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LINEAR ALGEBRA with Applications

Open Edition

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Adapted for

Emory University

Math 221

Linear Algebra

Sections 1 & 2

Lectured and adapted by

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Course page

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1. Systems of Linear Equations

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1.1 Solutions and Elementary Operations

Practical problems in many fields of study—such as biology, business, chemistry, computer science, economics, electronics, engineering, physics and the social sciences—can often be reduced to solving a system of linear equations. Linear algebra arose from attempts to find systematic methods for solving these systems, so it is natural to begin this book by studying linear equations.

If a, b, and c are real numbers, the graph of an equation of the form

$$ax + by = c$$

is a straight line (if a and b are not both zero), so such an equation is called a *linear* equation in the variables x and y. However, it is often convenient to write the variables as $x_1, x_2, \dots, x_n$, particularly when more than two variables are involved. An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is called a **linear equation** in the n variables $x_1, x_2, \dots, x_n$. Here $a_1, a_2, \dots, a_n$ denote real numbers (called the **coefficients** of $x_1, x_2, \dots, x_n$, respectively) and b is also a number (called the **constant term** of the equation). A finite collection of linear equations in the variables $x_1, x_2, \dots, x_n$ is called a **system of linear equations** in these variables. Hence,

$$2x_1 - 3x_2 + 5x_3 = 7$$

is a linear equation; the coefficients of x_1, x_2, and x_3 are 2, -3, and 5, and the constant term is 7. Note that each variable in a linear equation occurs to the first power only.

Given a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, a sequence $s_1, s_2, \dots, s_n$ of n numbers is called a **solution** to the equation if

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

that is, if the equation is satisfied when the substitutions $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ are made. A sequence of numbers is called **a solution to a system** of equations if it is a solution to every equation in the system.

For example, $x = -2$, $y = 5$, $z = 0$ and $x = 0$, $y = 4$, $z = -1$ are both solutions to the system

$$\begin{aligned} x + y + z &= 3 \\ 2x + y + 3z &= 1 \end{aligned}$$

A system may have no solution at all, or it may have a unique solution, or it may have an infinite family of solutions. For instance, the system $x + y = 2$, $x + y = 3$ has no solution because the sum of two numbers cannot be 2 and 3 simultaneously. A system that has no solution is called **inconsistent**; a system with at least one solution is called **consistent**. The system in the following example has infinitely many solutions.

Example 1.1.1

Show that, for arbitrary values of s and t,

$$\begin{aligned} x_1 &= t - s + 1 \\ x_2 &= t + s + 2 \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

is a solution to the system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & + & x_4 & = & -3 \\ 2x_1 & - & x_2 & + & 3x_3 & - & x_4 & = & 0 \end{array}$$

Solution. Simply substitute these values of x_1, x_2, x_3, and x_4 in each equation.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + x_4 &= (t - s + 1) - 2(t + s + 2) + 3s + t = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 &= 2(t - s + 1) - (t + s + 2) + 3s - t = 0 \end{aligned}$$

Because both equations are satisfied, it is a solution for all choices of s and t.

The quantities s and t in Example 1.1.1 are called **parameters**, and the set of solutions, described in this way, is said to be given in **parametric form** and is called the **general solution** to the system. It turns out that the solutions to *every* system of equations (if there *are* solutions) can be given in parametric form (that is, the variables x_1, x_2, ... are given in terms of new independent variables s, t, etc.). The following example shows how this happens in the simplest systems where only one equation is present.

Example 1.1.2

Describe all solutions to $3x - y + 2z = 6$ in parametric form.

Solution. Solving the equation for y in terms of x and z, we get $y = 3x + 2z - 6$. If s and t are arbitrary then, setting $x = s$, $z = t$, we get solutions

$$\begin{aligned} x &= s \\ y &= 3s + 2t - 6 \quad s \text{ and } t \text{ arbitrary} \\ z &= t \end{aligned}$$

Of course we could have solved for x: $x = \frac{1}{3}(y - 2z + 6)$. Then, if we take $y = p$, $z = q$, the solutions are represented as follows:

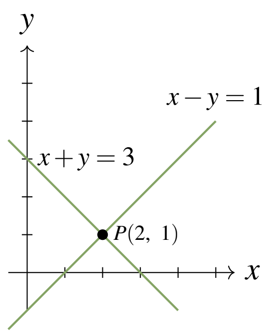
$$\begin{array}{rcll} x & = & \frac{1}{3}(p - 2q + 6) & \\ y & = & p & \qquad p \text{ and } q \text{ arbitrary} \\ z & = & q & \end{array}$$

The same family of solutions can "look" quite different!

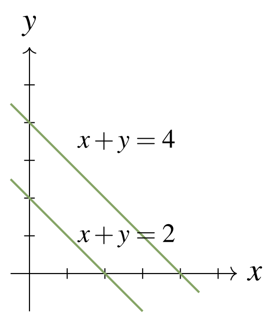
When only two variables are involved, the solutions to systems of linear equations can be described geometrically because the graph of a linear equation $ax + by = c$ is a straight line if a and b are not both zero. Moreover, a point $P(s, t)$ with coordinates s and t lies on the line if and only if $as + bt = c$—that is when $x = s$, $y = t$ is a solution to the equation. Hence the solutions to a *system* of linear equations correspond to the points $P(s, t)$ that lie on *all* the lines in question.

In particular, if the system consists of just one equation, there must be infinitely many solutions because there are infinitely many points on a line. If the system has two equations, there are three possibilities for the corresponding straight lines:

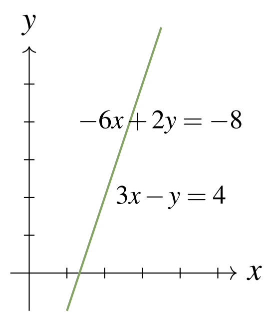
1. *The lines intersect at a single point. Then the system has a* unique solution *corresponding to that point.*
2. *The lines are parallel (and distinct) and so do not intersect. Then the system has* no solution.
3. *The lines are identical. Then the system has* infinitely many solutions—*one for each point on the (common) line.*



(a) Unique Solution
($x = 2$, $y = 1$)



(b) No Solution



(c) Infinitely many solutions
($x = t$, $y = 3t - 4$)

Figure 1.1.1

These three situations are illustrated in Figure 1.1.1. In each case the graphs of two specific lines are plotted and the corresponding equations are indicated. In the last case, the equations are $3x - y = 4$ and $-6x + 2y = -8$, which have identical graphs.

With three variables, the graph of an equation $ax + by + cz = d$ can be shown to be a plane (see Section 4.2) and so again provides a "picture" of the set of solutions. However, this graphical method has its limitations: When more than three variables are involved, no physical image of the graphs (called hyperplanes) is possible. It is necessary to turn to a more "algebraic" method of solution.

Before describing the method, we introduce a concept that simplifies the computations involved. Consider the following system

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & -1 \\ 2x_1 & & & - & x_3 & + & 2x_4 & = & 0 \\ 3x_1 & + & x_2 & + & 2x_3 & + & 5x_4 & = & 2 \end{array}$$

of three equations in four variables. The array of numbers[1]

$$\left[\begin{array}{rrrr|r} 3 & 2 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & 1 & 2 & 5 & 2 \end{array} \right]$$

occurring in the system is called the **augmented matrix** of the system. Each row of the matrix consists of the coefficients of the variables (in order) from the corresponding equation, together

[1]A rectangular array of numbers is called a **matrix**. Matrices will be discussed in more detail in Chapter 2.

with the constant term. For clarity, the constants are separated by a vertical line. The augmented matrix is just a different way of describing the system of equations. The array of coefficients of the variables

$$\left[\begin{array}{rrrr} 3 & 2 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \end{array}\right]$$

is called the **coefficient matrix** of the system and $\left[\begin{array}{r} -1 \\ 0 \\ 2 \end{array}\right]$ is called the **constant matrix** of the system.

Elementary Operations

The algebraic method for solving systems of linear equations is described as follows. Two such systems are said to be **equivalent** if they have the same set of solutions. A system is solved by writing a series of systems, one after the other, each equivalent to the previous system. Each of these systems has the same set of solutions as the original one; the aim is to end up with a system that is easy to solve. Each system in the series is obtained from the preceding system by a simple manipulation chosen so that it does not change the set of solutions.

As an illustration, we solve the system $x+2y=-2$, $2x+y=7$ in this manner. At each stage, the corresponding augmented matrix is displayed. The original system is

$$\begin{array}{rcr} x+2y &=& -2 \\ 2x+\ y &=& 7 \end{array} \qquad \left[\begin{array}{rr|r} 1 & 2 & -2 \\ 2 & 1 & 7 \end{array}\right]$$

First, subtract twice the first equation from the second. The resulting system is

$$\begin{array}{rcr} x+2y &=& -2 \\ -\,3y &=& 11 \end{array} \qquad \left[\begin{array}{rr|r} 1 & 2 & -2 \\ 0 & -3 & 11 \end{array}\right]$$

which is equivalent to the original (see Theorem 1.1.1). At this stage we obtain $y=-\frac{11}{3}$ by multiplying the second equation by $-\frac{1}{3}$. The result is the equivalent system

$$\begin{array}{rcr} x+2y &=& -2 \\ y &=& -\frac{11}{3} \end{array} \qquad \left[\begin{array}{rr|r} 1 & 2 & -2 \\ 0 & 1 & -\frac{11}{3} \end{array}\right]$$

Finally, we subtract twice the second equation from the first to get another equivalent system.

$$\begin{array}{rcr} x &=& \frac{16}{3} \\ y &=& -\frac{11}{3} \end{array} \qquad \left[\begin{array}{rr|r} 1 & 0 & \frac{16}{3} \\ 0 & 1 & -\frac{11}{3} \end{array}\right]$$

Now *this* system is easy to solve! And because it is equivalent to the original system, it provides the solution to that system.

Observe that, at each stage, a certain operation is performed on the system (and thus on the augmented matrix) to produce an equivalent system.

Definition 1.1 Elementary Operations

The following operations, called ***elementary operations****, can routinely be performed on systems of linear equations to produce equivalent systems.*

I. *Interchange two equations.*

II. *Multiply one equation by a* nonzero *number.*

III. *Add a multiple of one equation to a* different *equation.*

Theorem 1.1.1

Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

The proof is given at the end of this section.

Elementary operations performed on a system of equations produce corresponding manipulations of the *rows* of the augmented matrix. Thus, multiplying a row of a matrix by a number k means multiplying *every entry* of the row by k. Adding one row to another row means adding *each entry* of that row to the corresponding entry of the other row. Subtracting two rows is done similarly. Note that we regard two rows as equal when corresponding entries are the same.

In hand calculations (and in computer programs) we manipulate the rows of the augmented matrix rather than the equations. For this reason we restate these elementary operations for matrices.

Definition 1.2 Elementary Row Operations

The following are called ***elementary row operations*** *on a matrix.*

I. *Interchange two rows.*

II. *Multiply one row by a* nonzero *number.*

III. *Add a multiple of one row to a* different *row.*

In the illustration above, a series of such operations led to a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$$

where the asterisks represent arbitrary numbers. In the case of three equations in three variables, the goal is to produce a matrix of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

This does not always happen, as we will see in the next section. Here is an example in which it does happen.

Example 1.1.3

Find all solutions to the following system of equations.

$$\begin{aligned} 3x + 4y + z &= 1 \\ 2x + 3y \quad &= 0 \\ 4x + 3y - z &= -2 \end{aligned}$$

Solution. The augmented matrix of the original system is

$$\left[\begin{array}{rrr|r} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array}\right]$$

To create a 1 in the upper left corner we could multiply row 1 through by $\frac{1}{3}$. However, the 1 can be obtained without introducing fractions by subtracting row 2 from row 1. The result is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array}\right]$$

The upper left 1 is now used to "clean up" the first column, that is create zeros in the other positions in that column. First subtract 2 times row 1 from row 2 to obtain

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 4 & 3 & -1 & -2 \end{array}\right]$$

Next subtract 4 times row 1 from row 3. The result is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{array}\right]$$

This completes the work on column 1. We now use the 1 in the second position of the second row to clean up the second column by subtracting row 2 from row 1 and then adding row 2 to row 3. For convenience, both row operations are done in one step. The result is

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \end{array}\right]$$

Note that the last two manipulations *did not affect* the first column (the second row has a zero there), so our previous effort there has not been undermined. Finally we clean up the third column. Begin by multiplying row 3 by $-\frac{1}{7}$ to obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{8}{7} \end{array}\right]$$

Now subtract 3 times row 3 from row 1, and then add 2 times row 3 to row 2 to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{8}{7} \end{array}\right]$$

The corresponding equations are $x = -\frac{3}{7}$, $y = \frac{2}{7}$, and $z = \frac{8}{7}$, which give the (unique) solution.

Every elementary row operation can be **reversed** by another elementary row operation of the same type (called its **inverse**). To see how, we look at types I, II, and III separately:

Type I *Interchanging two rows is reversed by interchanging them again.*

Type II *Multiplying a row by a nonzero number* k *is reversed by multiplying by* $1/k$.

Type III *Adding* k *times row* p *to a different row* q *is reversed by adding* $-k$ *times row* p *to row* q *(in the new matrix). Note that* $p \neq q$ *is essential here.*

To illustrate the Type III situation, suppose there are four rows in the original matrix, denoted R_1, R_2, R_3, and R_4, and that k times R_2 is added to R_3. Then the reverse operation adds $-k$ times R_2, to R_3. The following diagram illustrates the effect of doing the operation first and then the reverse:

$$\left[\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\right] \rightarrow \left[\begin{array}{c} R_1 \\ R_2 \\ R_3 + kR_2 \\ R_4 \end{array}\right] \rightarrow \left[\begin{array}{c} R_1 \\ R_2 \\ (R_3 + kR_2) - kR_2 \\ R_4 \end{array}\right] = \left[\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\right]$$

The existence of inverses for elementary row operations and hence for elementary operations on a system of equations, gives:

Proof of Theorem 1.1.1. Suppose that a system of linear equations is transformed into a new system by a sequence of elementary operations. Then every solution of the original system is automatically a solution of the new system because adding equations, or multiplying an equation by a nonzero number, always results in a valid equation. In the same way, each solution of the new system must be a solution to the original system because the original system can be obtained from the new one by another series of elementary operations (the inverses of the originals). It follows that the original and new systems have the same solutions. This proves Theorem 1.1.1. □

Exercises for 1.1

Exercise 1.1.1 In each case verify that the following are solutions for all values of s and t.

a. $x = 19t - 35$, $y = 25 - 13t$, $z = t$ is a solution of
$$\begin{aligned} 2x + 3y + z &= 5 \\ 5x + 7y - 4z &= 0 \end{aligned}$$

b. $x_1 = 2s + 12t + 13$, $x_2 = s$, $x_3 = -s - 3t - 3$, $x_4 = t$ is a solution of
$$\begin{aligned} 2x_1 + 5x_2 + 9x_3 + 3x_4 &= -1 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$$

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b. $2(2s+12t+13)+5s+9(-s-3t-3)+3t=-1$; $(2s+12t+13)+2s+4(-s-3t-3)=1$

Exercise 1.1.2 Find all solutions to the following in parametric form in two ways.

a) $3x + y = 2$

b) $2x + 3y = 1$

c) $3x - y + 2z = 5$

d) $x - 2y + 5z = 1$

---

b. $x = t$, $y = \frac{1}{3}(1-2t)$ or $x = \frac{1}{2}(1-3s)$, $y = s$

d. $x = 1 + 2s - 5t$, $y = s$, $z = t$ or $x = s$, $y = t$, $z = \frac{1}{5}(1 - s + 2t)$

Exercise 1.1.3 Regarding $2x = 5$ as the equation $2x + 0y = 5$ in two variables, find all solutions in parametric form.

Exercise 1.1.4 Regarding $4x - 2y = 3$ as the equation $4x - 2y + 0z = 3$ in three variables, find all solutions in parametric form.

$x = \frac{1}{4}(3+2s)$, $y = s$, $z = t$

Exercise 1.1.5 Find all solutions to the general system $ax = b$ of one equation in one variable (a) when $a = 0$ and (b) when $a \neq 0$.

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a. No solution if $b \neq 0$. If $b = 0$, *any* x is a solution.

b. $x = \frac{b}{a}$

Exercise 1.1.6 Show that a system consisting of exactly one linear equation can have no solution, one solution, or infinitely many solutions. Give examples.

Exercise 1.1.7 Write the augmented matrix for each of the following systems of linear equations.

a) $$\begin{aligned} x - 3y &= 5 \\ 2x + y &= 1 \end{aligned}$$

b) $$\begin{aligned} x + 2y &= 0 \\ y &= 1 \end{aligned}$$

c) $$\begin{aligned} x - y + z &= 2 \\ x - z &= 1 \\ y + 2x &= 0 \end{aligned}$$

d) $$\begin{aligned} x + y &= 1 \\ y + z &= 0 \\ z - x &= 2 \end{aligned}$$

---

b. $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$

d. $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 2 \end{array}\right]$

Exercise 1.1.8 Write a system of linear equations that has each of the following augmented matrices.

a) $\left[\begin{array}{rrr|r} 1 & -1 & 6 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & -1 & 0 & 1 \end{array}\right]$

b) $\left[\begin{array}{rrr|r} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$

---

b. $$\begin{aligned} 2x - y \quad &= -1 \\ -3x + 2y + z &= 0 \\ y + z &= 3 \end{aligned}$$

or $$\begin{aligned} 2x_1 - x_2 \quad &= -1 \\ -3x_1 + 2x_2 + x_3 &= 0 \\ x_2 + x_3 &= 3 \end{aligned}$$

Exercise 1.1.9 Find the solution of each of the following systems of linear equations using augmented matrices.

a) $\begin{aligned} x - 3y &= 1 \\ 2x - 7y &= 3 \end{aligned}$ b) $\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= -1 \end{aligned}$

c) $\begin{aligned} 2x + 3y &= -1 \\ 3x + 4y &= 2 \end{aligned}$ d) $\begin{aligned} 3x + 4y &= 1 \\ 4x + 5y &= -3 \end{aligned}$

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b. $x = -3$, $y = 2$

d. $x = -17$, $y = 13$

Exercise 1.1.10 Find the solution of each of the following systems of linear equations using augmented matrices.

a) $\begin{aligned} x + y + 2z &= -1 \\ 2x + y + 3z &= 0 \\ -2y + z &= 2 \end{aligned}$ b) $\begin{aligned} 2x + y + z &= -1 \\ x + 2y + z &= 0 \\ 3x - 2z &= 5 \end{aligned}$

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b. $x = \frac{1}{9}$, $y = \frac{10}{9}$, $z = -\frac{7}{3}$

Exercise 1.1.11 Find all solutions (if any) of the following systems of linear equations.

a) $\begin{aligned} 3x - 2y &= 5 \\ -12x + 8y &= -20 \end{aligned}$ b) $\begin{aligned} 3x - 2y &= 5 \\ -12x + 8y &= 16 \end{aligned}$

---

b. No solution

Exercise 1.1.12 Show that the system

$$\left\{\begin{array}{rcrcrcl} x & + & 2y & - & z & = & a \\ 2x & + & y & + & 3z & = & b \\ x & - & 4y & + & 9z & = & c \end{array}\right.$$

is inconsistent unless $c = 2b - 3a$.

Exercise 1.1.13 By examining the possible positions of lines in the plane, show that two equations in two variables can have zero, one, or infinitely many solutions.

Exercise 1.1.14 In each case either show that the statement is true, or give an example[2] showing it is false.

a. If a linear system has n variables and m equations, then the augmented matrix has n rows.

b. A consistent linear system must have infinitely many solutions.

c. If a row operation is done to a consistent linear system, the resulting system must be consistent.

d. If a series of row operations on a linear system results in an inconsistent system, the original system is inconsistent.

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b. F. $x + y = 0$, $x - y = 0$ has a unique solution.

d. T. Theorem 1.1.1.

Exercise 1.1.15 Find a quadratic $a + bx + cx^2$ such that the graph of $y = a + bx + cx^2$ contains each of the points $(-1, 6)$, $(2, 0)$, and $(3, 2)$.

Exercise 1.1.16 Solve the system $\left\{\begin{array}{l} 3x + 2y = 5 \\ 7x + 5y = 1 \end{array}\right.$ by changing variables $\left\{\begin{array}{l} x = 5x' - 2y' \\ y = -7x' + 3y' \end{array}\right.$ and solving the resulting equations for x' and y'.

$x' = 5$, $y' = 1$, so $x = 23$, $y = -32$

Exercise 1.1.17 Find a, b, and c such that

$$\frac{x^2 - x + 3}{(x^2+2)(2x-1)} = \frac{ax+b}{x^2+2} + \frac{c}{2x-1}$$

[*Hint*: Multiply through by $(x^2 + 2)(2x - 1)$ and equate coefficients of powers of x.]

$a = -\frac{1}{9}$, $b = -\frac{5}{9}$, $c = \frac{11}{9}$

Exercise 1.1.18 A zookeeper wants to give an animal 42 mg of vitamin A and 65 mg of vitamin D per day. He has two supplements: the first contains 10% vitamin A and 25% vitamin D; the second contains 20% vitamin A and 25% vitamin D. How much

[2]Such an example is called a **counterexample**. For example, if the statement is that "all philosophers have beards", the existence of a non-bearded philosopher would be a counterexample proving that the statement is false. This is discussed again in Appendix **??**.

of each supplement should he give the animal each day?

Exercise 1.1.19 Workmen John and Joe earn a total of \$24.60 when John works 2 hours and Joe works 3 hours. If John works 3 hours and Joe works 2 hours, they get \$23.90. Find their hourly rates.

\$4.50, \$5.20

Exercise 1.1.20 A biologist wants to create a diet from fish and meal containing 183 grams of protein and 93 grams of carbohydrate per day. If fish contains 70% protein and 10% carbohydrate, and meal contains 30% protein and 60% carbohydrate, how much of each food is required each day?

1.2 Gaussian Elimination

The algebraic method introduced in the preceding section can be summarized as follows: Given a system of linear equations, use a sequence of elementary row operations to carry the augmented matrix to a "nice" matrix (meaning that the corresponding equations are easy to solve). In Example 1.1.3, this nice matrix took the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

The following definitions identify the nice matrices that arise in this process.

