets are preserved by onto transformations. More precisely, if $T : V \to W$ is a linear transformation, show that:

a. If T is one-to-one and $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ is independent in V, then $\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_n)\}$ is independent in W.

b. If T is onto and $V = \operatorname{span}\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$, then $W = \operatorname{span}\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_n)\}$.

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b. Given $\mathbf{w}$ in W, let $\mathbf{w} = T(\mathbf{v})$, $\mathbf{v}$ in V, and write $\mathbf{v} = r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n$. Then $\mathbf{w} = T(\mathbf{v}) = r_1T(\mathbf{v}_1) + \cdots + r_nT(\mathbf{v}_n)$.

Exercise 7.2.8 Given $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ in a vector space V, define $T : \mathbb{R}^n \to V$ by $T(r_1,\ \dots,\ r_n) = r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n$. Show that T is linear, and that:

a. T is one-to-one if and only if $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ is independent.

b. T is onto if and only if $V = \operatorname{span}\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$.

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b. $\operatorname{im} T = \{\sum_i r_i\mathbf{v}_i \mid r_i \text{ in } \mathbb{R}\} = \operatorname{span}\{\mathbf{v}_i\}$.

Exercise 7.2.9 Let $T : V \to V$ be a linear transformation where V is finite dimensional. Show that exactly one of (i) and (ii) holds: (i) $T(\mathbf{v}) = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$ in V; (ii) $T(\mathbf{x}) = \mathbf{v}$ has a solution $\mathbf{x}$ in V for every $\mathbf{v}$ in V.

Exercise 7.2.10 Let $T : \mathbf{M}_{nn} \to \mathbb{R}$ denote the trace map: $T(A) = \text{tr}\, A$ for all A in $\mathbf{M}_{nn}$. Show that $\dim(\ker T) = n^2 - 1$.

T is linear and onto. Hence $1 = \dim \mathbb{R} = \dim(\text{im}\, T) = \dim(\mathbf{M}_{nn}) - \dim(\ker T) = n^2 - \dim(\ker T)$.

Exercise 7.2.11 Show that the following are equivalent for a linear transformation $T : V \to W$.

1. $\ker T = V$
2. $\text{im}\, T = \{\mathbf{0}\}$
3. $T = 0$

Exercise 7.2.12 Let A and B be $m \times n$ and $k \times n$ matrices, respectively. Assume that $A\mathbf{x} = \mathbf{0}$ implies $B\mathbf{x} = \mathbf{0}$ for every n-column $\mathbf{x}$. Show that $\text{rank}\, A \geq \text{rank}\, B$.
[*Hint*: Theorem 7.2.4.]

The condition means $\ker(T_A) \subseteq \ker(T_B)$, so $\dim[\ker(T_A)] \leq \dim[\ker(T_B)]$. Then Theorem 7.2.4 gives $\dim[\text{im}(T_A)] \geq \dim[\text{im}(T_B)]$; that is, $\text{rank}\, A \geq \text{rank}\, B$.

Exercise 7.2.13 Let A be an $m \times n$ matrix of rank r. Thinking of $\mathbb{R}^n$ as rows, define $V = \{\mathbf{x} \text{ in } \mathbb{R}^m \mid \mathbf{x}A = \mathbf{0}\}$. Show that $\dim V = m - r$.

Exercise 7.2.14 Consider

$$V = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \middle| a + c = b + d \right\}$$

a. Consider $S : \mathbf{M}_{22} \to \mathbb{R}$ with $S \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = a + c - b - d$. Show that S is linear and onto and that V is a subspace of $\mathbf{M}_{22}$. Compute $\dim V$.

b. Consider $T : V \to \mathbb{R}$ with $T \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = a + c$. Show that T is linear and onto, and use this information to compute $\dim(\ker T)$.

Exercise 7.2.15 Define $T : \mathbf{P}_n \to \mathbb{R}$ by $T[p(x)] =$ the sum of all the coefficients of $p(x)$.

a. Use the dimension theorem to show that $\dim(\ker T) = n$.

b. Conclude that $\{x - 1,\ x^2 - 1,\ \dots,\ x^n - 1\}$ is a basis of $\ker T$.

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b. $B = \{x - 1, \dots, x^n - 1\}$ is independent (distinct degrees) and contained in $\ker T$. Hence B is a basis of $\ker T$ by **(a)**.

Exercise 7.2.16 Use the dimension theorem to prove Theorem 1.3.1: If A is an $m \times n$ matrix with $m < n$, the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n variables always has a nontrivial solution.

Exercise 7.2.17 Let B be an $n \times n$ matrix, and consider the subspaces $U = \{A \mid A \text{ in } \mathbf{M}_{mn},\ AB = 0\}$ and $V = \{AB \mid A \text{ in } \mathbf{M}_{mn}\}$. Show that $\dim U + \dim V = mn$.

Exercise 7.2.18 Let U and V denote, respectively, the spaces of even and odd polynomials in $\mathbf{P}_n$. Show that $\dim U + \dim V = n + 1$. [*Hint*: Consider $T : \mathbf{P}_n \to \mathbf{P}_n$ where $T[p(x)] = p(x) - p(-x)$.]

Exercise 7.2.19 Show that every polynomial $f(x)$ in $\mathbf{P}_{n-1}$ can be written as $f(x) = p(x+1) - p(x)$ for some polynomial $p(x)$ in $\mathbf{P}_n$. [*Hint*: Define $T : \mathbf{P}_n \to \mathbf{P}_{n-1}$ by $T[p(x)] = p(x+1) - p(x)$.]

Exercise 7.2.20 Let U and V denote the spaces of symmetric and skew-symmetric $n \times n$ matrices. Show that $\dim U + \dim V = n^2$.