Definition 1.3 Row-Echelon Form (Reduced)

A matrix is said to be in ***row-echelon form*** *(and will be called a* ***row-echelon matrix****) if it satisfies the following three conditions:*

1. *All* ***zero rows*** *(consisting entirely of zeros) are at the bottom.*
2. *The first nonzero entry from the left in each nonzero row is a* 1*, called the* ***leading 1*** *for that row.*
3. *Each leading* 1 *is to the right of all leading* 1*s in the rows above it.*

A row-echelon matrix is said to be in ***reduced row-echelon form*** *(and will be called a* ***reduced row-echelon matrix****) if, in addition, it satisfies the following condition:*

4. *Each leading* 1 *is the only nonzero entry in its column.*

The row-echelon matrices have a "staircase" form, as indicated by the following example (the asterisks indicate arbitrary numbers).

$$\left[\begin{array}{ccccccc} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading 1s proceed "down and to the right" through the matrix. Entries above and to the right of the leading 1s are arbitrary, but all entries below and to the left of them are zero. Hence, a matrix in row-echelon form is in reduced form if, in addition, the entries directly above each leading 1 are all zero. Note that a matrix in row-echelon form can, with a few more row operations, be carried to reduced form (use row operations to create zeros above each leading one in succession, beginning from the right).

Example 1.2.1

The following matrices are in row-echelon form (for any choice of numbers in $*$-positions).

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

The following, on the other hand, are in reduced row-echelon form.

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The choice of the positions for the leading 1s determines the (reduced) row-echelon form (apart from the numbers in $*$-positions).

The importance of row-echelon matrices comes from the following theorem.

Theorem 1.2.1

Every matrix can be brought to (reduced) row-echelon form by a sequence of elementary row operations.

In fact we can give a step-by-step procedure for actually finding a row-echelon matrix. Observe that while there are many sequences of row operations that will bring a matrix to row-echelon form, the one we use is systematic and is easy to program on a computer. Note that the algorithm deals with matrices in general, possibly with columns of zeros.

Gaussian[3] Algorithm[4]

Step 1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.

Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.

Step 3. Now multiply the new top row by $1/a$ to create a leading 1.

Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row, and all further row operations are carried out on the remaining rows.

Step 5. Repeat steps 1–4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

Observe that the gaussian algorithm is recursive: When the first leading 1 has been obtained, the procedure is repeated on the remaining rows of the matrix. This makes the algorithm easy to use on a computer. Note that the solution to Example 1.1.3 did not use the gaussian algorithm as written because the first leading 1 was not created by dividing row 1 by 3. The reason for this is that it avoids fractions. However, the general pattern is clear: Create the leading 1s from left to right, using each of them in turn to create zeros below it. Here are two more examples.

Example 1.2.2

Solve the following system of equations.

$$\begin{aligned} 3x + y - 4z &= -1 \\ x + 10z &= 5 \\ 4x + y + 6z &= 1 \end{aligned}$$

Solution. The corresponding augmented matrix is

$$\left[\begin{array}{rrr|r} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 4 & 1 & 6 & 1 \end{array}\right]$$

Create the first leading one by interchanging rows 1 and 2

$$\left[\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & 1 & 6 & 1 \end{array}\right]$$

Now subtract 3 times row 1 from row 2, and subtract 4 times row 1 from row 3. The result is

$$\left[\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 1 & -34 & -19 \end{array}\right]$$

Now subtract row 2 from row 3 to obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

This means that the following reduced system of equations

$$\begin{aligned} x + 10z &= 5 \\ y - 34z &= -16 \\ 0 &= -3 \end{aligned}$$

is equivalent to the original system. In other words, the two have the same solutions. But this last system clearly has no solution (the last equation requires that x, y and z satisfy $0x+0y+0z=-3$, and no such numbers exist). Hence the original system has no solution.

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[3]Carl Friedrich Gauss (1777–1855) ranks with Archimedes and Newton as one of the three greatest mathematicians of all time. He was a child prodigy and, at the age of 21, he gave the first proof that every polynomial has a complex root. In 1801 he published a timeless masterpiece, *Disquisitiones Arithmeticae*, in which he founded modern number theory. He went on to make ground-breaking contributions to nearly every branch of mathematics, often well before others rediscovered and published the results.

[4]The algorithm was known to the ancient Chinese.

Example 1.2.3

Solve the following system of equations.

$$\begin{aligned} x_1 - 2x_2 - \ \ x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + \ \ x_3 \qquad\ &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

Subtracting twice row 1 from row 2 and subtracting row 1 from row 3 gives

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

Now subtract row 2 from row 3 and multiply row 2 by $\frac{1}{3}$ to get

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is in row-echelon form, and we take it to reduced form by adding row 2 to row 1:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding reduced system of equations is

$$\begin{aligned} x_1 - 2x_2 \qquad + \ \ x_4 &= 2 \\ x_3 - 2x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

The leading ones are in columns 1 and 3 here, so the corresponding variables x_1 and x_3 are called leading variables. Because the matrix is in reduced row-echelon form, these equations can be used to solve for the leading variables in terms of the nonleading variables x_2 and x_4. More precisely, in the present example we set $x_2 = s$ and $x_4 = t$ where s and t are arbitrary, so these equations become

$$x_1 - 2s + t = 2 \quad \text{and} \quad x_3 - 2t = 1$$

Finally the solutions are given by

$$\begin{aligned} x_1 &= 2 + 2s - t \\ x_2 &= s \\ x_3 &= 1 + 2t \\ x_4 &= t \end{aligned}$$

where s and t are arbitrary.

The solution of Example 1.2.3 is typical of the general case. To solve a linear system, the augmented matrix is carried to reduced row-echelon form, and the variables corresponding to the leading ones are called **leading variables**. Because the matrix is in reduced form, each leading variable occurs in exactly one equation, so that equation can be solved to give a formula for the leading variable in terms of the nonleading variables. It is customary to call the nonleading variables "free" variables, and to label them by new variables s, t, $\dots$, called **parameters**. Hence, as in Example 1.2.3, every variable x_i is given by a formula in terms of the parameters s and t. Moreover, every choice of these parameters leads to a solution to the system, and every solution arises in this way. This procedure works in general, and has come to be called

Gaussian Elimination

To solve a system of linear equations proceed as follows:

1. *Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.*
2. *If a row* $\left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]$ *occurs, the system is inconsistent.*
3. *Otherwise, assign the nonleading variables (if any) as parameters, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.*

There is a variant of this procedure, wherein the augmented matrix is carried only to row-echelon form. The nonleading variables are assigned as parameters as before. Then the last equation (corresponding to the row-echelon form) is used to solve for the last leading variable in terms of the parameters. This last leading variable is then substituted into all the preceding equations. Then, the second last equation yields the second last leading variable, which is also substituted back. The process continues to give the general solution. This procedure is called **back-substitution**. This procedure can be shown to be numerically more efficient and so is important when solving very large systems.[5]

Example 1.2.4

Find a condition on the numbers a, b, and c such that the following system of equations is consistent. When that condition is satisfied, find all solutions (in terms of a, b, and c).

$$\begin{aligned} x_1 + 3x_2 + x_3 &= a \\ -x_1 - 2x_2 + x_3 &= b \\ 3x_1 + 7x_2 - x_3 &= c \end{aligned}$$

Solution. We use gaussian elimination except that now the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array}\right]$$

[5] With n equations where n is large, gaussian elimination requires roughly $n^3/2$ multiplications and divisions, whereas this number is roughly $n^3/3$ if back substitution is used.

has entries a, b, and c as well as known numbers. The first leading one is in place, so we create zeros below it in column 1:

$$\left[\begin{array}{rrr|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & c-3a \end{array}\right]$$

The second leading 1 has appeared, so use it to create zeros in the rest of column 2:

$$\left[\begin{array}{rrr|c} 1 & 0 & -5 & -2a-3b \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & c-a+2b \end{array}\right]$$

Now the whole solution depends on the number $c-a+2b = c-(a-2b)$. The last row corresponds to an equation $0 = c-(a-2b)$. If $c \neq a-2b$, there is *no* solution (just as in Example 1.2.2). Hence:

The system is consistent if and only if $c = a-2b$.

In this case the last matrix becomes

$$\left[\begin{array}{rrr|c} 1 & 0 & -5 & -2a-3b \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, if $c = a-2b$, taking $x_3 = t$ where t is a parameter gives the solutions

$$x_1 = 5t-(2a+3b) \quad x_2 = (a+b)-2t \quad x_3 = t.$$

Rank

It can be proven that the *reduced* row-echelon form of a matrix A is uniquely determined by A. That is, no matter which series of row operations is used to carry A to a reduced row-echelon matrix, the result will always be the same matrix. (A proof is given at the end of Section 2.5.) By contrast, this is not true for row-echelon matrices: Different series of row operations can carry the same matrix A to *different* row-echelon matrices. Indeed, the matrix $A = \left[\begin{array}{rrr} 1 & -1 & 4 \\ 2 & -1 & 2 \end{array}\right]$ can be carried (by one row operation) to the row-echelon matrix $\left[\begin{array}{rrr} 1 & -1 & 4 \\ 0 & 1 & -6 \end{array}\right]$, and then by another row operation to the (reduced) row-echelon matrix $\left[\begin{array}{rrr} 1 & 0 & -2 \\ 0 & 1 & -6 \end{array}\right]$. However, it *is* true that the number r of leading 1s must be the same in each of these row-echelon matrices (this will be proved in Chapter 5). Hence, the number r depends only on A and not on the way in which A is carried to row-echelon form.

Definition 1.4 Rank of a Matrix

The ***rank*** *of matrix A is the number of leading 1s in any row-echelon matrix to which A can be carried by row operations.*

Example 1.2.5

Compute the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$.

Solution. The reduction of A to row-echelon form is

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & -5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because this row-echelon matrix has two leading 1s, rank $A = 2$.

Suppose that rank $A = r$, where A is a matrix with m rows and n columns. Then $r \leq m$ because the leading 1s lie in different rows, and $r \leq n$ because the leading 1s lie in different columns. Moreover, the rank has a useful application to equations. Recall that a system of linear equations is called consistent if it has at least one solution.

Theorem 1.2.2

Suppose a system of m equations in n variables is ***consistent****, and that the rank of the augmented matrix is r.*

1. *The set of solutions involves exactly $n - r$ parameters.*
2. *If $r < n$, the system has infinitely many solutions.*
3. *If $r = n$, the system has a unique solution.*

Proof. The fact that the rank of the augmented matrix is r means there are exactly r leading variables, and hence exactly $n - r$ nonleading variables. These nonleading variables are all assigned as parameters in the gaussian algorithm, so the set of solutions involves exactly $n - r$ parameters. Hence if $r < n$, there is at least one parameter, and so infinitely many solutions. If $r = n$, there are no parameters and so a unique solution. □

Theorem 1.2.2 shows that, for any system of linear equations, exactly three possibilities exist:

1. *No solution. This occurs when a row* $\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ *occurs in the row-echelon form. This is the case where the system is inconsistent.*
2. *Unique solution. This occurs when* every *variable is a leading variable.*

3. *Infinitely many solutions. This occurs when the system is consistent and there is at least one nonleading variable, so at least one parameter is involved.*

Example 1.2.6

Suppose the matrix A in Example 1.2.5 is the augmented matrix of a system of $m = 3$ linear equations in $n = 3$ variables. As rank $A = r = 2$, the set of solutions will have $n - r = 1$ parameter. The reader can verify this fact directly.

Many important problems involve **linear inequalities** rather than **linear equations**. For example, a condition on the variables x and y might take the form of an inequality $2x - 5y \leq 4$ rather than an equality $2x - 5y = 4$. There is a technique (called the **simplex algorithm**) for finding solutions to a system of such inequalities that maximizes a function of the form $p = ax + by$ where a and b are fixed constants.

Exercises for 1.2

Exercise 1.2.1 Which of the following matrices are in reduced row-echelon form? Which are in row-echelon form?

a) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

---

b. No, no

d. No, yes

f. No, no

Exercise 1.2.2 Carry each of the following matrices to reduced row-echelon form.

a. $\begin{bmatrix} 0 & -1 & 2 & 1 & 2 & 1 & -1 \\ 0 & 1 & -2 & 2 & 7 & 2 & 4 \\ 0 & -2 & 4 & 3 & 7 & 1 & 0 \\ 0 & 3 & -6 & 1 & 6 & 4 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & -1 & 3 & 1 & 3 & 2 & 1 \\ 0 & -2 & 6 & 1 & -5 & 0 & -1 \\ 0 & 3 & -9 & 2 & 4 & 1 & -1 \\ 0 & 1 & -3 & -1 & 3 & 0 & 1 \end{bmatrix}$

---

b. $\begin{bmatrix} 0 & 1 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

Exercise 1.2.3 The augmented matrix of a system of linear equations has been carried to the following by row operations. In each case solve the system.

a. $\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

b. $\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

c. $\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

d. $\left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 4 & 6 & 2 \\ 0 & 1 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

---

b. $x_1 = 2r - 2s - t + 1,\ x_2 = r,\ x_3 = -5s + 3t - 1,\ x_4 = s,\ x_5 = -6t + 1,\ x_6 = t$

d. $x_1 = -4s - 5t - 4,\ x_2 = -2s + t - 2,\ x_3 = s,\ x_4 = 1,\ x_5 = t$

Exercise 1.2.4 Find all solutions (if any) to each of the following systems of linear equations.

a) $\begin{aligned} x - 2y &= 1 \\ 4y - x &= -2 \end{aligned}$

b) $\begin{aligned} 3x - y &= 0 \\ 2x - 3y &= 1 \end{aligned}$

c) $\begin{aligned} 2x + y &= 5 \\ 3x + 2y &= 6 \end{aligned}$

d) $\begin{aligned} 3x - y &= 2 \\ 2y - 6x &= -4 \end{aligned}$

e) $\begin{aligned} 3x - y &= 4 \\ 2y - 6x &= 1 \end{aligned}$

f) $\begin{aligned} 2x - 3y &= 5 \\ 3y - 2x &= 2 \end{aligned}$

---

b. $x = -\frac{1}{7},\ y = -\frac{3}{7}$

d. $x = \frac{1}{3}(t + 2),\ y = t$

f. No solution

Exercise 1.2.5 Find all solutions (if any) to each of the following systems of linear equations.

a) $\begin{aligned} x + y + 2z &= 8 \\ 3x - y + z &= 0 \\ -x + 3y + 4z &= -4 \end{aligned}$

b) $\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$

c) $\begin{aligned} x + y - z &= 10 \\ -x + 4y + 5z &= -5 \\ x + 6y + 3z &= 15 \end{aligned}$

d) $\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$

e) $\begin{aligned} 5x + y \phantom{{}+2z} &= 2 \\ 3x - y + 2z &= 1 \\ x + y - z &= 5 \end{aligned}$

f) $\begin{aligned} 3x - 2y + z &= -2 \\ x - y + 3z &= 5 \\ -x + y + z &= -1 \end{aligned}$

g) $\begin{aligned} x + y + z &= 2 \\ x \phantom{{}+y} + z &= 1 \\ 2x + 5y + 2z &= 7 \end{aligned}$

h) $\begin{aligned} x + 2y - 4z &= 10 \\ 2x - y + 2z &= 5 \\ x + y - 2z &= 7 \end{aligned}$

---

b. $x = -15t - 21,\ y = -11t - 17,\ z = t$

d. No solution

f. $x = -7,\ y = -9,\ z = 1$

h. $x = 4,\ y = 3 + 2t,\ z = t$

Exercise 1.2.6 Express the last equation of each system as a sum of multiples of the first two equations. [*Hint*: Label the equations, use the gaussian algorithm.]

a) $\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 3 \\ x_1 - 2x_2 + 2x_3 &= 2 \end{aligned}$

b) $\begin{aligned} x_1 + 2x_2 - 3x_3 &= -3 \\ x_1 + 3x_2 - 5x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -35 \end{aligned}$

---

b. Denote the equations as E_1, E_2, and E_3. Apply gaussian elimination to column 1 of the augmented matrix, and observe that $E_3 - E_1 = -4(E_2 - E_1)$. Hence $E_3 = 5E_1 - 4E_2$.

Exercise 1.2.7 Find all solutions to the following systems.

a. $\begin{aligned} 3x_1 + 8x_2 - 3x_3 - 14x_4 &= 2 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 1 \\ x_1 - 2x_2 + x_3 + 10x_4 &= 0 \\ x_1 + 5x_2 - 2x_3 - 12x_4 &= 1 \end{aligned}$

b. $\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$

c. $\begin{aligned} x_1 - x_2 + x_3 - 2x_4 &= 1 \\ -x_1 + x_2 + x_3 + x_4 &= -1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 2 \\ x_1 - x_2 + 2x_3 + x_4 &= 1 \end{aligned}$

d. $\begin{aligned} x_1 + x_2 + 2x_3 - x_4 &= 4 \\ 3x_2 - x_3 + 4x_4 &= 2 \\ x_1 + 2x_2 - 3x_3 + 5x_4 &= 0 \\ x_1 + x_2 - 5x_3 + 6x_4 &= -3 \end{aligned}$

---

b. $x_1 = 0$, $x_2 = -t$, $x_3 = 0$, $x_4 = t$

d. $x_1 = 1$, $x_2 = 1 - t$, $x_3 = 1 + t$, $x_4 = t$

Exercise 1.2.8 In each of the following, find (if possible) conditions on a and b such that the system has no solution, one solution, and infinitely many solutions.

a) $\begin{aligned} x - 2y &= 1 \\ ax + by &= 5 \end{aligned}$

b) $\begin{aligned} x + by &= -1 \\ ax + 2y &= 5 \end{aligned}$

c) $\begin{aligned} x - by &= -1 \\ x + ay &= 3 \end{aligned}$

d) $\begin{aligned} ax + y &= 1 \\ 2x + y &= b \end{aligned}$

---

b. If $ab \neq 2$, unique solution $x = \frac{-2-5b}{2-ab}$, $y = \frac{a+5}{2-ab}$. If $ab = 2$: no solution if $a \neq -5$; if $a = -5$, the solutions are $x = -1 + \frac{2}{5}t$, $y = t$.

d. If $a \neq 2$, unique solution $x = \frac{1-b}{a-2}$, $y = \frac{ab-2}{a-2}$. If $a = 2$, no solution if $b \neq 1$; if $b = 1$, the solutions are $x = \frac{1}{2}(1 - t)$, $y = t$.

Exercise 1.2.9 In each of the following, find (if possible) conditions on a, b, and c such that the system has no solution, one solution, or infinitely many solutions.

a) $\begin{aligned} 3x + y - z &= a \\ x - y + 2z &= b \\ 5x + 3y - 4z &= c \end{aligned}$

b) $\begin{aligned} 2x + y - z &= a \\ 2y + 3z &= b \\ x - z &= c \end{aligned}$

c) $\begin{aligned} -x + 3y + 2z &= -8 \\ x + z &= 2 \\ 3x + 3y + az &= b \end{aligned}$

d) $\begin{aligned} x + ay &= 0 \\ y + bz &= 0 \\ z + cx &= 0 \end{aligned}$

e) $\begin{aligned} 3x - y + 2z &= 3 \\ x + y - z &= 2 \\ 2x - 2y + 3z &= b \end{aligned}$

f) $\begin{aligned} x + ay - z &= 1 \\ -x + (a-2)y + z &= -1 \\ 2x + 2y + (a-2)z &= 1 \end{aligned}$

---

b. Unique solution $x = -2a + b + 5c$, $y = 3a - b - 6c$, $z = -2a + b + c$, for any a, b, c.

d. If $abc \neq -1$, unique solution $x = y = z = 0$; if $abc = -1$ the solutions are $x = abt$, $y = -bt$, $z = t$.

f. If $a = 1$, solutions $x = -t$, $y = t$, $z = -1$. If $a = 0$, there is no solution. If $a \neq 1$ and $a \neq 0$, unique solution $x = \frac{a-1}{a}$, $y = 0$, $z = \frac{-1}{a}$.

Exercise 1.2.10 Find the rank of each of the matrices in Exercise 1.2.1.

---

b. 1

d. 3

f. 1

Exercise 1.2.11 Find the rank of each of the following matrices.

a) $\left[\begin{array}{rrr} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{array}\right]$

b) $\left[\begin{array}{rrr} -2 & 3 & 3 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{array}\right]$

c) $\left[\begin{array}{rrrr} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{array}\right]$

d) $\left[\begin{array}{rrrr} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{array}\right]$

e) $\left[\begin{array}{cccc} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 1 & 2-a & -1 & -2a^2 \end{array}\right]$

f) $\left[\begin{array}{cccc} 1 & 1 & 2 & a^2 \\ 1 & 1-a & 2 & 0 \\ 2 & 2-a & 6-a & 4 \end{array}\right]$

---

b. 2

d. 3

f. 2 if $a = 0$ or $a = 2$; 3, otherwise.

Exercise 1.2.12 Consider a system of linear equations with augmented matrix A and coefficient matrix C. In each case either prove the statement or give an example showing that it is false.

a. If there is more than one solution, A has a row of zeros.

b. If A has a row of zeros, there is more than one solution.

c. If there is no solution, the reduced row-echelon form of C has a row of zeros.

d. If the row-echelon form of C has a row of zeros, there is no solution.

e. There is no system that is inconsistent for every choice of constants.

f. If the system is consistent for some choice of constants, it is consistent for every choice of constants.

Now assume that the augmented matrix A has 3 rows and 5 columns.

g. If the system is consistent, there is more than one solution.

h. The rank of A is at most 3.

i. If $\text{rank}\, A = 3$, the system is consistent.

j. If $\text{rank}\, C = 3$, the system is consistent.

---

b. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

d. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

f. False. $\begin{array}{r} 2x - y = 0 \\ -4x + 2y = 0 \end{array}$ is consistent but $\begin{array}{r} 2x - y = 1 \\ -4x + 2y = 1 \end{array}$ is not.

h. True, A has 3 rows, so there are at most 3 leading 1s.

Exercise 1.2.13 Find a sequence of row operations carrying

$$\left[\begin{array}{ccc} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ c_1 + a_1 & c_2 + a_2 & c_3 + a_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{array}\right] \text{ to } \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right]$$

Exercise 1.2.14 In each case, show that the reduced row-echelon form is as given.

a. $\left[\begin{array}{ccc} p & 0 & a \\ b & 0 & 0 \\ q & c & r \end{array}\right]$ with $abc \neq 0$; $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

b. $\left[\begin{array}{ccc} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{array}\right]$ where $c \neq a$ or $b \neq a$; $\left[\begin{array}{ccc} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right]$

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b. Since one of $b-a$ and $c-a$ is nonzero, then $\left[\begin{array}{ccc} 1 & a & b+c \\ 1 & b & c+a \\ 1 & b & c+a \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & b+c+a \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

Exercise 1.2.15 Show that $\left\{\begin{array}{l} az + by + cz = 0 \\ a_1x + b_1y + c_1z = 0 \end{array}\right.$ always has a solution other than $x = 0$, $y = 0$, $z = 0$.

Exercise 1.2.16 Find the circle $x^2 + y^2 + ax + by + c = 0$ passing through the following points.

a. $(-2,\ 1)$, $(5,\ 0)$, and $(4,\ 1)$

b. $(1,\ 1)$, $(5,\ -3)$, and $(-3,\ -3)$

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b. $x^2 + y^2 - 2x + 6y - 6 = 0$

Exercise 1.2.17 Three Nissans, two Fords, and four Chevrolets can be rented for \$106 per day. At the same rates two Nissans, four Fords, and three Chevrolets cost \$107 per day, whereas four Nissans, three Fords, and two Chevrolets cost \$102 per day. Find the rental rates for all three kinds of cars.

Exercise 1.2.18 A school has three clubs and each student is required to belong to exactly one club. One year the students switched club membership as follows: Club A. $\frac{4}{10}$ remain in A, $\frac{1}{10}$ switch to B, $\frac{5}{10}$ switch to C. Club B. $\frac{7}{10}$ remain in B, $\frac{2}{10}$ switch to A,

$\frac{1}{10}$ switch to C. Club C. $\frac{6}{10}$ remain in C, $\frac{2}{10}$ switch to A, $\frac{2}{10}$ switch to B. If the fraction of the student population in each club is unchanged, find each of these fractions. ______________

$\frac{5}{20}$ in A, $\frac{7}{20}$ in B, $\frac{8}{20}$ in C.

Exercise 1.2.19 Given points (p_1, q_1), (p_2, q_2), and (p_3, q_3) in the plane with p_1, p_2, and p_3 distinct, show that they lie on some curve with equation $y = a + bx + cx^2$. [*Hint*: Solve for a, b, and c.]

Exercise 1.2.20 The scores of three players in a tournament have been lost. The only information available is the total of the scores for players 1 and 2, the total for players 2 and 3, and the total for players 3 and 1.

a. Show that the individual scores can be rediscovered.

b. Is this possible with four players (knowing the totals for players 1 and 2, 2 and 3, 3 and 4, and 4 and 1)?

Exercise 1.2.21 A boy finds \$1.05 in dimes, nickels, and pennies. If there are 17 coins in all, how many coins of each type can he have?

Exercise 1.2.22 If a consistent system has more variables than equations, show that it has infinitely many solutions. [*Hint*: Use Theorem 1.2.2.]

1.3 Homogeneous Equations

A system of equations in the variables $x_1, x_2, \ldots, x_n$ is called **homogeneous** if all the constant terms are zero—that is, if each equation of the system has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Clearly $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is a solution to such a system; it is called the **trivial solution**. Any solution in which at least one variable has a nonzero value is called a **nontrivial solution**. Our chief goal in this section is to give a useful condition for a homogeneous system to have nontrivial solutions. The following example is instructive.

Example 1.3.1

Show that the following homogeneous system has nontrivial solutions.

$$\begin{array}{rl} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 2x_2 \qquad + x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 - x_4 &= 0 \end{array}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is outlined below.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 4 & -4 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The leading variables are x_1, x_2, and x_4, so x_3 is assigned as a parameter—say $x_3 = t$. Then the general solution is $x_1 = -t$, $x_2 = t$, $x_3 = t$, $x_4 = 0$. Hence, taking $t = 1$ (say), we get a nontrivial solution: $x_1 = -1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$.

The existence of a nontrivial solution in Example 1.3.1 is ensured by the presence of a parameter in the solution. This is due to the fact that there is a *nonleading* variable (x_3 in this case). But there *must* be a nonleading variable here because there are four variables and only three equations (and hence at *most* three leading variables). This discussion generalizes to a proof of the following fundamental theorem.

Theorem 1.3.1

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Proof. Suppose there are m equations in n variables where $n > m$, and let R denote the reduced row-echelon form of the augmented matrix. If there are r leading variables, there are $n - r$ nonleading variables, and so $n-r$ parameters. Hence, it suffices to show that $r<n$. But $r \leq m$ because R has r leading 1s and m rows, and $m<n$ by hypothesis. So $r \leq m<n$, which gives $r<n$. □

Note that the converse of Theorem 1.3.1 is not true: if a homogeneous system has nontrivial solutions, it need not have more variables than equations (the system $x_1+x_2=0$, $2x_1+2x_2=0$ has nontrivial solutions but $m=2=n$.)

Theorem 1.3.1 is very useful in applications. The next example provides an illustration from geometry.

Example 1.3.2

We call the graph of an equation $ax^2+bxy+cy^2+dx+ey+f=0$ a **conic** if the numbers a, b, and c are not all zero. Show that there is at least one conic through any five points in the plane that are not all on a line.

Solution. Let the coordinates of the five points be (p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4), and (p_5, q_5). The graph of $ax^2+bxy+cy^2+dx+ey+f=0$ passes through (p_i, q_i) if

$$ap_i^2+bp_iq_i+cq_i^2+dp_i+eq_i+f=0$$

This gives five equations, one for each i, linear in the six variables a, b, c, d, e, and f. Hence, there is a nontrivial solution by Theorem 1.3.1. If $a=b=c=0$, the five points all lie on the line with equation $dx+ey+f=0$, contrary to assumption. Hence, one of a, b, c is nonzero.

Linear Combinations and Basic Solutions

As for rows, two columns are regarded as **equal** if they have the same number of entries and corresponding entries are the same. Let $\mathbf{x}$ and $\mathbf{y}$ be columns with the same number of entries. As for elementary row operations, their **sum** $\mathbf{x}+\mathbf{y}$ is obtained by adding corresponding entries and, if k is a number, the **scalar product** $k\mathbf{x}$ is defined by multiplying each entry of $\mathbf{x}$ by k. More precisely:

$$\text{If } \mathbf{x}=\left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] \text{ and } \mathbf{y}=\left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right] \text{ then } \mathbf{x}+\mathbf{y}=\left[\begin{array}{c} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{array}\right] \text{ and } k\mathbf{x}=\left[\begin{array}{c} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{array}\right].$$

A sum of scalar multiples of several columns is called a **linear combination** of these columns. For example, $s\mathbf{x}+t\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$ for any choice of numbers s and t.

Example 1.3.3

If $\mathbf{x}=\left[\begin{array}{r} 3 \\ -2 \end{array}\right]$ and $\left[\begin{array}{r} -1 \\ 1 \end{array}\right]$ then $2\mathbf{x}+5\mathbf{y}=\left[\begin{array}{r} 6 \\ -4 \end{array}\right]+\left[\begin{array}{r} -5 \\ 5 \end{array}\right]=\left[\begin{array}{r} 1 \\ 1 \end{array}\right]$.

Example 1.3.4

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. If $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, determine whether $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

Solution. For $\mathbf{v}$, we must determine whether numbers r, s, and t exist such that $\mathbf{v} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$, that is, whether

$$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r+2s+3t \\ s+t \\ r+t \end{bmatrix}$$

Equating corresponding entries gives a system of linear equations $r+2s+3t=0$, $s+t=-1$, and $r+t=2$ for r, s, and t. By gaussian elimination, the solution is $r=2-k$, $s=-1-k$, and $t=k$ where k is a parameter. Taking $k=0$, we see that $\mathbf{v} = 2\mathbf{x} - \mathbf{y}$ is a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.
Turning to $\mathbf{w}$, we again look for r, s, and t such that $\mathbf{w} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$; that is,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r+2s+3t \\ s+t \\ r+t \end{bmatrix}$$

leading to equations $r+2s+3t=1$, $s+t=1$, and $r+t=1$ for real numbers r, s, and t. But this time there is *no* solution as the reader can verify, so $\mathbf{w}$ is *not* a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Our interest in linear combinations comes from the fact that they provide one of the best ways to describe the general solution of a homogeneous system of linear equations. When solving such a system with n variables $x_1, x_2, \dots, x_n$, write the variables as a column[6] matrix: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The trivial solution is denoted $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. As an illustration, the general solution in Example 1.3.1 is $x_1 = -t$, $x_2 = t$, $x_3 = t$, and $x_4 = 0$, where t is a parameter, and we would now express this by saying that the general solution is $\mathbf{x} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix}$, where t is arbitrary.

Now let $\mathbf{x}$ and $\mathbf{y}$ be two solutions to a homogeneous system with n variables. Then any linear

[6]The reason for using columns will be apparent later.

combination $s\mathbf{x}+t\mathbf{y}$ of these solutions turns out to be again a solution to the system. More generally:

Any linear combination of solutions to a homogeneous system is again a solution. (1.1)

In fact, suppose that a typical equation in the system is $a_1x_1+a_2x_2+\cdots+a_nx_n=0$, and suppose that $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ are solutions. Then $a_1x_1+a_2x_2+\cdots+a_nx_n=0$ and $a_1y_1+a_2y_2+\cdots+a_ny_n=0$. Hence $s\mathbf{x}+t\mathbf{y}=\begin{bmatrix} sx_1+ty_1 \\ sx_2+ty_2 \\ \vdots \\ sx_n+ty_n \end{bmatrix}$ is also a solution because

$$\begin{aligned}
a_1(sx_1+ty_1)&+a_2(sx_2+ty_2)+\cdots+a_n(sx_n+ty_n)\\
&=[a_1(sx_1)+a_2(sx_2)+\cdots+a_n(sx_n)]+[a_1(ty_1)+a_2(ty_2)+\cdots+a_n(ty_n)]\\
&=s(a_1x_1+a_2x_2+\cdots+a_nx_n)+t(a_1y_1+a_2y_2+\cdots+a_ny_n)\\
&=s(0)+t(0)\\
&=0
\end{aligned}$$

A similar argument shows that Statement 1.1 is true for linear combinations of more than two solutions.

The remarkable thing is that *every* solution to a homogeneous system is a linear combination of certain particular solutions and, in fact, these solutions are easily computed using the gaussian algorithm. Here is an example.

Example 1.3.5

Solve the homogeneous system with coefficient matrix

$$A=\begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 0 \\ -2 & 4 & 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the solutions are $x_1=2s+\frac{1}{5}t$, $x_2=s$, $x_3=\frac{3}{5}$, and $x_4=t$ by gaussian elimination. Hence we can write the general solution $\mathbf{x}$ in the matrix form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + \frac{1}{5}t \\ s \\ \frac{3}{5}t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s\mathbf{x}_1 + t\mathbf{x}_2.$$

Here $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix}$ are particular solutions determined by the gaussian algorithm.

The solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ in Example 1.3.5 are denoted as follows:

Definition 1.5 Basic Solutions

The gaussian algorithm systematically produces solutions to any homogeneous linear system, called ***basic solutions****, one for every parameter.*

Moreover, the algorithm gives a routine way to express *every* solution as a linear combination of basic solutions as in Example 1.3.5, where the general solution $\mathbf{x}$ becomes

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{5}t \begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \end{bmatrix}$$

Hence by introducing a new parameter $r = t/5$ we can multiply the original basic solution $\mathbf{x}_2$ by 5 and so eliminate fractions. For this reason:

Convention:
Any nonzero scalar multiple of a basic solution will still be called a basic solution.

In the same way, the gaussian algorithm produces basic solutions to *every* homogeneous system, one for each parameter (there are *no* basic solutions if the system has only the trivial solution). Moreover every solution is given by the algorithm as a linear combination of these basic solutions (as in Example 1.3.5). If A has rank r, Theorem 1.2.2 shows that there are exactly $n - r$ parameters, and so $n - r$ basic solutions. This proves:

Theorem 1.3.2

Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. *The system has exactly $n - r$ basic solutions, one for each parameter.*

2. *Every solution is a linear combination of these basic solutions.*

Example 1.3.6

Find basic solutions of the homogeneous system with coefficient matrix A, and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ -2 & 6 & 1 & 2 & -5 & 0 \\ 3 & -9 & -1 & 0 & 7 & 0 \\ -3 & 9 & 2 & 6 & -8 & 0 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the general solution is $x_1 = 3r - 2s - 2t$, $x_2 = r$, $x_3 = -6s + t$, $x_4 = s$, and $x_5 = t$ where r, s, and t are parameters. In matrix form this is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence basic solutions are

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Exercises for 1.3

Exercise 1.3.1 Consider the following statements about a system of linear equations with augmented matrix A. In each case either prove the statement or give an example for which it is false.

a. If the system is homogeneous, every solution is trivial.

b. If the system has a nontrivial solution, it cannot be homogeneous.

c. If there exists a trivial solution, the system is homogeneous.

d. If the system is consistent, it must be homogeneous.

Now assume that the system is homogeneous.

e. If there exists a nontrivial solution, there is no trivial solution.

f. If there exists a solution, there are infinitely many solutions.

g. If there exist nontrivial solutions, the row-echelon form of A has a row of zeros.

h. If the row-echelon form of A has a row of zeros, there exist nontrivial solutions.

i. If a row operation is applied to the system, the new system is also homogeneous.

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b. False. $A = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

d. False. $A = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$

f. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

h. False. $A = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Exercise 1.3.2 In each of the following, find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

a) $\begin{array}{rcl} x - 2y + z &=& 0 \\ x + ay - 3z &=& 0 \\ -x + 6y - 5z &=& 0 \end{array}$

b) $\begin{array}{rcl} x + 2y + z &=& 0 \\ x + 3y + 6z &=& 0 \\ 2x + 3y + az &=& 0 \end{array}$

c) $\begin{array}{rcl} x + y - z &=& 0 \\ ay - z &=& 0 \\ x + y + az &=& 0 \end{array}$

d) $\begin{array}{rcl} ax + y + z &=& 0 \\ x + y - z &=& 0 \\ x + y + az &=& 0 \end{array}$

---

b. $a = -3$, $x = 9t$, $y = -5t$, $z = t$

d. $a = 1$, $x = -t$, $y = t$, $z = 0$; or $a = -1$, $x = t$, $y = 0$, $z = t$

Exercise 1.3.3 Let $\mathbf{x} = \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right]$, $\mathbf{y} = \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]$, and $\mathbf{z} = \left[\begin{array}{r} 1 \\ 1 \\ -2 \end{array}\right]$. In each case, either write $\mathbf{v}$ as a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, or show that it is not such a linear combination.

a) $\mathbf{v} = \left[\begin{array}{r} 0 \\ 1 \\ -3 \end{array}\right]$

b) $\mathbf{v} = \left[\begin{array}{r} 4 \\ 3 \\ -4 \end{array}\right]$

c) $\mathbf{v} = \left[\begin{array}{r} 3 \\ 1 \\ 0 \end{array}\right]$

d) $\mathbf{v} = \left[\begin{array}{r} 3 \\ 0 \\ 3 \end{array}\right]$

---

b. Not a linear combination.

d. $\mathbf{v} = \mathbf{x} + 2\mathbf{y} - \mathbf{z}$

Exercise 1.3.4 In each case, either express $\mathbf{y}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, or show that it is not such a linear combination. Here:

$$\mathbf{a}_1 = \left[\begin{array}{r} -1 \\ 3 \\ 0 \\ 1 \end{array}\right], \quad \mathbf{a}_2 = \left[\begin{array}{r} 3 \\ 1 \\ 2 \\ 0 \end{array}\right], \quad \text{and } \mathbf{a}_3 = \left[\begin{array}{r} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

a) $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}$ b) $\mathbf{y} = \begin{bmatrix} -1 \\ 9 \\ 2 \\ 6 \end{bmatrix}$

---

b. $\mathbf{y} = 2\mathbf{a}_1 - \mathbf{a}_2 + 4\mathbf{a}_3$.

Exercise 1.3.5 For each of the following homogeneous systems, find a set of basic solutions and express the general solution as a linear combination of these basic solutions.

a. $$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 + 2x_3 \quad + x_5 &= 0 \\ 2x_1 + 4x_2 - 2x_3 + 3x_4 + x_5 &= 0 \end{aligned}$$

b. $$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ -x_1 - 2x_2 + 2x_3 \quad + x_5 &= 0 \\ -x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 &= 0 \end{aligned}$$

c. $$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ 2x_1 + 3x_2 - x_3 + 2x_4 + x_5 &= 0 \\ 4x_1 + 5x_2 - 2x_3 + 5x_4 + 2x_5 &= 0 \end{aligned}$$

d. $$\begin{aligned} x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 &= 0 \\ 2x_1 + 2x_2 - 4x_3 - 4x_4 + x_5 &= 0 \\ x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 0 \\ -2x_1 - 4x_2 + 8x_3 + 10x_4 + x_5 &= 0 \end{aligned}$$

---

b. $r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

d. $s \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Exercise 1.3.6

a. Does Theorem 1.3.1 imply that the system $\left\{ \begin{array}{l} -z + 3y = 0 \\ 2x - 6y = 0 \end{array} \right.$ has nontrivial solutions? Explain.

b. Show that the converse to Theorem 1.3.1 is not true. That is, show that the existence of nontrivial solutions does *not* imply that there are more variables than equations.

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b. The system in (a) has nontrivial solutions.

Exercise 1.3.7 In each case determine how many solutions (and how many parameters) are possible for a homogeneous system of four linear equations in six variables with augmented matrix A. Assume that A has nonzero entries. Give all possibilities.

a) Rank $A = 2$. b) Rank $A = 1$.

c) A has a row of zeros.

d) The row-echelon form of A has a row of zeros.

---

b. By Theorem 1.2.2, there are $n - r = 6 - 1 = 5$ parameters and thus infinitely many solutions.

d. If R is the row-echelon form of A, then R has a row of zeros and 4 rows in all. Hence R has $r = \text{rank } A = 1$, 2, or 3. Thus there are $n - r = 6 - r = 5$, 4, or 3 parameters and thus infinitely many solutions.

Exercise 1.3.8 The graph of an equation $ax + by + cz = 0$ is a plane through the origin (provided that not all of a, b, and c are zero). Use Theorem 1.3.1 to show that two planes through the origin have a point in common other than the origin $(0, 0, 0)$.

Exercise 1.3.9

a. Show that there is a line through any pair of points in the plane. [*Hint*: Every line has equation $ax + by + c = 0$, where a, b, and c are not all zero.]

b. Generalize and show that there is a plane $ax + by + cz + d = 0$ through any three points in space.

---

b. That the graph of $ax+by+cz=d$ contains three points leads to 3 linear equations homogeneous in variables a, b, c, and d. Apply Theorem 1.3.1.

Exercise 1.3.10 The graph of

$$a(x^2+y^2)+bx+cy+d=0$$

is a circle if $a \neq 0$. Show that there is a circle through any three points in the plane that are not all on a line.

Exercise 1.3.11 Consider a homogeneous system of linear equations in n variables, and suppose that the augmented matrix has rank r. Show that the system has nontrivial solutions if and only if $n > r$.

There are $n-r$ parameters (Theorem 1.2.2), so there are nontrivial solutions if and only if $n-r>0$.

Exercise 1.3.12 If a consistent (possibly non-homogeneous) system of linear equations has more variables than equations, prove that it has more than one solution.

Supplementary Exercises for Chapter 1

Exercise 1.1 We show in Chapter 4 that the graph of an equation $ax+by+cz=d$ is a plane in space when not all of a, b, and c are zero.

a. By examining the possible positions of planes in space, show that three equations in three variables can have zero, one, or infinitely many solutions.

b. Can two equations in three variables have a unique solution? Give reasons for your answer.

---

b. No. If the corresponding planes are parallel and distinct, there is no solution. Otherwise they either coincide or have a whole common line of solutions, that is, at least one parameter.

Exercise 1.2 Find all solutions to the following systems of linear equations.

a.
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + 5x_2 - 2x_3 + x_4 &= 1 \\ -3x_1 - 7x_2 + 7x_3 - 5x_4 &= 7 \\ x_1 + 3x_2 - 4x_3 + 3x_4 &= -5 \end{aligned}$$

b.
$$\begin{aligned} x_1 + 4x_2 - x_3 + x_4 &= 2 \\ 3x_1 + 2x_2 + x_3 + 2x_4 &= 5 \\ x_1 - 6x_2 + 3x_3 &= 1 \\ x_1 + 14x_2 - 5x_3 + 2x_4 &= 3 \end{aligned}$$

---

b. $x_1 = \frac{1}{10}(-6s-6t+16)$, $x_2 = \frac{1}{10}(4s-t+1)$, $x_3 = s$, $x_4 = t$

Exercise 1.3 In each case find (if possible) conditions on a, b, and c such that the system has zero, one, or infinitely many solutions.

a)
$$\begin{aligned} x + 2y - 4z &= 4 \\ 3x - y + 13z &= 2 \\ 4x + y + a^2z &= a+3 \end{aligned}$$

b)
$$\begin{aligned} x + y + 3z &= a \\ ax + y + 5z &= 4 \\ x + ay + 4z &= a \end{aligned}$$

---

b. If $a=1$, no solution. If $a=2$, $x=2-2t$, $y=-t$, $z=t$. If $a\neq 1$ and $a\neq 2$, the unique solution is $x=\frac{8-5a}{3(a-1)}$, $y=\frac{-2-a}{3(a-1)}$, $z=\frac{a+2}{3}$

Exercise 1.4 Show that any two rows of a matrix can be interchanged by elementary row transformations of the other two types.

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \rightarrow \begin{bmatrix} R_1+R_2 \\ R_2 \end{bmatrix} \rightarrow \begin{bmatrix} R_1+R_2 \\ -R_1 \end{bmatrix} \rightarrow \begin{bmatrix} R_2 \\ -R_1 \end{bmatrix} \rightarrow \begin{bmatrix} R_2 \\ R_1 \end{bmatrix}$$

Exercise 1.5 If $ad\neq bc$, show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has reduced row-echelon form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Exercise 1.6 Find a, b, and c so that the system

$$\begin{aligned} x + ay + cz &= 0 \\ bx + cy - 3z &= 1 \\ ax + 2y + bz &= 5 \end{aligned}$$

has the solution $x=3$, $y=-1$, $z=2$.

$a=1$, $b=2$, $c=-1$

Exercise 1.7 Solve the system

$$\begin{aligned} x + 2y + 2z &= -3 \\ 2x + y + z &= -4 \\ x - y + iz &= i \end{aligned}$$

where $i^2=-1$. [See Appendix **??**.]

Exercise 1.8 Show that the *real* system

$$\begin{cases} x + y + z = 5 \\ 2x - y - z = 1 \\ -3x + 2y + 2z = 0 \end{cases}$$

has a *complex* solution: $x=2$, $y=i$, $z=3-i$ where $i^2=-1$. Explain. What happens when such a real system has a unique solution?

The (real) solution is $x = 2$, $y = 3 - t$, $z = t$ where t is a parameter. The given complex solution occurs when $t = 3 - i$ is complex. If the real system has a unique solution, that solution is real because the coefficients and constants are all real.

Exercise 1.9 A man is ordered by his doctor to take 5 units of vitamin A, 13 units of vitamin B, and 23 units of vitamin C each day. Three brands of vitamin pills are available, and the number of units of each vitamin per pill are shown in the accompanying table.

	Vitamin		
Brand	**A**	**B**	**C**
1	1	2	4
2	1	1	3
3	0	1	1

a. Find all combinations of pills that provide exactly the required amount of vitamins (no partial pills allowed).

b. If brands 1, 2, and 3 cost 3¢, 2¢, and 5¢ per pill, respectively, find the least expensive treatment.