Define $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ by $T(A) = A - A^T$ for all A in $\mathbf{M}_{nn}$. Then $\ker T = U$ and $\text{im}\, T = V$ by Example 7.2.3, so the dimension theorem gives $n^2 = \dim \mathbf{M}_{nn} = \dim(U) + \dim(V)$.

Exercise 7.2.21 Assume that B in $\mathbf{M}_{nn}$ satisfies $B^k = 0$ for some $k \geq 1$. Show that every matrix in $\mathbf{M}_{nn}$ has the form $BA - A$ for some A in $\mathbf{M}_{nn}$. [*Hint*: Show that $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ is linear and one-to-one where
$T(A) = BA - A$ for each A.]

Exercise 7.2.22 Fix a column $\mathbf{y} \neq \mathbf{0}$ in $\mathbb{R}^n$ and let $U = \{A \text{ in } \mathbf{M}_{nn} \mid A\mathbf{y} = \mathbf{0}\}$. Show that $\dim U = n(n-1)$.

Define $T : \mathbf{M}_{nn} \to \mathbb{R}^n$ by $T(A) = A\mathbf{y}$ for all A in $\mathbf{M}_{nn}$. Then T is linear with $\ker T = U$, so it is enough to show that T is onto (then $\dim U = n^2 - \dim(\text{im}\, T) = n^2 - n$). We have $T(0) = \mathbf{0}$. Let $\mathbf{y} = \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \right]^T \neq \mathbf{0}$ in $\mathbb{R}^n$. If $y_k \neq \mathbf{0}$

let $\mathbf{c}_k = y_k^{-1}\mathbf{y}$, and let $\mathbf{c}_j = \mathbf{0}$ if $j \neq k$. If $A = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right]$, then $T(A) = A\mathbf{y} = y_1\mathbf{c}_1 + \cdots + y_k\mathbf{c}_k + \cdots + y_n\mathbf{c}_n = \mathbf{y}$. This shows that T is onto, as required.

Exercise 7.2.23 If B in $\mathbf{M}_{mn}$ has rank r, let $U = \{A$ in $\mathbf{M}_{nn} \mid BA = 0\}$ and $W = \{BA \mid A$ in $\mathbf{M}_{nn}\}$. Show that $\dim U = n(n-r)$ and $\dim W = nr$. [*Hint*: Show that U consists of all matrices A whose columns are in the null space of B. Use Example 7.2.7.]

Exercise 7.2.24 Let $T : V \to V$ be a linear transformation where $\dim V = n$. If $\ker T \cap \operatorname{im} T = \{\mathbf{0}\}$, show that every vector $\mathbf{v}$ in V can be written $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u}$ in $\ker T$ and $\mathbf{w}$ in $\operatorname{im} T$. [*Hint*: Choose bases $B \subseteq \ker T$ and $D \subseteq \operatorname{im} T$, and use Exercise 6.3.33.]

Exercise 7.2.25 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator of rank 1, where $\mathbb{R}^n$ is written as rows. Show that there exist numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ such that $T(X) = XA$ for all rows X in $\mathbb{R}^n$, where

$$A = \left[\begin{array}{cccc} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{array}\right]$$

[*Hint*: $\operatorname{im} T = \mathbb{R}\mathbf{w}$ for $\mathbf{w} = (b_1, \dots, b_n)$ in $\mathbb{R}^n$.]

Exercise 7.2.26 Prove Theorem 7.2.5.

Exercise 7.2.27 Let $T : V \to \mathbb{R}$ be a nonzero linear transformation, where $\dim V = n$. Show that there is a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of V so that $T(r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + \cdots + r_n\mathbf{e}_n) = r_1$.

Exercise 7.2.28 Let $f \neq 0$ be a fixed polynomial of degree $m \geq 1$. If p is any polynomial, recall that $(p \circ f)(x) = p[f(x)]$. Define $T_f : P_n \to P_{n+m}$ by $T_f(p) = p \circ f$.

a. Show that T_f is linear.

b. Show that T_f is one-to-one.

Exercise 7.2.29 Let U be a subspace of a finite dimensional vector space V.

a. Show that $U = \ker T$ for some linear operator $T : V \to V$.

b. Show that $U = \operatorname{im} S$ for some linear operator $S : V \to V$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

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b. By Lemma 6.4.2, let $\{\mathbf{u}_1, \dots, \mathbf{u}_m, \dots, \mathbf{u}_n\}$ be a basis of V where $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a basis of U. By Theorem 7.1.3 there is a linear transformation $S : V \to V$ such that $S(\mathbf{u}_i) = \mathbf{u}_i$ for $1 \leq i \leq m$, and $S(\mathbf{u}_i) = \mathbf{0}$ if $i > m$. Because each $\mathbf{u}_i$ is in $\operatorname{im} S$, $U \subseteq \operatorname{im} S$. But if $S(\mathbf{v})$ is in $\operatorname{im} S$, write $\mathbf{v} = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m + \cdots + r_n\mathbf{u}_n$. Then $S(\mathbf{v}) = r_1S(\mathbf{u}_1) + \cdots + r_mS(\mathbf{u}_m) = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m$ is in U. So $\operatorname{im} S \subseteq U$.

Exercise 7.2.30 Let V and W be finite dimensional vector spaces.

a. Show that $\dim W \leq \dim V$ if and only if there exists an onto linear transformation $T : V \to W$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

b. Show that $\dim W \geq \dim V$ if and only if there exists a one-to-one linear transformation $T : V \to W$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

Exercise 7.2.31 Let A and B be $n \times n$ matrices, and assume that $AXB = 0$, $X \in \mathbf{M}_{nn}$, implies $X = 0$. Show that A and B are both invertible. [Hint: Dimension Theorem.]