---

b. 5 of brand 1, 0 of brand 2, 3 of brand 3

Exercise 1.10 A restaurant owner plans to use x tables seating 4, y tables seating 6, and z tables seating 8, for a total of 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables, and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x, y, and z.

Exercise 1.11

a. Show that a matrix with two rows and two columns that is in reduced row-echelon form must have one of the following forms:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$$

[*Hint*: The leading 1 in the first row must be in column 1 or 2 or not exist.]

b. List the seven reduced row-echelon forms for matrices with two rows and three columns.

c. List the four reduced row-echelon forms for matrices with three rows and two columns.

Exercise 1.12 An amusement park charges $7 for adults, $2 for youths, and $0.50 for children. If 150 people enter and pay a total of $100, find the numbers of adults, youths, and children. [*Hint*: These numbers are nonnegative *integers*.]

Exercise 1.13 Solve the following system of equations for x and y.

$$\begin{aligned} x^2 + xy - y^2 &= 1 \\ 2x^2 - xy + 3y^2 &= 13 \\ x^2 + 3xy + 2y^2 &= 0 \end{aligned}$$

[*Hint*: These equations are linear in the new variables $x_1 = x^2$, $x_2 = xy$, and $x_3 = y^2$.]

2. Matrix Algebra

Contents

In the study of systems of linear equations in Chapter 1, we found it convenient to manipulate the augmented matrix of the system. Our aim was to reduce it to row-echelon form (using elementary row operations) and hence to write down all solutions to the system. In the present chapter we consider matrices for their own sake. While some of the motivation comes from linear equations, it turns out that matrices can be multiplied and added and so form an algebraic system somewhat analogous to the real numbers. This "matrix algebra" is useful in ways that are quite different from the study of linear equations. For example, the geometrical transformations obtained by rotating the euclidean plane about the origin can be viewed as multiplications by certain 2×2 matrices. These "matrix transformations" are an important tool in geometry and, in turn, the geometry provides a "picture" of the matrices. Furthermore, matrix algebra has many other applications, some of which will be explored in this chapter. This subject is quite old and was first studied systematically in 1858 by Arthur Cayley.[1]

[1]Arthur Cayley (1821-1895) showed his mathematical talent early and graduated from Cambridge in 1842 as senior wrangler. With no employment in mathematics in view, he took legal training and worked as a lawyer while continuing to do mathematics, publishing nearly 300 papers in fourteen years. Finally, in 1863, he accepted the Sadlerian professorship in Cambridge and remained there for the rest of his life, valued for his administrative and teaching skills as well as for his scholarship. His mathematical achievements were of the first rank. In addition to originating matrix theory and the theory of determinants, he did fundamental work in group theory, in higher-dimensional geometry, and in the theory of invariants. He was one of the most prolific mathematicians of all time and produced 966 papers.

2.1 Matrix Addition, Scalar Multiplication, and Transposition

A rectangular array of numbers is called a **matrix** (the plural is **matrices**), and the numbers are called the **entries** of the matrix. Matrices are usually denoted by uppercase letters: A, B, C, and so on. Hence,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

are matrices. Clearly matrices come in various shapes depending on the number of **rows** and **columns**. For example, the matrix A shown has 2 rows and 3 columns. In general, a matrix with m rows and n columns is referred to as an $m \times n$ ***matrix*** or as having **size** $m \times n$. Thus matrices A, B, and C above have sizes 2×3, 2×2, and 3×1, respectively. A matrix of size $1 \times n$ is called a **row matrix**, whereas one of size $m \times 1$ is called a **column matrix**. Matrices of size $n \times n$ for some n are called **square** matrices.

Each entry of a matrix is identified by the row and column in which it lies. The rows are numbered from the top down, and the columns are numbered from left to right. Then the **(i, j)-entry** of a matrix is the number lying simultaneously in row i and column j. For example,

$$\text{The } (1, 2)\text{-entry of } \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ is } -1.$$

$$\text{The } (2, 3)\text{-entry of } \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \text{ is } 6.$$

A special notation is commonly used for the entries of a matrix. If A is an $m \times n$ matrix, and if the (i, j)-entry of A is denoted as a_{ij}, then A is displayed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

This is usually denoted simply as $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. Thus a_{ij} is the entry in row i and column j of A. For example, a 3×4 matrix in this notation is written

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

It is worth pointing out a convention regarding rows and columns: *Rows are mentioned before columns.* For example:

- *If a matrix has size $m \times n$, it has m rows and n columns.*
- *If we speak of the (i, j)-entry of a matrix, it lies in row i and column j.*

- *If an entry is denoted a_{ij}, the first subscript i refers to the row and the second subscript j to the column in which a_{ij} lies.*

Two points (x_1, y_1) and (x_2, y_2) in the plane are equal if and only if[2] they have the same coordinates, that is $x_1 = x_2$ and $y_1 = y_2$. Similarly, two matrices A and B are called **equal** (written $A = B$) if and only if:

1. *They have the same size.*

2. *Corresponding entries are equal.*

If the entries of A and B are written in the form $A = \left[a_{ij}\right]$, $B = \left[b_{ij}\right]$, described earlier, then the second condition takes the following form:

$$A = \left[a_{ij}\right] = \left[b_{ij}\right] \text{ means } a_{ij} = b_{ij} \text{ for all } i \text{ and } j$$

Example 2.1.1

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ discuss the possibility that $A = B$, $B = C$, $A = C$.

Solution. $A = B$ is impossible because A and B are of different sizes: A is 2×2 whereas B is 2×3. Similarly, $B = C$ is impossible. But $A = C$ is possible provided that corresponding entries are equal: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ means $a = 1$, $b = 0$, $c = -1$, and $d = 2$.

Matrix Addition

Definition 2.1 Matrix Addition

*If A and B are matrices of the same size, their **sum** $A + B$ is the matrix formed by adding corresponding entries.*

If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, this takes the form

$$A + B = \left[a_{ij} + b_{ij}\right]$$

Note that addition is *not* defined for matrices of different sizes.

[2] If p and q are statements, we say that p implies q if q is true whenever p is true. Then "p if and only if q" means that both p implies q and q implies p. See Appendix **??** for more on this.

Example 2.1.2

If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 6 \end{bmatrix}$, compute $A+B$.

Solution.

$$A+B = \begin{bmatrix} 2+1 & 1+1 & 3-1 \\ -1+2 & 2+0 & 0+6 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

Example 2.1.3

Find a, b, and c if $\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$.

Solution. Add the matrices on the left side to obtain

$$\begin{bmatrix} a+c & b+a & c+b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

Because corresponding entries must be equal, this gives three equations: $a+c=3$, $b+a=2$, and $c+b=-1$. Solving these yields $a=3$, $b=-1$, $c=0$.

If A, B, and C are any matrices *of the same size*, then

$$A+B = B+A \qquad \text{(commutative law)}$$
$$A+(B+C) = (A+B)+C \qquad \text{(associative law)}$$

In fact, if $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, then the $(i,\ j)$-entries of $A+B$ and $B+A$ are, respectively, $a_{ij}+b_{ij}$ and $b_{ij}+a_{ij}$. Since these are equal for all i and j, we get

$$A+B = \begin{bmatrix} a_{ij}+b_{ij} \end{bmatrix} = \begin{bmatrix} b_{ij}+a_{ij} \end{bmatrix} = B+A$$

The associative law is verified similarly.

The $m \times n$ matrix in which every entry is zero is called the $m \times n$ **zero matrix** and is denoted as 0 (or 0_{mn} if it is important to emphasize the size). Hence,

$$0+X = X$$

holds for all $m \times n$ matrices X. The **negative** of an $m \times n$ matrix A (written $-A$) is defined to be the $m \times n$ matrix obtained by multiplying each entry of A by -1. If $A = \left[a_{ij}\right]$, this becomes $-A = \left[-a_{ij}\right]$. Hence,

$$A+(-A) = 0$$

holds for all matrices A where, of course, 0 is the zero matrix of the same size as A.

A closely related notion is that of subtracting matrices. If A and B are two $m \times n$ matrices, their **difference** $A-B$ is defined by

$$A-B = A+(-B)$$

Note that if $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, then

$$A-B = \left[a_{ij}\right] + \left[-b_{ij}\right] = \left[a_{ij}-b_{ij}\right]$$

is the $m \times n$ matrix formed by *subtracting* corresponding entries.

Example 2.1.4

Let $A = \left[\begin{array}{rrr} 3 & -1 & 0 \\ 1 & 2 & -4 \end{array} \right]$, $B = \left[\begin{array}{rrr} 1 & -1 & 1 \\ -2 & 0 & 6 \end{array} \right]$, $C = \left[\begin{array}{rrr} 1 & 0 & -2 \\ 3 & 1 & 1 \end{array} \right]$. Compute $-A$, $A-B$, and $A+B-C$.

Solution.

$$
\begin{aligned}
-A &= \left[\begin{array}{rrr} -3 & 1 & 0 \\ -1 & -2 & 4 \end{array} \right] \\
A - B &= \left[\begin{array}{ccc} 3-1 & -1-(-1) & 0-1 \\ 1-(-2) & 2-0 & -4-6 \end{array} \right] = \left[\begin{array}{rrr} 2 & 0 & -1 \\ 3 & 2 & -10 \end{array} \right] \\
A + B - C &= \left[\begin{array}{ccc} 3+1-1 & -1-1-0 & 0+1-(-2) \\ 1-2-3 & 2+0-1 & -4+6-1 \end{array} \right] = \left[\begin{array}{rrr} 3 & -2 & 3 \\ -4 & 1 & 1 \end{array} \right]
\end{aligned}
$$

Example 2.1.5

Solve $\left[\begin{array}{rr} 3 & 2 \\ -1 & 1 \end{array} \right] + X = \left[\begin{array}{rr} 1 & 0 \\ -1 & 2 \end{array} \right]$ where X is a matrix.

Solution. We solve a numerical equation $a + x = b$ by subtracting the number a from both sides to obtain $x = b - a$. This also works for matrices. To solve $\left[\begin{array}{rr} 3 & 2 \\ -1 & 1 \end{array} \right] + X = \left[\begin{array}{rr} 1 & 0 \\ -1 & 2 \end{array} \right]$ simply subtract the matrix $\left[\begin{array}{rr} 3 & 2 \\ -1 & 1 \end{array} \right]$ from both sides to get

$$
X = \left[\begin{array}{rr} 1 & 0 \\ -1 & 2 \end{array} \right] - \left[\begin{array}{rr} 3 & 2 \\ -1 & 1 \end{array} \right] = \left[\begin{array}{cc} 1-3 & 0-2 \\ -1-(-1) & 2-1 \end{array} \right] = \left[\begin{array}{rr} -2 & -2 \\ 0 & 1 \end{array} \right]
$$

The reader should verify that this matrix X does indeed satisfy the original equation.

The solution in Example 2.1.5 solves the single matrix equation $A + X = B$ directly via matrix subtraction: $X = B - A$. This ability to work with matrices as entities lies at the heart of matrix algebra.

It is important to note that the sizes of matrices involved in some calculations are often determined by the context. For example, if

$$
A + C = \left[\begin{array}{rrr} 1 & 3 & -1 \\ 2 & 0 & 1 \end{array} \right]
$$

then A and C must be the same size (so that $A + C$ makes sense), and that size must be 2×3 (so that the sum is 2×3). For simplicity we shall often omit reference to such facts when they are clear from the context.

Scalar Multiplication

In gaussian elimination, multiplying a row of a matrix by a number k means multiplying *every* entry of that row by k.

Definition 2.2 Matrix Scalar Multiplication

*More generally, if A is any matrix and k is any number, the **scalar multiple** kA is the matrix obtained from A by multiplying each entry of A by k.*

If $A = \left[a_{ij}\right]$, this is

$$kA = \left[ka_{ij}\right]$$

Thus $1A = A$ and $(-1)A = -A$ for any matrix A.

The term *scalar* arises here because the set of numbers from which the entries are drawn is usually referred to as the set of scalars. We have been using real numbers as scalars, but we could equally well have been using complex numbers.

Example 2.1.6

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$ compute $5A$, $\frac{1}{2}B$, and $3A - 2B$.

Solution.

$$5A = \begin{bmatrix} 15 & -5 & 20 \\ 10 & 0 & 30 \end{bmatrix}, \quad \frac{1}{2}B = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{3}{2} & 1 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 9 & -3 & 12 \\ 6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -2 \\ 0 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 14 \\ 6 & -6 & 14 \end{bmatrix}$$

If A is any matrix, note that kA is the same size as A for all scalars k. We also have

$$0A = 0 \quad \text{and} \quad k0 = 0$$

because the zero matrix has every entry zero. In other words, $kA = 0$ if either $k = 0$ or $A = 0$. The converse of this statement is also true, as Example 2.1.7 shows.

Example 2.1.7

If $kA = 0$, show that either $k = 0$ or $A = 0$.

Solution. Write $A = \left[a_{ij}\right]$ so that $kA = 0$ means $ka_{ij} = 0$ for all i and j. If $k = 0$, there is nothing to do. If $k \neq 0$, then $ka_{ij} = 0$ implies that $a_{ij} = 0$ for all i and j; that is, $A = 0$.

For future reference, the basic properties of matrix addition and scalar multiplication are listed in Theorem 2.1.1.

Theorem 2.1.1

Let A, B, and C denote arbitrary $m \times n$ matrices where m and n are fixed. Let k and p denote arbitrary real numbers. Then

1. $A+B=B+A$.
2. $A+(B+C)=(A+B)+C$.
3. *There is an* $m \times n$ *matrix* 0, *such that* $0+A=A$ *for each* A.
4. *For each* A *there is an* $m \times n$ *matrix,* $-A$, *such that* $A+(-A)=0$.
5. $k(A+B)=kA+kB$.
6. $(k+p)A=kA+pA$.
7. $(kp)A=k(pA)$.
8. $1A=A$.

Proof. Properties 1–4 were given previously. To check Property 5, let $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ denote matrices of the same size. Then $A+B=\left[a_{ij}+b_{ij}\right]$, as before, so the $(i,\ j)$-entry of $k(A+B)$ is

$$k(a_{ij}+b_{ij})=ka_{ij}+kb_{ij}$$

But this is just the $(i,\ j)$-entry of $kA+kB$, and it follows that $k(A+B)=kA+kB$. The other Properties can be similarly verified; the details are left to the reader. □

The Properties in Theorem 2.1.1 enable us to do calculations with matrices in much the same way that numerical calculations are carried out. To begin, Property 2 implies that the sum

$$(A+B)+C=A+(B+C)$$

is the same no matter how it is formed and so is written as $A+B+C$. Similarly, the sum

$$A+B+C+D$$

is independent of how it is formed; for example, it equals both $(A+B)+(C+D)$ and $A+[B+(C+D)]$. Furthermore, property 1 ensures that, for example,

$$B+D+A+C=A+B+C+D$$

In other words, the *order* in which the matrices are added does not matter. A similar remark applies to sums of five (or more) matrices.

Properties 5 and 6 in Theorem 2.1.1 are called **distributive laws** for scalar multiplication, and they extend to sums of more than two terms. For example,

$$k(A+B-C)=kA+kB-kC$$

$$(k+p-m)A=kA+pA-mA$$

Similar observations hold for more than three summands. These facts, together with properties 7 and 8, enable us to simplify expressions by collecting like terms, expanding, and taking common factors in exactly the same way that algebraic expressions involving variables and real numbers are manipulated. The following example illustrates these techniques.

Example 2.1.8

Simplify $2(A+3C)-3(2C-B)-3\left[2(2A+B-4C)-4(A-2C)\right]$ where A, B, and C are all matrices of the same size.

Solution. The reduction proceeds as though A, B, and C were variables.

$$\begin{aligned}
&2(A+3C)-3(2C-B)-3\left[2(2A+B-4C)-4(A-2C)\right]\\
&\quad = 2A+6C-6C+3B-3\left[4A+2B-8C-4A+8C\right]\\
&\quad = 2A+3B-3\left[2B\right]\\
&\quad = 2A-3B
\end{aligned}$$

Transpose of a Matrix

Many results about a matrix A involve the *rows* of A, and the corresponding result for columns is derived in an analogous way, essentially by replacing the word *row* by the word *column* throughout. The following definition is made with such applications in mind.

Definition 2.3 Transpose of a Matrix

If A is an $m \times n$ matrix, the ***transpose*** *of A, written A^T, is the $n \times m$ matrix whose rows are just the columns of A in the same order.*

In other words, the first row of A^T is the first column of A (that is it consists of the entries of column 1 in order). Similarly the second row of A^T is the second column of A, and so on.

Example 2.1.9

Write down the transpose of each of the following matrices.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution.

$$A^T = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \quad B^T = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad \text{and } D^T = D.$$

If $A = \left[a_{ij}\right]$ is a matrix, write $A^T = \left[b_{ij}\right]$. Then b_{ij} is the jth element of the ith row of A^T and so is the jth element of the ith *column* of A. This means $b_{ij} = a_{ji}$, so the definition of A^T can be stated as follows:

$$\text{If } A = \left[a_{ij}\right], \text{ then } A^T = \left[a_{ji}\right]. \tag{2.1}$$

This is useful in verifying the following properties of transposition.

Theorem 2.1.2

Let A and B denote matrices of the same size, and let k denote a scalar.

1. *If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.*
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$.
4. $(A+B)^T = A^T + B^T$.

Proof. Property 1 is part of the definition of A^T, and Property 2 follows from (2.1). As to Property 3: If $A = \left[a_{ij}\right]$, then $kA = \left[ka_{ij}\right]$, so (2.1) gives

$$(kA)^T = \left[ka_{ji}\right] = k\left[a_{ji}\right] = kA^T$$

Finally, if $B = \left[b_{ij}\right]$, then $A+B = \left[c_{ij}\right]$ where $c_{ij} = a_{ij} + b_{ij}$ Then (2.1) gives Property 4:

$$(A+B)^T = \left[c_{ij}\right]^T = \left[c_{ji}\right] = \left[a_{ji} + b_{ji}\right] = \left[a_{ji}\right] + \left[b_{ji}\right] = A^T + B^T$$

□

There is another useful way to think of transposition. If $A = \left[a_{ij}\right]$ is an $m \times n$ matrix, the elements $a_{11}, a_{22}, a_{33}, \dots$ are called the **main diagonal** of A. Hence the main diagonal extends down and to the right from the upper left corner of the matrix A; it is shaded in the following examples:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$$

Thus forming the transpose of a matrix A can be viewed as "flipping" A about its main diagonal, or as "rotating" A through 180° about the line containing the main diagonal. This makes Property 2 in Theorem 2.1.2 transparent.

Example 2.1.10

Solve for A if $\left(2A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$.

Solution. Using Theorem 2.1.2, the left side of the equation is

$$\left(2A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right)^T = 2\left(A^T\right)^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^T = 2A - 3\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Hence the equation becomes

$$2A - 3\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

Thus $2A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 5 & 5 \end{bmatrix}$, so finally $A = \frac{1}{2}\begin{bmatrix} 5 & 0 \\ 5 & 5 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Note that Example 2.1.10 can also be solved by first transposing both sides, then solving for A^T, and so obtaining $A = (A^T)^T$. The reader should do this.

The matrix $D = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ in Example 2.1.9 has the property that $D = D^T$. Such matrices are important; a matrix A is called **symmetric** if $A = A^T$. A symmetric matrix A is necessarily square (if A is $m \times n$, then A^T is $n \times m$, so $A = A^T$ forces $n = m$). The name comes from the fact that these matrices exhibit a symmetry about the main diagonal. That is, entries that are directly across the main diagonal from each other are equal.

For example, $\begin{bmatrix} a & b & c \\ b' & d & e \\ c' & e' & f \end{bmatrix}$ is symmetric when $b = b'$, $c = c'$, and $e = e'$.

Example 2.1.11

If A and B are symmetric $n \times n$ matrices, show that $A + B$ is symmetric.

Solution. We have $A^T = A$ and $B^T = B$, so, by Theorem 2.1.2, we have $(A+B)^T = A^T + B^T = A + B$. Hence $A + B$ is symmetric.

Example 2.1.12

Suppose a square matrix A satisfies $A = 2A^T$. Show that necessarily $A = 0$.

Solution. If we iterate the given equation, Theorem 2.1.2 gives

$$A = 2A^T = 2\left[2A^T\right]^T = 2\left[2(A^T)^T\right] = 4A$$

Subtracting A from both sides gives $3A = 0$, so $A = \frac{1}{3}(0) = 0$.

Exercises for 2.1

Exercise 2.1.1 Find a, b, c, and d if

a. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c-3d & -d \\ 2a+d & a+b \end{bmatrix}$

b. $\begin{bmatrix} a-b & b-c \\ c-d & d-a \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$

c. $3\begin{bmatrix} a \\ b \end{bmatrix} + 2\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & c \\ d & a \end{bmatrix}$

---

b. $(a\ b\ c\ d) = (-2,\ -4,\ -6,\ 0) + t(1,\ 1,\ 1,\ 1)$, t arbitrary

d. $a = b = c = d = t$, t arbitrary

Exercise 2.1.2 Compute the following:

a) $\begin{bmatrix} 3 & 2 & 1 \\ 5 & 1 & 0 \end{bmatrix} - 5\begin{bmatrix} 3 & 0 & -2 \\ 1 & -1 & 2 \end{bmatrix}$

b) $3\begin{bmatrix} 3 \\ -1 \end{bmatrix} - 5\begin{bmatrix} 6 \\ 2 \end{bmatrix} + 7\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

c) $\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 2 & -3 \\ -1 & -2 \end{bmatrix}$

d) $\begin{bmatrix} 3 & -1 & 2 \end{bmatrix} - 2\begin{bmatrix} 9 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 11 & -6 \end{bmatrix}$

e) $\begin{bmatrix} 1 & -5 & 4 & 0 \\ 2 & 1 & 0 & 6 \end{bmatrix}^T$

f) $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}^T$

g) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^T$

h) $3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^T - 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

---

b. $\begin{bmatrix} -14 \\ -20 \end{bmatrix}$

d. $(-12,\ 4,\ -12)$

f. $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$

h. $\begin{bmatrix} 4 & -1 \\ -1 & -6 \end{bmatrix}$

Exercise 2.1.3 Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Compute the following (where possible).

a) $3A - 2B$

b) $5C$

c) $3E^T$

d) $B + D$

e) $4A^T - 3C$

f) $(A + C)^T$

g) $2B - 3E$

h) $A - D$

i) $(B - 2E)^T$

---

b. $\begin{bmatrix} 15 & -5 \\ 10 & 0 \end{bmatrix}$

d. Impossible

f. $\begin{bmatrix} 5 & 2 \\ 0 & -1 \end{bmatrix}$

h. Impossible

Exercise 2.1.4 Find A if:

a. $5A - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3A - \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$

b. $3A - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5A - 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

---

b. $\begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$

Exercise 2.1.5 Find A in terms of B if:

a) $A+B=3A+2B$ b) $2A-B=5(A+2B)$

---

b. $A=-\frac{11}{3}B$

Exercise 2.1.6 If X, Y, A, and B are matrices of the same size, solve the following systems of equations to obtain X and Y in terms of A and B.

a) $5X+3Y=A$, $2X+Y=B$ b) $4X+3Y=A$, $5X+4Y=B$

---

b. $X=4A-3B$, $Y=4B-5A$

Exercise 2.1.7 Find all matrices X and Y such that:

a) $3X-2Y=\begin{bmatrix} 3 & -1 \end{bmatrix}$ b) $2X-5Y=\begin{bmatrix} 1 & 2 \end{bmatrix}$

---

b. $Y=(s,\ t)$, $X=\frac{1}{2}(1+5s,\ 2+5t)$; s and t arbitrary

Exercise 2.1.8 Simplify the following expressions where A, B, and C are matrices.

a. $2[9(A-B)+7(2B-A)]$
$-2[3(2B+A)-2(A+3B)-5(A+B)]$

b. $5[3(A-B+2C)-2(3C-B)-A]$
$+2[3(3A-B+C)+2(B-2A)-2C]$

---

b. $20A-7B+2C$

Exercise 2.1.9 If A is any 2×2 matrix, show that:

a. $A=a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}+d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for some numbers a, b, c, and d.

b. $A=p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+q\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}+r\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}+s\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for some numbers p, q, r, and s.

---

b. If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $(p,\ q,\ r,\ s)=\frac{1}{2}(2d,\ a+b-c-d,\ a-b+c-d,\ -a+b+c+d)$.

Exercise 2.1.10 Let $A=\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$, and $C=\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. If $rA+sB+tC=0$ for some scalars r, s, and t, show that necessarily $r=s=t=0$.

Exercise 2.1.11

a. If $Q+A=A$ holds for every $m\times n$ matrix A, show that $Q=0_{mn}$.

b. If A is an $m\times n$ matrix and $A+A'=0_{mn}$, show that $A'=-A$.

---

b. If $A+A'=0$ then $-A=-A+0=-A+(A+A')=(-A+A)+A'=0+A'=A'$

Exercise 2.1.12 If A denotes an $m\times n$ matrix, show that $A=-A$ if and only if $A=0$.

Exercise 2.1.13 A square matrix is called a **diagonal** matrix if all the entries off the main diagonal are zero. If A and B are diagonal matrices, show that the following matrices are also diagonal.

a) $A+B$ b) $A-B$

c) kA for any number k

---

b. Write $A=\text{diag}(a_1,\ \dots,\ a_n)$, where $a_1,\ \dots,\ a_n$ are the main diagonal entries. If $B=\text{diag}(b_1,\ \dots,\ b_n)$ then $kA=\text{diag}(ka_1,\ \dots,\ ka_n)$.

Exercise 2.1.14 In each case determine all s and t such that the given matrix is symmetric:

a) $\begin{bmatrix} 1 & s \\ -2 & t \end{bmatrix}$ b) $\begin{bmatrix} s & t \\ st & 1 \end{bmatrix}$

c) $\begin{bmatrix} s & 2s & st \\ t & -1 & s \\ t & s^2 & s \end{bmatrix}$ d) $\begin{bmatrix} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{bmatrix}$

---

b. $s = 1$ or $t = 0$

d. $s = 0$, and $t = 3$

Exercise 2.1.15 In each case find the matrix A.

a. $\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$

b. $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$

c. $(2A - 3\begin{bmatrix} 1 & 2 & 0 \end{bmatrix})^T = 3A^T + \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$

d. $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

---

b. $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 7 \\ -\frac{9}{2} & -5 \end{bmatrix}$

Exercise 2.1.16 Let A and B be symmetric (of the same size). Show that each of the following is symmetric.

a) $(A - B)$ b) kA for any scalar k

---

b. $A = A^T$, so using Theorem 2.1.2, $(kA)^T = kA^T = kA$.

Exercise 2.1.17 Show that $A + A^T$ and AA^T are symmetric for *any* square matrix A.

Exercise 2.1.18 If A is a square matrix and $A = kA^T$ where $k \neq \pm 1$, show that $A = 0$.

Exercise 2.1.19 In each case either show that the statement is true or give an example showing it is false.

a. If $A + B = A + C$, then B and C have the same size.

b. If $A + B = 0$, then $B = 0$.

c. If the (3, 1)-entry of A is 5, then the (1, 3)-entry of A^T is -5.

d. A and A^T have the same main diagonal for every matrix A.

e. If B is symmetric and $A^T = 3B$, then $A = 3B$.

f. If A and B are symmetric, then $kA + mB$ is symmetric for any scalars k and m.

---

b. False. Take $B = -A$ for any $A \neq 0$.

d. True. Transposing fixes the main diagonal.

f. True. $(kA + mB)^T = (kA)^T + (mB)^T = kA^T + mB^T = kA + mB$

Exercise 2.1.20 A square matrix W is called **skew-symmetric** if $W^T = -W$. Let A be any square matrix.

a. Show that $A - A^T$ is skew-symmetric.

b. Find a symmetric matrix S and a skew-symmetric matrix W such that $A = S + W$.

c. Show that S and W in part (b) are uniquely determined by A.

---

c. Suppose $A = S + W$, where $S = S^T$ and $W = -W^T$. Then $A^T = S^T + W^T = S - W$, so $A + A^T = 2S$ and $A - A^T = 2W$. Hence $S = \frac{1}{2}(A + A^T)$ and $W = \frac{1}{2}(A - A^T)$ are uniquely determined by A.

Exercise 2.1.21 If W is skew-symmetric (Exercise 2.1.20), show that the entries on the main diagonal are zero.

Exercise 2.1.22 Prove the following parts of Theorem 2.1.1.

a) $(k+p)A = kA + pA$ b) $(kp)A = k(pA)$

---

b. If $A = [a_{ij}]$ then $(kp)A = [(kp)a_{ij}] = [k(pa_{ij})] = k[pa_{ij}] = k(pA)$.

Exercise 2.1.23 Let A, A_1, A_2, $\dots$, A_n denote matrices of the same size. Use induction on n to verify the following extensions of properties 5 and 6 of Theorem 2.1.1.

a. $k(A_1 + A_2 + \cdots + A_n) = kA_1 + kA_2 + \cdots + kA_n$ for any number k

b. $(k_1 + k_2 + \cdots + k_n)A = k_1A + k_2A + \cdots + k_nA$ for any numbers k_1, k_2, $\dots$, k_n

Exercise 2.1.24 Let A be a square matrix. If $A = pB^T$ and $B = qA^T$ for some matrix B and numbers p and q, show that either $A = 0 = B$ or $pq = 1$. [*Hint*: Example 2.1.7.]

2.2 Matrix-Vector Multiplication

Up to now we have used matrices to solve systems of linear equations by manipulating the rows of the augmented matrix. In this section we introduce a different way of describing linear systems that makes more use of the coefficient matrix of the system and leads to a useful way of "multiplying" matrices.

Vectors

It is a well-known fact in analytic geometry that two points in the plane with coordinates (a_1, a_2) and (b_1, b_2) are equal if and only if $a_1 = b_1$ and $a_2 = b_2$. Moreover, a similar condition applies to points (a_1, a_2, a_3) in space. We extend this idea as follows.

An ordered sequence $(a_1, a_2, \dots, a_n)$ of real numbers is called an **ordered n-tuple**. The word "ordered" here reflects our insistence that two ordered n-tuples are equal if and only if corresponding entries are the same. In other words,

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n) \quad \text{if and only if} \quad a_1 = b_1, a_2 = b_2, \dots, \text{ and } a_n = b_n.$$

Thus the ordered 2-tuples and 3-tuples are just the ordered pairs and triples familiar from geometry.

Definition 2.4 The set $\mathbb{R}^n$ of ordered n-tuples of real numbers

Let $\mathbb{R}$ denote the set of all real numbers. The set of all *ordered n-tuples from $\mathbb{R}$ has a special notation:*

$\mathbb{R}^n$ denotes the set of all ordered n-tuples of real numbers.

There are two commonly used ways to denote the n-tuples in $\mathbb{R}^n$: As rows $(r_1, r_2, \dots, r_n)$ or columns $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$; the notation we use depends on the context. In any event they are called **vectors** or **n-vectors** and will be denoted using bold type such as $\mathbf{x}$ or $\mathbf{v}$. For example, an $m \times n$ matrix A will be written as a row of columns:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \text{ where } \mathbf{a}_j \text{ denotes column } j \text{ of } A \text{ for each } j.$$

If $\mathbf{x}$ and $\mathbf{y}$ are two n-vectors in $\mathbb{R}^n$, it is clear that their matrix sum $\mathbf{x}+\mathbf{y}$ is also in $\mathbb{R}^n$ as is the scalar multiple $k\mathbf{x}$ for any real number k. We express this observation by saying that $\mathbb{R}^n$ is **closed** under addition and scalar multiplication. In particular, all the basic properties in Theorem 2.1.1 are true of these n-vectors. These properties are fundamental and will be used frequently below without comment. As for matrices in general, the $n \times 1$ zero matrix is called the **zero n-vector** in $\mathbb{R}^n$ and, if $\mathbf{x}$ is an n-vector, the n-vector $-\mathbf{x}$ is called the **negative $\mathbf{x}$**.

Of course, we have already encountered these n-vectors in Section 1.3 as the solutions to systems of linear equations with n variables. In particular we defined the notion of a linear combination of vectors and showed that a linear combination of solutions to a homogeneous system is again a solution. Clearly, a linear combination of n-vectors in $\mathbb{R}^n$ is again in $\mathbb{R}^n$, a fact that we will be using.

Matrix-Vector Multiplication

Given a system of linear equations, the left sides of the equations depend only on the coefficient matrix A and the column $\mathbf{x}$ of variables, and not on the constants. This observation leads to a fundamental idea in linear algebra: We view the left sides of the equations as the "product" $A\mathbf{x}$ of the matrix A and the vector $\mathbf{x}$. This simple change of perspective leads to a completely new way of viewing linear systems—one that is very useful and will occupy our attention throughout this book.

To motivate the definition of the "product" $A\mathbf{x}$, consider first the following system of two equations in three variables:

$$\begin{aligned} ax_1 + \ bx_2 + \ cx_3 &= b_1 \\ a'x_1 + b'x_2 + c'x_3 &= b_1 \end{aligned} \tag{2.2}$$

and let $A = \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ denote the coefficient matrix, the variable matrix, and the constant matrix, respectively. The system (2.2) can be expressed as a single vector equation

$$\begin{bmatrix} ax_1 + \ bx_2 + \ cx_3 \\ a'x_1 + b'x_2 + c'x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

which in turn can be written as follows:

$$x_1 \begin{bmatrix} a \\ a' \end{bmatrix} + x_2 \begin{bmatrix} b \\ b' \end{bmatrix} + x_3 \begin{bmatrix} c \\ c' \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Now observe that the vectors appearing on the left side are just the columns

$$\mathbf{a}_1 = \begin{bmatrix} a \\ a' \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} b \\ b' \end{bmatrix}, \ \text{and } \mathbf{a}_3 = \begin{bmatrix} c \\ c' \end{bmatrix}$$

of the coefficient matrix A. Hence the system (2.2) takes the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b} \tag{2.3}$$

This shows that the system (2.2) has a solution if and only if the constant matrix $\mathbf{b}$ is a linear combination[3] of the columns of A, and that in this case the entries of the solution are the coefficients x_1, x_2, and x_3 in this linear combination.

Moreover, this holds in general. If A is any $m \times n$ matrix, it is often convenient to view A as a row of columns. That is, if $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of A, we write

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

and say that $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ is *given in terms of its columns.*

[3]Linear combinations were introduced in Section 1.3 to describe the solutions of homogeneous systems of linear equations. They will be used extensively in what follows.

Now consider any system of linear equations with $m \times n$ coefficient matrix A. If $\mathbf{b}$ is the constant matrix of the system, and if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the matrix of variables then, exactly as above, the system can be written as a single vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{2.4}$$

Example 2.2.1

Write the system $\left\{\begin{aligned} 3x_1 + 2x_2 - 4x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 3 \\ x_2 - 5x_3 &= -1 \end{aligned}\right.$ in the form given in (2.4).

Solution.

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

As mentioned above, we view the left side of (2.4) as the *product* of the matrix A and the vector $\mathbf{x}$. This basic idea is formalized in the following definition:

Definition 2.5 Matrix-Vector Multiplication

*Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be an $m \times n$ matrix, written in terms of its columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any n-vector, the **product** $A\mathbf{x}$ is defined to be the m-vector given by:*

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

In other words, if A is $m \times n$ and $\mathbf{x}$ is an n-vector, the product $A\mathbf{x}$ is the linear combination of the columns of A where the coefficients are the entries of $\mathbf{x}$ (in order).

Note that if A is an $m \times n$ matrix, the product $A\mathbf{x}$ is only defined if $\mathbf{x}$ is an n-vector and then the vector $A\mathbf{x}$ is an m-vector because this is true of each column $\mathbf{a}_j$ of A. But in this case the *system* of linear equations with coefficient matrix A and constant vector $\mathbf{b}$ takes the form of a *single* matrix equation

$$A\mathbf{x} = \mathbf{b}$$

The following theorem combines Definition 2.5 and equation (2.4) and summarizes the above discussion. Recall that a system of linear equations is said to be *consistent* if it has at least one solution.

Theorem 2.2.1

1. *Every system of linear equations has the form* $A\mathbf{x} = \mathbf{b}$ *where* A *is the coefficient matrix,* $\mathbf{b}$ *is the constant matrix, and* $\mathbf{x}$ *is the matrix of variables.*

2. *The system* $A\mathbf{x} = \mathbf{b}$ *is consistent if and only if* $\mathbf{b}$ *is a linear combination of the columns of* A*.*

3. *If* $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ *are the columns of* A *and if* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, *then* $\mathbf{x}$ *is a solution to the linear system* $A\mathbf{x} = \mathbf{b}$ *if and only if* $x_1, x_2, \dots, x_n$ *are a solution of the vector equation*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

A system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ as in (1) of Theorem 2.2.1 is said to be written in **matrix form**. This is a useful way to view linear systems as we shall see.

Theorem 2.2.1 transforms the problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ into the problem of expressing the constant matrix B as a linear combination of the columns of the coefficient matrix A. Such a change in perspective is very useful because one approach or the other may be better in a particular situation; the importance of the theorem is that there is a choice.

Example 2.2.2

If $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, compute $A\mathbf{x}$.

Solution. By Definition 2.5: $A\mathbf{x} = 2\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix}$.

Example 2.2.3

Given columns $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in $\mathbb{R}^3$, write $2\mathbf{a}_1 - 3\mathbf{a}_2 + 5\mathbf{a}_3 + \mathbf{a}_4$ in the form $A\mathbf{x}$ where A is a matrix and $\mathbf{x}$ is a vector.

Solution. Here the column of coefficients is $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 5 \\ 1 \end{bmatrix}$. Hence Definition 2.5 gives

$$A\mathbf{x} = 2\mathbf{a}_1 - 3\mathbf{a}_2 + 5\mathbf{a}_3 + \mathbf{a}_4$$

where $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ is the matrix with $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ as its columns.

Example 2.2.4

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ be the 3×4 matrix given in terms of its columns $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$, and $\mathbf{a}_4 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. In each case below, either express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$, or show that it is not such a linear combination. Explain what your answer means for the corresponding system $A\mathbf{x} = \mathbf{b}$ of linear equations.

a. $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

b. $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Solution. By Theorem 2.2.1, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it has a solution). So in each case we carry the augmented matrix $[A|\mathbf{b}]$ of the system $A\mathbf{x} = \mathbf{b}$ to reduced form.

a. Here $\left[\begin{array}{rrrr|r} 2 & 1 & 3 & 3 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ -1 & 1 & -3 & 0 & 3 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$, so the system $A\mathbf{x} = \mathbf{b}$ has no solution in this case. Hence $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$.

b. Now $\left[\begin{array}{rrrr|r} 2 & 1 & 3 & 3 & 4 \\ 0 & 1 & -1 & 1 & 2 \\ -1 & 1 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, so the system $A\mathbf{x} = \mathbf{b}$ is consistent.

Thus $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in this case. In fact the general solution is $x_1 = 1 - 2s - t$, $x_2 = 2 + s - t$, $x_3 = s$, and $x_4 = t$ where s and t are arbitrary parameters. Hence $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ for *any* choice of s and t. If we take $s = 0$ and $t = 0$, this becomes $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{b}$, whereas taking $s = 1 = t$ gives $-2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = \mathbf{b}$.

Example 2.2.5

Taking A to be the zero matrix, we have $0\mathbf{x} = \mathbf{0}$ for all vectors $\mathbf{x}$ by Definition 2.5 because every column of the zero matrix is zero. Similarly, $A\mathbf{0} = \mathbf{0}$ for all matrices A because every entry of the zero vector is zero.

Example 2.2.6

If $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $I\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^3$.

Solution. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then Definition 2.5 gives

$$I\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}$$

The matrix I in Example 2.2.6 is called the 3×3 **identity matrix**, and we will encounter such matrices again in Example 2.2.11 below. Before proceeding, we develop some algebraic properties of matrix-vector multiplication that are used extensively throughout linear algebra.

Theorem 2.2.2

Let A and B be $m \times n$ matrices, and let $\mathbf{x}$ and $\mathbf{y}$ be n-vectors in $\mathbb{R}^n$. Then:

1. $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
2. $A(a\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$ *for all scalars a.*
3. $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$.

Proof. We prove (3); the other verifications are similar and are left as exercises. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ be given in terms of their columns. Since adding two matrices is the same as adding their columns, we have

$$A + B = \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 + \mathbf{b}_2 & \cdots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix}$$

If we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ Definition 2.5 gives

$$\begin{aligned} (A + B)\mathbf{x} &= x_1(\mathbf{a}_1 + \mathbf{b}_1) + x_2(\mathbf{a}_2 + \mathbf{b}_2) + \cdots + x_n(\mathbf{a}_n + \mathbf{b}_n) \\ &= (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n) + (x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n) \\ &= A\mathbf{x} + B\mathbf{x} \end{aligned}$$

□

Theorem 2.2.2 allows matrix-vector computations to be carried out much as in ordinary arithmetic. For example, for any $m \times n$ matrices A and B and any n-vectors $\mathbf{x}$ and $\mathbf{y}$, we have:

$$A(2\mathbf{x} - 5\mathbf{y}) = 2A\mathbf{x} - 5A\mathbf{y} \quad \text{and} \quad (3A - 7B)\mathbf{x} = 3A\mathbf{x} - 7B\mathbf{x}$$

We will use such manipulations throughout the book, often without mention.

Linear Equations

Theorem 2.2.2 also gives a useful way to describe the solutions to a system

$$A\mathbf{x} = \mathbf{b}$$

of linear equations. There is a related system

$$A\mathbf{x} = \mathbf{0}$$

called the **associated homogeneous system**, obtained from the original system $A\mathbf{x} = \mathbf{b}$ by replacing all the constants by zeros. Suppose $\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$ (that is $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_0 = \mathbf{0}$). Then $\mathbf{x}_1 + \mathbf{x}_0$ is another solution to $A\mathbf{x} = \mathbf{b}$. Indeed, Theorem 2.2.2 gives

$$A(\mathbf{x}_1 + \mathbf{x}_0) = A\mathbf{x}_1 + A\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

This observation has a useful converse.

Theorem 2.2.3

Suppose $\mathbf{x}_1$ is any particular solution to the system $A\mathbf{x} = \mathbf{b}$ of linear equations. Then every solution $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1$$

for some solution $\mathbf{x}_0$ of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof. Suppose $\mathbf{x}_2$ is also a solution to $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{x}_2 = \mathbf{b}$. Write $\mathbf{x}_0 = \mathbf{x}_2 - \mathbf{x}_1$. Then $\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1$ and, using Theorem 2.2.2, we compute

$$A\mathbf{x}_0 = A(\mathbf{x}_2 - \mathbf{x}_1) = A\mathbf{x}_2 - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Hence $\mathbf{x}_0$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. □

Note that gaussian elimination provides one such representation.

Example 2.2.7

Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 - x_2 - 3x_3 + 4x_4 &= 6 \\ x_1 - 2x_3 + x_4 &= 4 \end{aligned}$$

Solution. Gaussian elimination gives $x_1 = 4 + 2s - t$, $x_2 = 2 + s + 2t$, $x_3 = s$, and $x_4 = t$ where s and t are arbitrary parameters. Hence the general solution can be written

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4+2s-t \\ 2+s+2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \left(s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Thus $\mathbf{x}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution (where $s = 0 = t$), and $\mathbf{x}_0 = s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ gives *all* solutions to the associated homogeneous system. (To see why this is so, carry out the gaussian elimination again but with all the constants set equal to zero.)

The following useful result is included with no proof.

Theorem 2.2.4

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$. Write $\text{rank}\, A = r$.

1. $\text{rank} \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ *is either r or $r+1$.*
2. *The system is consistent if and only if* $\text{rank} \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r$.
3. *The system is inconsistent if and only if* $\text{rank} \left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r+1$.

The Dot Product

Definition 2.5 is not always the easiest way to compute a matrix-vector product $A\mathbf{x}$ because it requires that the columns of A be explicitly identified. There is another way to find such a product which uses the matrix A as a whole with no reference to its columns, and hence is useful in practice. The method depends on the following notion.

Definition 2.6 Dot Product in $\mathbb{R}^n$

If $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are two ordered n-tuples, their ***dot product*** *is defined to be the number*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

obtained by multiplying corresponding entries and adding the results.

To see how this relates to matrix products, let A denote a 3×4 matrix and let $\mathbf{x}$ be a 4-vector.

Writing

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

in the notation of Section 2.1, we compute

$$\begin{aligned} A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + x_4 \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \end{bmatrix} \end{aligned}$$

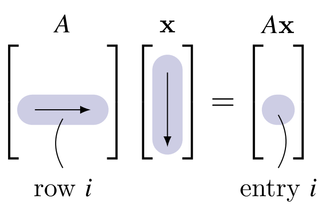
From this we see that each entry of $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$. This computation goes through in general, and we record the result in Theorem 2.2.5.

Theorem 2.2.5: Dot Product Rule

Let A be an $m \times n$ matrix and let $\mathbf{x}$ be an n-vector. Then each entry of the vector $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$.

This result is used extensively throughout linear algebra.

If A is $m \times n$ and $\mathbf{x}$ is an n-vector, the computation of $A\mathbf{x}$ by the dot product rule is simpler than using Definition 2.5 because the computation can be carried out directly with no explicit reference to the columns of A (as in Definition 2.5). The first entry of $A\mathbf{x}$ is the dot product of row 1 of A with $\mathbf{x}$. In hand calculations this is computed by going *across* row one of A, going *down* the column $\mathbf{x}$, multiplying corresponding entries, and adding the results. The other entries of $A\mathbf{x}$ are computed in the same way using the other rows of A with the column $\mathbf{x}$.



In general, compute entry i of $A\mathbf{x}$ as follows (see the diagram):

> Go *across* row i of A and *down* column $\mathbf{x}$, multiply corresponding entries, and add the results.

As an illustration, we rework Example 2.2.2 using the dot product rule instead of Definition 2.5.

Example 2.2.8

If $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, compute $A\mathbf{x}$.

Solution. The entries of $A\mathbf{x}$ are the dot products of the rows of A with $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 & + & (-1)1 & + & 3 \cdot 0 & + & 5(-2) \\ 0 \cdot 2 & + & 2 \cdot 1 & + & (-3)0 & + & 1(-2) \\ (-3)2 & + & 4 \cdot 1 & + & 1 \cdot 0 & + & 2(-2) \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix}$$

Of course, this agrees with the outcome in Example 2.2.2.

Example 2.2.9

Write the following system of linear equations in the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} 5x_1 - x_2 + 2x_3 + x_4 - 3x_5 &= 8 \\ x_1 + x_2 + 3x_3 - 5x_4 + 2x_5 &= -2 \\ -x_1 + x_2 - 2x_3 + - 3x_5 &= 0 \end{aligned}$$

Solution. Write $A = \begin{bmatrix} 5 & -1 & 2 & 1 & -3 \\ 1 & 1 & 3 & -5 & 2 \\ -1 & 1 & -2 & 0 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. Then the dot product rule gives $A\mathbf{x} = \begin{bmatrix} 5x_1 - x_2 + 2x_3 + x_4 - 3x_5 \\ x_1 + x_2 + 3x_3 - 5x_4 + 2x_5 \\ -x_1 + x_2 - 2x_3 - 3x_5 \end{bmatrix}$, so the entries of $A\mathbf{x}$ are the left sides of the equations in the linear system. Hence the system becomes $A\mathbf{x} = \mathbf{b}$ because matrices are equal if and only corresponding entries are equal.

Example 2.2.10

If A is the zero $m \times n$ matrix, then $A\mathbf{x} = \mathbf{0}$ for each n-vector $\mathbf{x}$.

Solution. For each k, entry k of $A\mathbf{x}$ is the dot product of row k of A with $\mathbf{x}$, and this is zero because row k of A consists of zeros.

Definition 2.7 The Identity Matrix

For each $n > 2$, the ***identity matrix*** I_n *is the $n \times n$ matrix with 1s on the main diagonal (upper left to lower right), and zeros elsewhere.*

The first few identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \dots$$

In Example 2.2.6 we showed that $I_3\mathbf{x} = \mathbf{x}$ for each 3-vector $\mathbf{x}$ using Definition 2.5. The following result shows that this holds in general, and is the reason for the name.

Example 2.2.11

For each $n \geq 2$ we have $I_n\mathbf{x} = \mathbf{x}$ for each n-vector $\mathbf{x}$ in $\mathbb{R}^n$.

Solution. We verify the case $n = 4$. Given the 4-vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ the dot product rule gives

$$I_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1+0+0+0 \\ 0+x_2+0+0 \\ 0+0+x_3+0 \\ 0+0+0+x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}$$

In general, $I_n\mathbf{x} = \mathbf{x}$ because entry k of $I_n\mathbf{x}$ is the dot product of row k of I_n with $\mathbf{x}$, and row k of I_n has 1 in position k and zeros elsewhere.

Example 2.2.12

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be any $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. If $\mathbf{e}_j$ denotes column j of the $n \times n$ identity matrix I_n, then $A\mathbf{e}_j = \mathbf{a}_j$ for each $j = 1, 2, \dots, n$.

Solution. Write $\mathbf{e}_j = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$ where $t_j = 1$, but $t_i = 0$ for all $i \neq j$. Then Theorem 2.2.5 gives

$$A\mathbf{e}_j = t_1\mathbf{a}_1 + \cdots + t_j\mathbf{a}_j + \cdots + t_n\mathbf{a}_n = 0 + \cdots + \mathbf{a}_j + \cdots + 0 = \mathbf{a}_j$$

Example 2.2.12 will be referred to later; for now we use it to prove:

Theorem 2.2.6

Let A and B be $m \times n$ matrices. If $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, then $A = B$.

Proof. Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ and in terms of their columns. It is enough to show that $\mathbf{a}_k = \mathbf{b}_k$ holds for all k. But we are assuming that $A\mathbf{e}_k = B\mathbf{e}_k$, which gives $\mathbf{a}_k = \mathbf{b}_k$ by Example 2.2.12. □

We have introduced matrix-vector multiplication as a new way to think about systems of linear equations. But it has several other uses as well. It turns out that many geometric operations can be described using matrix multiplication, and we now investigate how this happens. As a bonus, this description provides a geometric "picture" of a matrix by revealing the effect on a vector when it is multiplied by A. This "geometric view" of matrices is a fundamental tool in understanding them.

Transformations

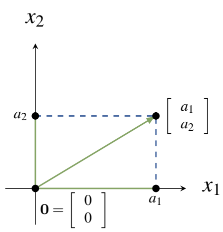


Figure 2.2.1

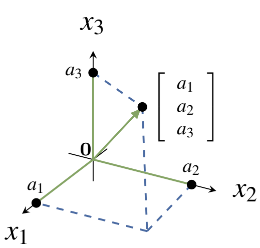


Figure 2.2.2

The set $\mathbb{R}^2$ has a geometrical interpretation as the euclidean plane where a vector $\left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]$ in $\mathbb{R}^2$ represents the point (a_1, a_2) in the plane (see Figure 2.2.1). In this way we regard $\mathbb{R}^2$ as the set of all points in the plane. Accordingly, we will refer to vectors in $\mathbb{R}^2$ as points, and denote their coordinates as a column rather than a row. To enhance this geometrical interpretation of the vector $\left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]$, it is denoted graphically by an arrow from the origin $\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$ to the vector as in Figure 2.2.1.

Similarly we identify $\mathbb{R}^3$ with 3-dimensional space by writing a point (a_1, a_2, a_3) as the vector $\left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right]$ in $\mathbb{R}^3$, again represented by an arrow[4] from the origin to the point as in Figure 2.2.2. In this way the terms "point" and "vector" mean the same thing in the plane or in space.

We begin by describing a particular geometrical transformation of the plane $\mathbb{R}^2$.

Example 2.2.13

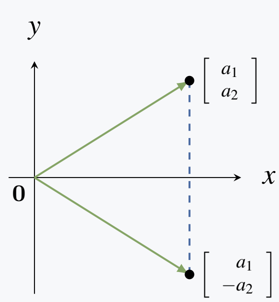


Figure 2.2.3

Consider the transformation of $\mathbb{R}^2$ given by *reflection* in the x axis. This operation carries the vector $\left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]$ to its reflection $\left[\begin{array}{c} a_1 \\ -a_2 \end{array}\right]$ as in Figure 2.2.3. Now observe that

$$\left[\begin{array}{c} a_1 \\ -a_2 \end{array}\right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] \left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]$$

so reflecting $\left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]$ in the x axis can be achieved by

[4]This "arrow" representation of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ will be used extensively in Chapter 4.

multiplying by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

If we write $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, Example 2.2.13 shows that reflection in the x axis carries each vector $\mathbf{x}$ in $\mathbb{R}^2$ to the vector $A\mathbf{x}$ in $\mathbb{R}^2$. It is thus an example of a function

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{where} \quad T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^2$$

As such it is a generalization of the familiar functions $f : \mathbb{R} \to \mathbb{R}$ that carry a *number* x to another real *number* $f(x)$.

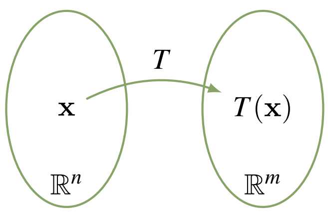


Figure 2.2.4

More generally, functions $T : \mathbb{R}^n \to \mathbb{R}^m$ are called **transformations** from $\mathbb{R}^n$ to $\mathbb{R}^m$. Such a transformation T is a rule that assigns to every vector $\mathbf{x}$ in $\mathbb{R}^n$ a uniquely determined vector $T(\mathbf{x})$ in $\mathbb{R}^m$ called the **image** of $\mathbf{x}$ under T. We denote this state of affairs by writing

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{or} \quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

The transformation T can be visualized as in Figure 2.2.4.

To describe a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we must specify the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ for every $\mathbf{x}$ in $\mathbb{R}^n$. This is referred to as **defining** T, or as specifying the **action** of T. Saying that the action *defines* the transformation means that we regard two transformations $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ as **equal** if they have the **same action**; more formally

$$S = T \quad \text{if and only if} \quad S(\mathbf{x}) = T(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Again, this what we mean by $f = g$ where $f,\ g : \mathbb{R} \to \mathbb{R}$ are ordinary functions.

Functions $f : \mathbb{R} \to \mathbb{R}$ are often described by a formula, examples being $f(x) = x^2 + 1$ and $f(x) = \sin x$. The same is true of transformations; here is an example.

Example 2.2.14

The formula $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$ defines a transformation $\mathbb{R}^4 \to \mathbb{R}^3$.

Example 2.2.13 suggests that matrix multiplication is an important way of defining transformations $\mathbb{R}^n \to \mathbb{R}^m$. If A is any $m \times n$ matrix, multiplication by A gives a transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{defined by} \quad T_A(\mathbf{x}) = A\mathbf{x} \text{ for every } \mathbf{x} \text{ in } \mathbb{R}^n$$

Definition 2.8 Matrix Transformation T_A

T_A is called the ***matrix transformation induced*** *by A.*

Thus Example 2.2.13 shows that reflection in the x axis is the matrix transformation $\mathbb{R}^2 \to \mathbb{R}^2$ induced by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Also, the transformation $R : \mathbb{R}^4 \to \mathbb{R}^3$ in Example 2.2.13 is the matrix transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ because } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$$

Example 2.2.15

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation about the origin through $\frac{\pi}{2}$ radians (that is, 90°)[5]. Show that $R_{\frac{\pi}{2}}$ is induced by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution.

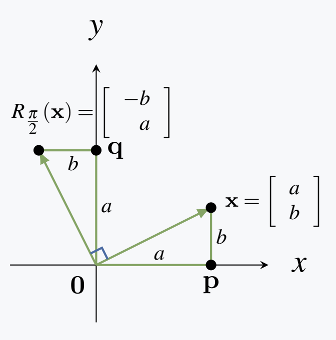


Figure 2.2.5

The effect of $R_{\frac{\pi}{2}}$ is to rotate the vector $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ counterclockwise through $\frac{\pi}{2}$ to produce the vector $R_{\frac{\pi}{2}}(\mathbf{x})$ shown in Figure 2.2.5. Since triangles $\mathbf{0px}$ and $\mathbf{0q}R_{\frac{\pi}{2}}(\mathbf{x})$ are identical, we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = \begin{bmatrix} -b \\ a \end{bmatrix}$. But $\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In other words, $R_{\frac{\pi}{2}}$ is the matrix transformation induced by A.

If A is the $m \times n$ zero matrix, then A induces the transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{given by} \quad T(\mathbf{x}) = A\mathbf{x} = \mathbf{0} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

This is called the **zero transformation**, and is denoted $T = 0$.

Another important example is the **identity transformation**

$$1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{given by} \quad 1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

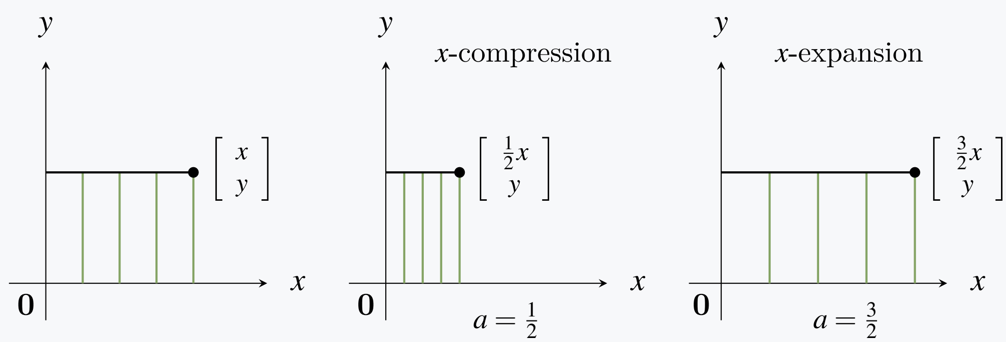
That is, the action of $1_{\mathbb{R}^n}$ on $\mathbf{x}$ is to do nothing to it. If I_n denotes the $n \times n$ identity matrix, we showed in Example 2.2.11 that $I_n\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Hence $1_{\mathbb{R}^n}(\mathbf{x}) = I_n\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$; that is, the identity matrix I_n induces the identity transformation.

Here are two more examples of matrix transformations with a clear geometric description.

[5] *Radian measure* for angles is based on the fact that 360° equals 2π radians. Hence π radians $= 180^\circ$ and $\frac{\pi}{2}$ radians $= 90^\circ$.

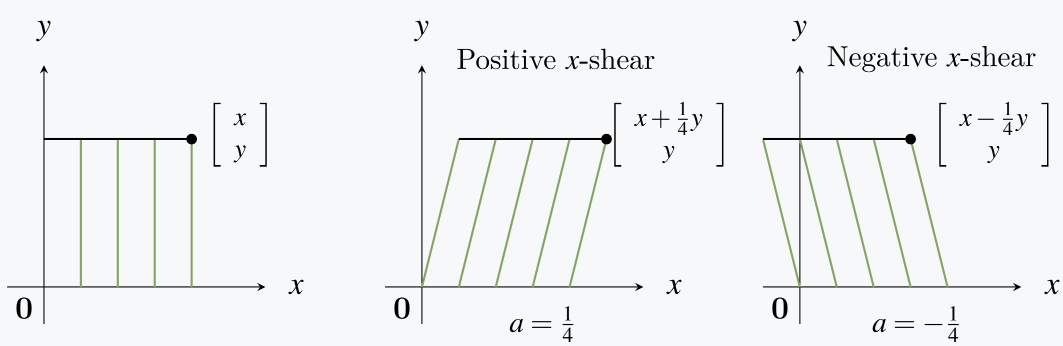
Example 2.2.16

If $a > 0$, the matrix transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is called an ***x*-expansion** of $\mathbb{R}^2$ if $a > 1$, and an ***x*-compression** if $0 < a < 1$. The reason for the names is clear in the diagram below. Similarly, if $b > 0$ the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ gives rise to ***y*-expansions** and ***y*-compressions**.



Example 2.2.17

If a is a number, the matrix transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is called an ***x*-shear** of $\mathbb{R}^2$ (**positive** if $a > 0$ and **negative** if $a < 0$). Its effect is illustrated below when $a = \frac{1}{4}$ and $a = -\frac{1}{4}$.



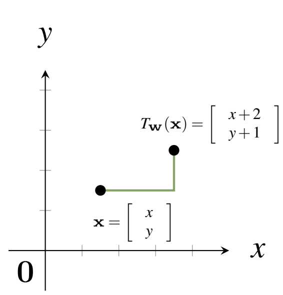


Figure 2.2.6

We hasten to note that there are important geometric transformations that are *not* matrix transformations. For example, if $\mathbf{w}$ is a fixed column in $\mathbb{R}^n$, define the transformation $T_\mathbf{w} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_\mathbf{w}(\mathbf{x}) = \mathbf{x} + \mathbf{w} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Then $T_\mathbf{w}$ is called **translation** by $\mathbf{w}$. In particular, if $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in

$\mathbb{R}^2$, the effect of $T_{\mathbf{w}}$ on $\left[\begin{array}{c} x \\ y \end{array}\right]$ is to translate it two units to the right and one unit up (see Figure 2.2.6).

The translation $T_{\mathbf{w}}$ is not a matrix transformation unless $\mathbf{w} = \mathbf{0}$. Indeed, if $T_{\mathbf{w}}$ were induced by a matrix A, then $A\mathbf{x} = T_{\mathbf{w}}(\mathbf{x}) = \mathbf{x} + \mathbf{w}$ would hold for every $\mathbf{x}$ in $\mathbb{R}^n$. In particular, taking $\mathbf{x} = \mathbf{0}$ gives $\mathbf{w} = A\mathbf{0} = \mathbf{0}$.

Exercises for 2.2

Exercise 2.2.1 In each case find a system of equations that is equivalent to the given vector equation. (Do not solve the system.)

a. $x_1 \left[\begin{array}{r} 2 \\ -3 \\ 0 \end{array}\right] + x_2 \left[\begin{array}{r} 1 \\ 1 \\ 4 \end{array}\right] + x_3 \left[\begin{array}{r} 2 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{r} 5 \\ 6 \\ -3 \end{array}\right]$

b. $x_1 \left[\begin{array}{r} 1 \\ 0 \\ 1 \\ 0 \end{array}\right] + x_2 \left[\begin{array}{r} -3 \\ 8 \\ 2 \\ 1 \end{array}\right] + x_3 \left[\begin{array}{r} -3 \\ 0 \\ 2 \\ 2 \end{array}\right] + x_4 \left[\begin{array}{r} 3 \\ 2 \\ 0 \\ -2 \end{array}\right] = \left[\begin{array}{r} 5 \\ 1 \\ 2 \\ 0 \end{array}\right]$

---

b. $\begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & - & 3x_3 & + & 3x_4 & = & 5 \\ & & 8x_2 & & & + & 2x_4 & = & 1 \\ x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ & & x_2 & + & 2x_3 & - & 5x_4 & = & 0 \end{array}$

Exercise 2.2.2 In each case find a vector equation that is equivalent to the given system of equations. (Do not solve the equation.)

a. $\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 3x_3 & = & 5 \\ -3x_1 & + & x_2 & + & x_3 & = & -6 \\ 5x_1 & - & 8x_2 & & & = & 9 \end{array}$

b. $\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & x_4 & = & 5 \\ -x_1 & & & + & x_3 & - & 2x_4 & = & -3 \\ 2x_1 & - & 2x_2 & + & 7x_3 & & & = & 8 \\ 3x_1 & - & 4x_2 & + & 9x_3 & - & 2x_4 & = & 12 \end{array}$

---

b. $x_1 \left[\begin{array}{r} 1 \\ -1 \\ 2 \\ 3 \end{array}\right] + x_2 \left[\begin{array}{r} -2 \\ 0 \\ -2 \\ -4 \end{array}\right] + x_3 \left[\begin{array}{r} -1 \\ 1 \\ 7 \\ 9 \end{array}\right] + x_4 \left[\begin{array}{r} 1 \\ -2 \\ 0 \\ -2 \end{array}\right] = \left[\begin{array}{r} 5 \\ -3 \\ 8 \\ 12 \end{array}\right]$

Exercise 2.2.3 In each case compute $A\mathbf{x}$ using: (i) Definition 2.5. (ii) Theorem 2.2.5.

a. $A = \left[\begin{array}{rrr} 3 & -2 & 0 \\ 5 & -4 & 1 \end{array}\right]$ and $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]$.

b. $A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 0 & -4 & 5 \end{array}\right]$ and $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]$.

c. $A = \left[\begin{array}{rrrr} -2 & 0 & 5 & 4 \\ 1 & 2 & 0 & 3 \\ -5 & 6 & -7 & 8 \end{array}\right]$ and $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right]$.

d. $A = \left[\begin{array}{rrrr} 3 & -4 & 1 & 6 \\ 0 & 2 & 1 & 5 \\ -8 & 7 & -3 & 0 \end{array}\right]$ and $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right]$.

---

b. $A\mathbf{x} = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 0 & -4 & 5 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = x_1 \left[\begin{array}{r} 1 \\ 0 \end{array}\right] + x_2 \left[\begin{array}{r} 2 \\ -4 \end{array}\right] + x_3 \left[\begin{array}{r} 3 \\ 5 \end{array}\right] = \left[\begin{array}{c} x_1 + 2x_2 + 3x_3 \\ -4x_2 + 5x_3 \end{array}\right]$

d. $A\mathbf{x} = \left[\begin{array}{rrrr} 3 & -4 & 1 & 6 \\ 0 & 2 & 1 & 5 \\ -8 & 7 & -3 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right]$

$$= x_1 \begin{bmatrix} 3 \\ 0 \\ -8 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3x_1 - 4x_2 + x_3 + 6x_4 \\ 2x_2 + x_3 + 5x_4 \\ -8x_1 + 7x_2 - 3x_3 \end{bmatrix}$$

Exercise 2.2.4 Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ be the 3×4 matrix given in terms of its columns $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, and $\mathbf{a}_4 = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$. In each case either express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$, or show that it is not such a linear combination. Explain what your answer means for the corresponding system $A\mathbf{x} = \mathbf{b}$ of linear equations.

a) $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

b) $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

---

b. To solve $A\mathbf{x} = \mathbf{b}$ the reduction is

$$\left[\begin{array}{rrrr|r} 1 & 3 & 2 & 0 & 4 \\ 1 & 0 & -1 & -3 & 1 \\ -1 & 2 & 3 & 5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & -1 & -3 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So the general solution is $\begin{bmatrix} 1+s+3t \\ 1-s-t \\ s \\ t \end{bmatrix}$.

Hence $(1+s+3t)\mathbf{a}_1 + (1-s-t)\mathbf{a}_2 + s\mathbf{a}_3 + t\mathbf{a}_4 = \mathbf{b}$ for any choice of s and t. If $s = t = 0$, we get $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{b}$; if $s = 1$ and $t = 0$, we have $2\mathbf{a}_1 + \mathbf{a}_3 = \mathbf{b}$.

Exercise 2.2.5 In each case, express every solution of the system as a sum of a specific solution plus a solution of the associated homogeneous system.

a) $\begin{aligned} x + y + z &= 2 \\ 2x + y \quad &= 3 \\ x - y - 3z &= 0 \end{aligned}$

b) $\begin{aligned} x - y - 4z &= -4 \\ x + 2y + 5z &= 2 \\ x + y + 2z &= 0 \end{aligned}$

c) $\begin{aligned} x_1 + x_2 - x_3 \quad - 5x_5 &= 2 \\ x_2 + x_3 \quad - 4x_5 &= -1 \\ x_2 + x_3 + x_4 - x_5 &= -1 \\ 2x_1 \quad - 4x_3 + x_4 + x_5 &= 6 \end{aligned}$

d) $\begin{aligned} 2x_1 + x_2 - x_3 - x_4 &= -1 \\ 3x_1 + x_2 + x_3 - 2x_4 &= -2 \\ -x_1 - x_2 + 2x_3 + x_4 &= 2 \\ -2x_1 - x_2 \quad + 2x_4 &= 3 \end{aligned}$

---

b. $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$

d. $\begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$

Exercise 2.2.6 If $\mathbf{x}_0$ and $\mathbf{x}_1$ are solutions to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, use Theorem 2.2.2 to show that $s\mathbf{x}_0 + t\mathbf{x}_1$ is also a solution for any scalars s and t (called a **linear combination** of $\mathbf{x}_0$ and $\mathbf{x}_1$).

We have $A\mathbf{x}_0 = \mathbf{0}$ and $A\mathbf{x}_1 = \mathbf{0}$ and so $A(s\mathbf{x}_0 + t\mathbf{x}_1) = s(A\mathbf{x}_0) + t(A\mathbf{x}_1) = s \cdot \mathbf{0} + t \cdot \mathbf{0} = \mathbf{0}$.

Exercise 2.2.7 Assume that $A \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \mathbf{0} = A \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$. Show that $\mathbf{x}_0 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Find a two-parameter family of solutions to $A\mathbf{x} = \mathbf{b}$.

Exercise 2.2.8 In each case write the system in the form $A\mathbf{x} = \mathbf{b}$, use the gaussian algorithm to solve the system, and express the solution as a particular solution plus a linear combination of basic solutions to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

a. $\begin{aligned} x_1 - 2x_2 + x_3 + 4x_4 - x_5 &= 8 \\ -2x_1 + 4x_2 + x_3 - 2x_4 - 4x_5 &= -1 \\ 3x_1 - 6x_2 + 8x_3 + 4x_4 - 13x_5 &= 1 \\ 8x_1 - 16x_2 + 7x_3 + 12x_4 - 6x_5 &= 11 \end{aligned}$

b.
$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= -4 \\ -3x_1 + 6x_2 - 2x_3 - 3x_4 - 11x_5 &= 11 \\ -2x_1 + 4x_2 - x_3 + x_4 - 8x_5 &= 7 \\ -x_1 + 2x_2 + 3x_4 - 5x_5 &= 3 \end{aligned}$$

---

b. $\mathbf{x} = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \left(s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$.

Exercise 2.2.9 Given vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, find a vector $\mathbf{b}$ that is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. Justify your answer. [*Hint*: Part (2) of Theorem 2.2.1.]

Exercise 2.2.10 In each case either show that the statement is true, or give an example showing that it is false.

a. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

b. If $A\mathbf{x}$ has a zero entry, then A has a row of zeros.

c. If $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} \neq \mathbf{0}$, then $A = 0$.

d. Every linear combination of vectors in $\mathbb{R}^n$ can be written in the form $A\mathbf{x}$.

e. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if $\mathbf{b} = 3\mathbf{a}_1 - 2\mathbf{a}_2$, then the system $A\mathbf{x} = \mathbf{b}$ has a solution.

f. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if the system $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{b} = s\mathbf{a}_1 + t\mathbf{a}_2$ for some s, t.

g. If A is $m \times n$ and $m < n$, then $A\mathbf{x} = \mathbf{b}$ has a solution for every column $\mathbf{b}$.

h. If $A\mathbf{x} = \mathbf{b}$ has a solution for some column $\mathbf{b}$, then it has a solution for every column $\mathbf{b}$.

i. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

j. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns. If $\mathbf{a}_3 = s\mathbf{a}_1 + t\mathbf{a}_2$, then $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} s \\ t \\ -1 \end{bmatrix}$.

---

b. False. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

d. True. The linear combination $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ equals $A\mathbf{x}$ where $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ by Theorem 2.2.1.

f. False. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, then

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \neq s \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for any } s \text{ and } t.$$

h. False. If $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, there is a solution for $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ but not for $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Exercise 2.2.11 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation. In each case show that T is induced by a matrix and find the matrix.

a. T is a reflection in the y axis.

b. T is a reflection in the line $y = x$.

c. T is a reflection in the line $y = -x$.

d. T is a clockwise rotation through $\frac{\pi}{2}$.

---

b. Here $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

d. Here $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Exercise 2.2.12 The **projection** $P : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$. Show that P is induced by a matrix and find the matrix.

Exercise 2.2.13 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation. In each case show that T is induced by a matrix and find the matrix.

a. T is a reflection in the $x-y$ plane.

b. T is a reflection in the $y-z$ plane.

---

b. Here

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so the matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercise 2.2.14 Fix $a > 0$ in $\mathbb{R}$, and define $T_a : \mathbb{R}^4 \to \mathbb{R}^4$ by $T_a(\mathbf{x}) = a\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^4$. Show that T is induced by a matrix and find the matrix. [T is called a **dilation** if $a > 1$ and a **contraction** if $a < 1$.]

Exercise 2.2.15 Let A be $m \times n$ and let $\mathbf{x}$ be in $\mathbb{R}^n$. If A has a row of zeros, show that $A\mathbf{x}$ has a zero entry.

Exercise 2.2.16 If a vector $\mathbf{b}$ is a linear combination of the columns of A, show that the system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it has at least one solution.)

Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ in terms of its columns. If $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$ where the x_i are scalars, then $A\mathbf{x} = \mathbf{b}$ by Theorem 2.2.1 where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$. That is, $\mathbf{x}$ is a solution to the system $A\mathbf{x} = \mathbf{b}$.

Exercise 2.2.17 If a system $A\mathbf{x} = \mathbf{b}$ is inconsistent (no solution), show that $\mathbf{b}$ is not a linear combination of the columns of A.

Exercise 2.2.18 Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

a. Show that $\mathbf{x}_1 + \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

b. Show that $t\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{0}$ for any scalar t.

---

b. By Theorem 2.2.3, $A(t\mathbf{x}_1) = t(A\mathbf{x}_1) = t \cdot \mathbf{0} = \mathbf{0}$; that is, $t\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{0}$.

Exercise 2.2.19 Suppose $\mathbf{x}_1$ is a solution to the system $A\mathbf{x} = \mathbf{b}$. If $\mathbf{x}_0$ is any nontrivial solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, show that $\mathbf{x}_1 + t\mathbf{x}_0$, t a scalar, is an infinite one parameter family of solutions to $A\mathbf{x} = \mathbf{b}$. [*Hint*: Example 2.1.7 Section 2.1.]

Exercise 2.2.20 Let A and B be matrices of the same size. If $\mathbf{x}$ is a solution to both the system $A\mathbf{x} = \mathbf{0}$ and the system $B\mathbf{x} = \mathbf{0}$, show that $\mathbf{x}$ is a solution to the system $(A + B)\mathbf{x} = \mathbf{0}$.

Exercise 2.2.21 If A is $m \times n$ and $A\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$ in $\mathbb{R}^n$, show that $A = 0$ is the zero matrix. [*Hint*: Consider $A\mathbf{e}_j$ where $\mathbf{e}_j$ is the jth column of I_n; that is, $\mathbf{e}_j$ is the vector in $\mathbb{R}^n$ with 1 as entry j and every other entry 0.]

Exercise 2.2.22 Prove part (1) of Theorem 2.2.2.

If A is $m \times n$ and $\mathbf{x}$ and $\mathbf{y}$ are n-vectors, we must show that $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$. Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\dots$, $\mathbf{a}_n$, and write $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$. Then $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 & x_2 + y_2 & \cdots & x_n + y_n \end{bmatrix}^T$, so Definition 2.1 and Theorem 2.1.1 give $A(\mathbf{x} + \mathbf{y}) = (x_1 + y_1)\mathbf{a}_1 + (x_2 + y_2)\mathbf{a}_2 + \cdots + (x_n + y_n)\mathbf{a}_n = (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n) + (y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \cdots + y_n\mathbf{a}_n) = A\mathbf{x} + A\mathbf{y}$.

Exercise 2.2.23 Prove part (2) of Theorem 2.2.2.

2.3 Matrix Multiplication

In Section 2.2 matrix-vector products were introduced. If A is an $m \times n$ matrix, the product $A\mathbf{x}$ was defined for any n-column $\mathbf{x}$ in $\mathbb{R}^n$ as follows: If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ where the $\mathbf{a}_j$ are the columns of A, and if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, Definition 2.5 reads

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \tag{2.5}$$

This was motivated as a way of describing systems of linear equations with coefficient matrix A. Indeed every such system has the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is the column of constants.

In this section we extend this matrix-vector multiplication to a way of multiplying matrices in general, and then investigate matrix algebra for its own sake. While it shares several properties of ordinary arithmetic, it will soon become clear that matrix arithmetic is different in a number of ways.

Matrix multiplication is closely related to composition of transformations.

Composition and Matrix Multiplication

Sometimes two transformations "link" together as follows:

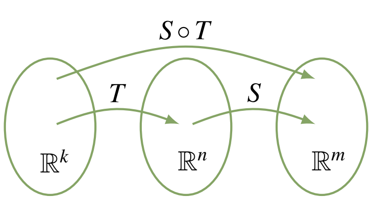
$$\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$$

In this case we can apply T first and then apply S, and the result is a new transformation

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m$$

called the **composite** of S and T, defined by

$$(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^k$$



The action of $S \circ T$ can be described as "first T then S" (note the order!)[6]. This new transformation is described in the diagram. The reader will have encountered composition of ordinary functions: For example, consider $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$ where $f(x) = x^2$ and $g(x) = x + 1$ for all x in $\mathbb{R}$. Then

$$(f \circ g)(x) = f[g(x)] = f(x+1) = (x+1)^2$$
$$(g \circ f)(x) = g[f(x)] = g(x^2) = x^2 + 1$$

[6]When reading the notation $S \circ T$, we read S first and then T even though the action is "first T then S". This annoying state of affairs results because we write $T(\mathbf{x})$ for the effect of the transformation T on $\mathbf{x}$, with T on the left. If we wrote this instead as $(\mathbf{x})T$, the confusion would not occur. However the notation $T(\mathbf{x})$ is well established.

for all x in $\mathbb{R}$.

Our concern here is with matrix transformations. Suppose that A is an $m \times n$ matrix and B is an $n \times k$ matrix, and let $\mathbb{R}^k \xrightarrow{T_B} \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$ be the matrix transformations induced by B and A respectively, that is:

$$T_B(\mathbf{x}) = B\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^k \quad \text{and} \quad T_A(\mathbf{y}) = A\mathbf{y} \text{ for all } \mathbf{y} \text{ in } \mathbb{R}^n$$

Write $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right]$ where $\mathbf{b}_j$ denotes column j of B for each j. Hence each $\mathbf{b}_j$ is an n-vector (B is $n \times k$) so we can form the matrix-vector product $A\mathbf{b}_j$. In particular, we obtain an $m \times k$ matrix

$$\left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array}\right]$$

with columns $A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_k$. Now compute $(T_A \circ T_B)(\mathbf{x})$ for any $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_k \end{array}\right]$ in $\mathbb{R}^k$:

$$\begin{array}{rcll}
(T_A \circ T_B)(\mathbf{x}) & = & T_A\left[T_B(\mathbf{x})\right] & \text{Definition of } T_A \circ T_B \\
& = & A(B\mathbf{x}) & A \text{ and } B \text{ induce } T_A \text{ and } T_B \\
& = & A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k) & \text{Equation 2.5 above} \\
& = & A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_k\mathbf{b}_k) & \text{Theorem 2.2.2} \\
& = & x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_k(A\mathbf{b}_k) & \text{Theorem 2.2.2} \\
& = & \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array}\right]\mathbf{x} & \text{Equation 2.5 above}
\end{array}$$

Because $\mathbf{x}$ was an arbitrary vector in $\mathbb{R}^n$, this shows that $T_A \circ T_B$ is the matrix transformation induced by the matrix $\left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array}\right]$. This motivates the following definition.

Definition 2.9 Matrix Multiplication

Let A be an $m \times n$ matrix, let B be an $n \times k$ matrix, and write $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right]$ where $\mathbf{b}_j$ is column j of B for each j. The product matrix AB is the $m \times k$ matrix defined as follows:

$$AB = A\left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array}\right]$$

Thus the product matrix AB is given in terms of its columns $A\mathbf{b}_1, A\mathbf{b}_2, \dots, A\mathbf{b}_n$: Column j of AB is the matrix-vector product $A\mathbf{b}_j$ of A and the corresponding column $\mathbf{b}_j$ of B. Note that each such product $A\mathbf{b}_j$ makes sense by Definition 2.5 because A is $m \times n$ and each $\mathbf{b}_j$ is in $\mathbb{R}^n$ (since B has n rows). Note also that if B is a column matrix, this definition reduces to Definition 2.5 for matrix-vector multiplication.

Given matrices A and B, Definition 2.9 and the above computation give

$$A(B\mathbf{x}) = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array}\right]\mathbf{x} = (AB)\mathbf{x}$$

for all $\mathbf{x}$ in $\mathbb{R}^k$. We record this for reference.

Theorem 2.3.1

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. Then the product matrix AB is $m \times k$ and satisfies

$$A(B\mathbf{x}) = (AB)\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^k$$

Here is an example of how to compute the product AB of two matrices using Definition 2.9.

Example 2.3.1

Compute AB if $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix}$.

Solution. The columns of B are $\mathbf{b}_1 = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}$, so Definition 2.5 gives

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 67 \\ 78 \\ 55 \end{bmatrix} \text{ and } A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \\ 10 \end{bmatrix}$$

Hence Definition 2.9 above gives $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{bmatrix}$.

Example 2.3.2

If A is $m \times n$ and B is $n \times k$, Theorem 2.3.1 gives a simple formula for the composite of the matrix transformations T_A and T_B:

$$T_A \circ T_B = T_{AB}$$

Solution. Given any $\mathbf{x}$ in $\mathbb{R}^k$,

$$\begin{aligned} (T_A \circ T_B)(\mathbf{x}) &= T_A[T_B(\mathbf{x})] \\ &= A[B\mathbf{x}] \\ &= (AB)\mathbf{x} \\ &= T_{AB}(\mathbf{x}) \end{aligned}$$

While Definition 2.9 is important, there is another way to compute the matrix product AB that gives a way to calculate each individual entry. In Section 2.2 we defined the dot product of two n-tuples to be the sum of the products of corresponding entries. We went on to show (Theorem 2.2.5) that if A is an $m \times n$ matrix and $\mathbf{x}$ is an n-vector, then entry j of the product $A\mathbf{x}$ is the dot product of row j of A with $\mathbf{x}$. This observation was called the "dot product rule" for matrix-vector multiplication, and the next theorem shows that it extends to matrix multiplication in general.

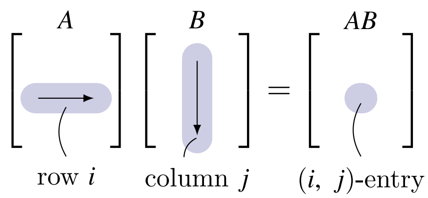
Theorem 2.3.2: Dot Product Rule

Let A and B be matrices of sizes $m \times n$ and $n \times k$, respectively. Then the $(i,\ j)$-entry of AB is the dot product of row i of A with column j of B.

Proof. Write $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{array} \right]$ in terms of its columns. Then $A\mathbf{b}_j$ is column j of AB for each j. Hence the $(i,\ j)$-entry of AB is entry i of $A\mathbf{b}_j$, which is the dot product of row i of A with $\mathbf{b}_j$. This proves the theorem. □

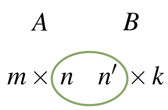
Thus to compute the $(i,\ j)$-entry of AB, proceed as follows (see the diagram):

Go *across* row i of A, and *down* column j of B, multiply corresponding entries, and add the results.



Note that this requires that the rows of A must be the same length as the columns of B. The following rule is useful for remembering this and for deciding the size of the product matrix AB.

Compatibility Rule



Let A and B denote matrices. If A is $m \times n$ and B is $n' \times k$, the product AB can be formed if and only if $n = n'$. In this case the size of the product matrix AB is $m \times k$, and we say that AB is **defined**, or that A and B are **compatible** for multiplication.

The diagram provides a useful mnemonic for remembering this. We adopt the following convention:

Convention

Whenever a product of matrices is written, it is tacitly assumed that the sizes of the factors are such that the product is defined.

To illustrate the dot product rule, we recompute the matrix product in Example 2.3.1.

Example 2.3.3

Compute AB if $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix}$.

Solution. Here A is 3×3 and B is 3×2, so the product matrix AB is defined and will be of size 3×2. Theorem 2.3.2 gives each entry of AB as the dot product of the corresponding row of A with the corresponding column of B_j that is,

$$AB = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 8 + 3 \cdot 7 + 5 \cdot 6 & 2 \cdot 9 + 3 \cdot 2 + 5 \cdot 1 \\ 1 \cdot 8 + 4 \cdot 7 + 7 \cdot 6 & 1 \cdot 9 + 4 \cdot 2 + 7 \cdot 1 \\ 0 \cdot 8 + 1 \cdot 7 + 8 \cdot 6 & 0 \cdot 9 + 1 \cdot 2 + 8 \cdot 1 \end{bmatrix} = \begin{bmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{bmatrix}$$

Of course, this agrees with Example 2.3.1.

Example 2.3.4

Compute the (1, 3)- and (2, 4)-entries of AB where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}.$$

Then compute AB.

Solution. The (1, 3)-entry of AB is the dot product of row 1 of A and column 3 of B (highlighted in the following display), computed by multiplying corresponding entries and adding the results.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} \quad (1,\ 3)\text{-entry} = 3 \cdot 6 + (-1) \cdot 3 + 2 \cdot 5 = 25$$

Similarly, the (2, 4)-entry of AB involves row 2 of A and column 4 of B.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} \quad (2,\ 4)\text{-entry} = 0 \cdot 0 + 1 \cdot 4 + 4 \cdot 8 = 36$$

Since A is 2×3 and B is 3×4, the product is 2×4.

$$AB = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix}$$

Example 2.3.5

If $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, compute A^2, AB, BA, and B^2 when they are defined.[7]

Solution. Here, A is a 1×3 matrix and B is a 3×1 matrix, so A^2 and B^2 are not defined. However, the compatibility rule reads

$$\begin{array}{cc} A & B \\ 1 \times 3 & 3 \times 1 \end{array} \quad \text{and} \quad \begin{array}{cc} B & A \\ 3 \times 1 & 1 \times 3 \end{array}$$

so both AB and BA can be formed and these are 1×1 and 3×3 matrices, respectively.

$$AB = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 3 \cdot 6 + 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 31 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 3 & 5 \cdot 2 \\ 6 \cdot 1 & 6 \cdot 3 & 6 \cdot 2 \\ 4 \cdot 1 & 4 \cdot 3 & 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 10 \\ 6 & 18 & 12 \\ 4 & 12 & 8 \end{bmatrix}$$

Unlike numerical multiplication, matrix products AB and BA *need not be equal*. In fact they need not even be the same size, as Example 2.3.5 shows. It turns out to be rare that $AB = BA$ (although it is by no means impossible), and A and B are said to **commute** when this happens.

Example 2.3.6

Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2, AB, BA.

[7] As for numbers, we write $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, etc. Note that A^2 is defined if and only if A is of size $n \times n$ for some n.

Solution. $A^2 = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $A^2 = 0$ can occur even if $A \neq 0$. Next,

$$AB = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ 2 & -8 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -6 & -9 \end{bmatrix}$$

Hence $AB \neq BA$, even though AB and BA are the same size.

Example 2.3.7

If A is any matrix, then $IA = A$ and $AI = A$, and where I denotes an identity matrix of a size so that the multiplications are defined.

Solution. These both follow from the dot product rule as the reader should verify. For a more formal proof, write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ where $\mathbf{a}_j$ is column j of A. Then Definition 2.9 and Example 2.2.11 give

$$IA = \begin{bmatrix} I\mathbf{a}_1 & I\mathbf{a}_2 & \cdots & I\mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = A$$

If $\mathbf{e}_j$ denotes column j of I, then $A\mathbf{e}_j = \mathbf{a}_j$ for each j by Example 2.2.12. Hence Definition 2.9 gives:

$$AI = A\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{e}_1 & A\mathbf{e}_2 & \cdots & A\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = A$$

The following theorem collects several results about matrix multiplication that are used everywhere in linear algebra.

Theorem 2.3.3

Assume that a is any scalar, and that A, B, and C are matrices of sizes such that the indicated matrix products are defined. Then:

1. *$IA = A$ and $AI = A$ where I denotes an identity matrix.*
2. *$A(BC) = (AB)C$.*
3. *$A(B+C) = AB + AC$.*
4. *$(B+C)A = BA + CA$.*
5. *$a(AB) = (aA)B = A(aB)$.*
6. *$(AB)^T = B^T A^T$.*

Proof. Condition (1) is Example 2.3.7; we prove (2), (4), and (6) and leave (3) and (5) as exercises.

1. If $C = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_k \end{bmatrix}$ in terms of its columns, then $BC = \begin{bmatrix} B\mathbf{c}_1 & B\mathbf{c}_2 & \cdots & B\mathbf{c}_k \end{bmatrix}$ by Defi-

nition 2.9, so

$$\begin{aligned} A(BC) &= \left[\begin{array}{cccc} A(B\mathbf{c}_1) & A(B\mathbf{c}_2) & \cdots & A(B\mathbf{c}_k) \end{array}\right] && \text{Definition 2.9} \\ &= \left[\begin{array}{cccc} (AB)\mathbf{c}_1 & (AB)\mathbf{c}_2 & \cdots & (AB)\mathbf{c}_k) \end{array}\right] && \text{Theorem 2.3.1} \\ &= (AB)C && \text{Definition 2.9} \end{aligned}$$

4. We know (Theorem 2.2.2) that $(B+C)\mathbf{x} = B\mathbf{x} + C\mathbf{x}$ holds for every column $\mathbf{x}$. If we write $A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right]$ in terms of its columns, we get

$$\begin{aligned} (B+C)A &= \left[\begin{array}{cccc} (B+C)\mathbf{a}_1 & (B+C)\mathbf{a}_2 & \cdots & (B+C)\mathbf{a}_n \end{array}\right] && \text{Definition 2.9} \\ &= \left[\begin{array}{cccc} B\mathbf{a}_1 + C\mathbf{a}_1 & B\mathbf{a}_2 + C\mathbf{a}_2 & \cdots & B\mathbf{a}_n + C\mathbf{a}_n \end{array}\right] && \text{Theorem 2.2.2} \\ &= \left[\begin{array}{cccc} B\mathbf{a}_1 & B\mathbf{a}_2 & \cdots & B\mathbf{a}_n \end{array}\right] + \left[\begin{array}{cccc} C\mathbf{a}_1 & C\mathbf{a}_2 & \cdots & C\mathbf{a}_n \end{array}\right] && \text{Adding Columns} \\ &= BA + CA && \text{Definition 2.9} \end{aligned}$$

6. As in Section 2.1, write $A = [a_{ij}]$ and $B = [b_{ij}]$, so that $A^T = [a'_{ij}]$ and $B^T = [b'_{ij}]$ where $a'_{ij} = a_{ji}$ and $b'_{ji} = b_{ij}$ for all i and j. If c_{ij} denotes the $(i,\ j)$-entry of $B^T A^T$, then c_{ij} is the dot product of row i of B^T with column j of A^T. Hence

$$\begin{aligned} c_{ij} = b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \cdots + b'_{im}a'_{mj} &= b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{mi}a_{jm} \\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jm}b_{mi} \end{aligned}$$

But this is the dot product of row j of A with column i of B; that is, the $(j,\ i)$-entry of AB; that is, the $(i,\ j)$-entry of $(AB)^T$. This proves (6). □

Property 2 in Theorem 2.3.3 is called the **associative law** of matrix multiplication. It asserts that the equation $A(BC) = (AB)C$ holds for all matrices (if the products are defined). Hence this product is the same no matter how it is formed, and so is written simply as ABC. This extends: The product $ABCD$ of four matrices can be formed several ways—for example, $(AB)(CD)$, $[A(BC)]D$, and $A[B(CD)]$—but the associative law implies that they are all equal and so are written as $ABCD$. A similar remark applies in general: Matrix products can be written unambiguously with no parentheses.

However, a note of caution about matrix multiplication must be taken: The fact that AB and BA need *not* be equal means that the *order* of the factors is important in a product of matrices. For example $ABCD$ and $ADCB$ may *not* be equal.

Warning

If the order of the factors in a product of matrices is changed, the product matrix may change (or may not be defined). Ignoring this warning is a source of many errors by students of linear algebra!

Properties 3 and 4 in Theorem 2.3.3 are called **distributive laws**. They assert that $A(B+C) = AB+AC$ and $(B+C)A = BA+CA$ hold whenever the sums and products are defined. These rules extend to more than two terms and, together with Property 5, ensure that many manipulations familiar from ordinary algebra extend to matrices. For example

$$\begin{aligned} A(2B-3C+D-5E) &= 2AB-3AC+AD-5AE \\ (A+3C-2D)B &= AB+3CB-2DB \end{aligned}$$

Note again that the warning is in effect: For example $A(B-C)$ need *not* equal $AB-CA$. These rules make possible a lot of simplification of matrix expressions.

Example 2.3.8

Simplify the expression $A(BC-CD)+A(C-B)D-AB(C-D)$.

Solution.

$$\begin{aligned} A(BC-CD)+A(C-B)D-AB(C-D) &= A(BC)-A(CD)+(AC-AB)D-(AB)C+(AB)D \\ &= ABC-ACD+ACD-ABD-ABC+ABD \\ &= 0 \end{aligned}$$

Example 2.3.9 and Example 2.3.10 below show how we can use the properties in Theorem 2.3.2 to deduce other facts about matrix multiplication. Matrices A and B are said to **commute** if $AB = BA$.

Example 2.3.9

Suppose that A, B, and C are $n \times n$ matrices and that both A and B commute with C; that is, $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Solution. Showing that AB commutes with C means verifying that $(AB)C = C(AB)$. The computation uses the associative law several times, as well as the given facts that $AC = CA$ and $BC = CB$.

$$(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$$

Example 2.3.10

Show that $AB = BA$ if and only if $(A-B)(A+B) = A^2-B^2$.

Solution. The following *always* holds:

$$(A-B)(A+B) = A(A+B)-B(A+B) = A^2+AB-BA-B^2 \tag{2.6}$$

Hence if $AB = BA$, then $(A-B)(A+B) = A^2-B^2$ follows. Conversely, if this last equation holds, then equation (2.6) becomes

$$A^2-B^2 = A^2+AB-BA-B^2$$

This gives $0 = AB - BA$, and $AB = BA$ follows.

In Section 2.2 we saw (in Theorem 2.2.1) that every system of linear equations has the form

$$A\mathbf{x} = \mathbf{b}$$

where A is the coefficient matrix, $\mathbf{x}$ is the column of variables, and $\mathbf{b}$ is the constant matrix. Thus the *system* of linear equations becomes a single matrix equation. Matrix multiplication can yield information about such a system.

Example 2.3.11

Consider a system $A\mathbf{x} = \mathbf{b}$ of linear equations where A is an $m \times n$ matrix. Assume that a matrix C exists such that $CA = I_n$. If the system $A\mathbf{x} = \mathbf{b}$ *has* a solution, show that this solution must be $C\mathbf{b}$. Give a condition guaranteeing that $C\mathbf{b}$ *is in fact* a solution.

Solution. Suppose that $\mathbf{x}$ is any solution to the system, so that $A\mathbf{x} = \mathbf{b}$. Multiply both sides of this matrix equation by C to obtain, successively,

$$C(A\mathbf{x}) = C\mathbf{b}, \quad (CA)\mathbf{x} = C\mathbf{b}, \quad I_n\mathbf{x} = C\mathbf{b}, \quad \mathbf{x} = C\mathbf{b}$$

This shows that *if* the system has a solution $\mathbf{x}$, then that solution must be $\mathbf{x} = C\mathbf{b}$, as required. But it does *not* guarantee that the system *has* a solution. However, if we write $\mathbf{x}_1 = C\mathbf{b}$, then

$$A\mathbf{x}_1 = A(C\mathbf{b}) = (AC)\mathbf{b}$$

Thus $\mathbf{x}_1 = C\mathbf{b}$ will be a solution if the condition $AC = I_m$ is satisfied.

The ideas in Example 2.3.11 lead to important information about matrices; this will be pursued in the next section.

Block Multiplication

Definition 2.10 Block Partition of a Matrix

It is often useful to consider matrices whose entries are themselves matrices (called ***blocks****). A matrix viewed in this way is said to be* ***partitioned into blocks****.*

For example, writing a matrix B in the form

$$B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array} \right] \text{ where the } \mathbf{b}_j \text{ are the columns of } B$$

is such a block partition of B. Here is another example.

Consider the matrices

$$A = \left[\begin{array}{rr|rrr} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array}\right] = \left[\begin{array}{cc} I_2 & 0_{23} \\ P & Q \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{rr} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array}\right] = \left[\begin{array}{c} X \\ Y \end{array}\right]$$

where the blocks have been labelled as indicated. This is a natural way to partition A into blocks in view of the blocks I_2 and 0_{23} that occur. This notation is particularly useful when we are multiplying the matrices A and B because the product AB can be computed in block form as follows:

$$AB = \left[\begin{array}{cc} I & 0 \\ P & Q \end{array}\right] \left[\begin{array}{c} X \\ Y \end{array}\right] = \left[\begin{array}{c} IX + 0Y \\ PX + QY \end{array}\right] = \left[\begin{array}{c} X \\ PX + QY \end{array}\right] = \left[\begin{array}{rr} 4 & -2 \\ 5 & 6 \\ \hline 30 & 8 \\ 8 & 27 \end{array}\right]$$

This is easily checked to be the product AB, computed in the conventional manner.

In other words, *we can compute the product* AB *by ordinary matrix multiplication, using blocks as entries.* The only requirement is that the blocks be **compatible**. That is, *the sizes of the blocks must be such that all (matrix) products of blocks that occur make sense.* This means that the number of columns in each block of A must equal the number of rows in the corresponding block of B.

Theorem 2.3.4: Block Multiplication

If matrices A and B are partitioned compatibly into blocks, the product AB can be computed by matrix multiplication using blocks as entries.

We omit the proof.

We have been using two cases of block multiplication. If $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right]$ is a matrix where the $\mathbf{b}_j$ are the columns of B, and if the matrix product AB is defined, then we have

$$AB = A\left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array}\right]$$

This is Definition 2.9 and is a block multiplication where $A = [A]$ has only one block. As another illustration,

$$B\mathbf{x} = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_k \end{array}\right] = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k$$

where $\mathbf{x}$ is any $k \times 1$ column matrix (this is Definition 2.5).

It is not our intention to pursue block multiplication in detail here. However, we give one more example because it will be used below.

Theorem 2.3.5

Suppose matrices $A = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$ *and* $A_1 = \begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix}$ *are partitioned as shown where* B *and* B_1 *are square matrices of the same size, and* C *and* C_1 *are also square of the same size. These are compatible partitionings and block multiplication gives*

$$AA_1 = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix} \begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix} = \begin{bmatrix} BB_1 & BX_1 + XC_1 \\ 0 & CC_1 \end{bmatrix}$$

Example 2.3.12

Obtain a formula for A^k where $A = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}$ is square and I is an identity matrix.

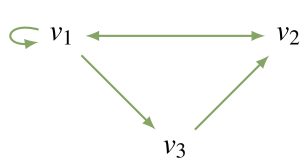
Solution. We have $A^2 = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I^2 & IX + X0 \\ 0 & 0^2 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} = A$. Hence $A^3 = AA^2 = AA = A^2 = A$. Continuing in this way, we see that $A^k = A$ for every $k \geq 1$.

Block multiplication has theoretical uses as we shall see. However, it is also useful in computing products of matrices in a computer with limited memory capacity. The matrices are partitioned into blocks in such a way that each product of blocks can be handled. Then the blocks are stored in auxiliary memory and their products are computed one by one.

Directed Graphs

The study of directed graphs illustrates how matrix multiplication arises in ways other than the study of linear equations or matrix transformations.

A **directed graph** consists of a set of points (called **vertices**) connected by arrows (called **edges**). For example, the vertices could represent cities and the edges available flights. If the graph has n vertices $v_1, v_2, \dots, v_n$, the **adjacency** matrix $A = [a_{ij}]$ is the $n \times n$ matrix whose (i, j)-entry a_{ij} is 1 if there is an edge from v_j to v_i (note the order), and zero otherwise. For example, the adjacency matrix of the directed graph shown is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.



A **path of length** r (or an r-**path**) from vertex j to vertex i is a sequence of r edges leading from v_j to v_i. Thus $v_1 \to v_2 \to v_1 \to v_1 \to v_3$ is a 4-path from v_1 to v_3 in the given graph. The edges are just the paths of length 1, so the (i, j)-entry a_{ij} of the adjacency matrix A is the number of 1-paths from v_j to v_i. This observation has an important extension:

Theorem 2.3.6

If A is the adjacency matrix of a directed graph with n vertices, then the $(i,\ j)$-entry of A^r is the number of r-paths $v_j \to v_i$.

As an illustration, consider the adjacency matrix A in the graph shown. Then

$$A = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right], \quad A^2 = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right], \quad \text{and} \quad A^3 = \left[\begin{array}{ccc} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{array}\right]$$

Hence, since the $(2,\ 1)$-entry of A^2 is 2, there are two 2-paths $v_1 \to v_2$ (in fact they are $v_1 \to v_1 \to v_2$ and $v_1 \to v_3 \to v_2$). Similarly, the $(2,\ 3)$-entry of A^2 is zero, so there are *no* 2-paths $v_3 \to v_2$, as the reader can verify. The fact that no entry of A^3 is zero shows that it is possible to go from any vertex to any other vertex in exactly three steps.

To see why Theorem 2.3.6 is true, observe that it asserts that

$$\text{the } (i,\ j)\text{-entry of } A^r \text{ equals the number of } r\text{-paths } v_j \to v_i \tag{2.7}$$

holds for each $r \geq 1$. We proceed by induction on r (see Appendix **??**). The case $r = 1$ is the definition of the adjacency matrix. So assume inductively that (2.7) is true for some $r \geq 1$; we must prove that (2.7) also holds for $r+1$. But every $(r+1)$-path $v_j \to v_i$ is the result of an r-path $v_j \to v_k$ for some k, followed by a 1-path $v_k \to v_i$. Writing $A = \left[a_{ij}\right]$ and $A^r = \left[b_{ij}\right]$, there are b_{kj} paths of the former type (by induction) and a_{ik} of the latter type, and so there are $a_{ik}b_{kj}$ such paths in all. Summing over k, this shows that there are

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \quad (r+1)\text{-paths } v_j \to v_i$$

But this sum is the dot product of the ith row $\left[\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \end{array}\right]$ of A with the jth column $\left[\begin{array}{cccc} b_{1j} & b_{2j} & \cdots & b_{nj} \end{array}\right]^T$ of A^r. As such, it is the $(i,\ j)$-entry of the matrix product $A^rA = A^{r+1}$. This shows that (2.7) holds for $r+1$, as required.

Exercises for 2.3

Exercise 2.3.1 Compute the following matrix products.

a) $\left[\begin{array}{rr} 1 & 3 \\ 0 & -2 \end{array}\right]\left[\begin{array}{rr} 2 & -1 \\ 0 & 1 \end{array}\right]$

b) $\left[\begin{array}{rrr} 1 & -1 & 2 \\ 2 & 0 & 4 \end{array}\right]\left[\begin{array}{rrr} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{array}\right]$

c) $\left[\begin{array}{rrr} 5 & 0 & -7 \\ 1 & 5 & 9 \end{array}\right]\left[\begin{array}{r} 3 \\ 1 \\ -1 \end{array}\right]$

d) $\left[\begin{array}{rrr} 1 & 3 & -3 \end{array}\right]\left[\begin{array}{rr} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{array}\right]$

e) $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{rr} 3 & -2 \\ 5 & -7 \\ 9 & 7 \end{array}\right]$

f) $\left[\begin{array}{rrr} 1 & -1 & 3 \end{array}\right] \left[\begin{array}{r} 2 \\ 1 \\ -8 \end{array}\right]$

g) $\left[\begin{array}{r} 2 \\ 1 \\ -7 \end{array}\right] \left[\begin{array}{rrr} 1 & -1 & 3 \end{array}\right]$

h) $\left[\begin{array}{rr} 3 & 1 \\ 5 & 2 \end{array}\right] \left[\begin{array}{rr} 2 & -1 \\ -5 & 3 \end{array}\right]$

i) $\left[\begin{array}{rrr} 2 & 3 & 1 \\ 5 & 7 & 4 \end{array}\right] \left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right]$

j) $\left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right] \left[\begin{array}{ccc} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{array}\right]$

---

b. $\left[\begin{array}{rrr} -1 & -6 & -2 \\ 0 & 6 & 10 \end{array}\right]$

d. $\left[\begin{array}{rr} -3 & -15 \end{array}\right]$

f. $[-23]$

h. $\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right]$

j. $\left[\begin{array}{ccc} aa' & 0 & 0 \\ 0 & bb' & 0 \\ 0 & 0 & cc' \end{array}\right]$

Exercise 2.3.2 In each of the following cases, find all possible products A^2, AB, AC, and so on.

a. $A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ -1 & 0 & 0 \end{array}\right]$, $B = \left[\begin{array}{rr} 1 & -2 \\ \frac{1}{2} & 3 \end{array}\right]$, $C = \left[\begin{array}{rr} -1 & 0 \\ 2 & 5 \\ 0 & 5 \end{array}\right]$

b. $A = \left[\begin{array}{rrr} 1 & 2 & 4 \\ 0 & 1 & -1 \end{array}\right]$, $B = \left[\begin{array}{rr} -1 & 6 \\ 1 & 0 \end{array}\right]$, $C = \left[\begin{array}{rr} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{array}\right]$

---

b. $BA = \left[\begin{array}{rrr} -1 & 4 & -10 \\ 1 & 2 & 4 \end{array}\right]$, $B^2 = \left[\begin{array}{rr} 7 & -6 \\ -1 & 6 \end{array}\right]$, $CB = \left[\begin{array}{rr} -2 & 12 \\ 2 & -6 \\ 1 & 6 \end{array}\right]$

$AC = \left[\begin{array}{rr} 4 & 10 \\ -2 & -1 \end{array}\right]$, $CA = \left[\begin{array}{rrr} 2 & 4 & 8 \\ -1 & -1 & -5 \\ 1 & 4 & 2 \end{array}\right]$

Exercise 2.3.3 Find a, b, a_1, and b_1 if:

a. $\left[\begin{array}{cc} a & b \\ a_1 & b_1 \end{array}\right] \left[\begin{array}{rr} 3 & -5 \\ -1 & 2 \end{array}\right] = \left[\begin{array}{rr} 1 & -1 \\ 2 & 0 \end{array}\right]$

b. $\left[\begin{array}{rr} 2 & 1 \\ -1 & 2 \end{array}\right] \left[\begin{array}{cc} a & b \\ a_1 & b_1 \end{array}\right] = \left[\begin{array}{rr} 7 & 2 \\ -1 & 4 \end{array}\right]$

---

b. $(a,\ b,\ a_1,\ b_1) = (3,\ 0,\ 1,\ 2)$

Exercise 2.3.4 Verify that $A^2 - A - 6I = 0$ if:

a) $\left[\begin{array}{rr} 3 & -1 \\ 0 & -2 \end{array}\right]$ b) $\left[\begin{array}{rr} 2 & 2 \\ 2 & -1 \end{array}\right]$

---

b. $A^2 - A - 6I = \left[\begin{array}{rr} 8 & 2 \\ 2 & 5 \end{array}\right] - \left[\begin{array}{rr} 2 & 2 \\ 2 & -1 \end{array}\right] - \left[\begin{array}{rr} 6 & 0 \\ 0 & 6 \end{array}\right] = \left[\begin{array}{rr} 0 & 0 \\ 0 & 0 \end{array}\right]$

Exercise 2.3.5 Given $A = \left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \end{array}\right]$, $B = \left[\begin{array}{rrr} 1 & 0 & -2 \\ 3 & 1 & 0 \end{array}\right]$, $C = \left[\begin{array}{rr} 1 & 0 \\ 2 & 1 \\ 5 & 8 \end{array}\right]$, and $D = \left[\begin{array}{rrr} 3 & -1 & 2 \\ 1 & 0 & 5 \end{array}\right]$, verify the following facts from Theorem 2.3.1.

a) $A(B-D) = AB - AD$ b) $A(BC) = (AB)C$
c) $(CD)^T = D^T C^T$

---

b. $A(BC) = \left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \end{array}\right] \left[\begin{array}{rr} -9 & -16 \\ 5 & 1 \end{array}\right] = \left[\begin{array}{rr} -14 & -17 \\ 5 & 1 \end{array}\right] = \left[\begin{array}{rrr} -2 & -1 & -2 \\ 3 & 1 & 0 \end{array}\right] \left[\begin{array}{rr} 1 & 0 \\ 2 & 1 \\ 5 & 8 \end{array}\right] = (AB)C$

Exercise 2.3.6 Let A be a 2×2 matrix.

a. If A commutes with $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for some a and b.

b. If A commutes with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, show that $A = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ for some a and c.

c. Show that A commutes with *every* 2×2 matrix if and only if $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some a.

---

b. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, compare entries an AE and EA.

Exercise 2.3.7

a. If A^2 can be formed, what can be said about the size of A?

b. If AB and BA can both be formed, describe the sizes of A and B.

c. If ABC can be formed, A is 3×3, and C is 5×5, what size is B?

---

b. $m \times n$ and $n \times m$ for some m and n

Exercise 2.3.8

a. Find two 2×2 matrices A such that $A^2 = 0$.

b. Find three 2×2 matrices A such that (i) $A^2 = I$; (ii) $A^2 = A$.

c. Find 2×2 matrices A and B such that $AB = 0$ but $BA \neq 0$.

---

b. i. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Exercise 2.3.9 Write $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and let A be $3 \times n$ and B be $m \times 3$.

a. Describe PA in terms of the rows of A.

b. Describe BP in terms of the columns of B.

Exercise 2.3.10 Let A, B, and C be as in Exercise 2.3.5. Find the (3, 1)-entry of CAB using exactly six numerical multiplications.

Exercise 2.3.11 Compute AB, using the indicated block partitioning.

$$A = \left[\begin{array}{rr|rr} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \left[\begin{array}{rr|r} 1 & 2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right]$$

Exercise 2.3.12 In each case give formulas for all powers A, A^2, A^3, $\dots$ of A using the block decomposition indicated.

a. $A = \left[\begin{array}{r|rr} 1 & 0 & 0 \\ \hline 1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right]$

b. $A = \left[\begin{array}{rr|rr} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

---

b. $A^{2k} = \left[\begin{array}{rr|rr} 1 & -2k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ for $k = 0, 1, 2, \dots$,

$$A^{2k+1} = A^{2k}A = \left[\begin{array}{rr|rr} 1 & -(2k+1) & 2 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

for $k = 0, 1, 2, \dots$

Exercise 2.3.13 Compute the following using block multiplication (all blocks are $k \times k$).

a) $\begin{bmatrix} I & X \\ -Y & I \end{bmatrix}\begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$ b) $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$

c) $\begin{bmatrix} I & X \end{bmatrix}\begin{bmatrix} I & X \end{bmatrix}^T$ d) $\begin{bmatrix} I & X^T \end{bmatrix}\begin{bmatrix} -X & I \end{bmatrix}^T$

e) $\begin{bmatrix} I & X \\ 0 & -I \end{bmatrix}^n$ any $n \geq 1$

f) $\begin{bmatrix} 0 & X \\ I & 0 \end{bmatrix}^n$ any $n \geq 1$

---

b. $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_{2k}$

d. 0_k

f. $\begin{bmatrix} X^m & 0 \\ 0 & X^m \end{bmatrix}$ if $n = 2m$; $\begin{bmatrix} 0 & X^{m+1} \\ X^m & 0 \end{bmatrix}$ if $n = 2m+1$

Exercise 2.3.14 Let A denote an $m \times n$ matrix.

a. If $AX = 0$ for every $n \times 1$ matrix X, show that $A = 0$.

b. If $YA = 0$ for every $1 \times m$ matrix Y, show that $A = 0$.

---

b. If Y is row i of the identity matrix I, then YA is row i of $IA = A$.

Exercise 2.3.15

a. If $U = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, and $AU = 0$, show that $A = 0$.

b. Let U be such that $AU = 0$ implies that $A = 0$. If $PU = QU$, show that $P = Q$.

Exercise 2.3.16 Simplify the following expressions where A, B, and C represent matrices.

a. $A(3B-C)+(A-2B)C+2B(C+2A)$

b. $A(B+C-D)+B(C-A+D)-(A+B)C$
$+(A-B)D$

c. $AB(BC-CB)+(CA-AB)BC+CA(A-B)C$

d. $(A-B)(C-A)+(C-B)(A-C)+(C-A)^2$

---

b. $AB-BA$

d. 0

Exercise 2.3.17 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \neq 0$, show that A factors in the form $A = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} y & z \\ 0 & w \end{bmatrix}$.

Exercise 2.3.18 If A and B commute with C, show that the same is true of:

a) $A+B$ b) kA, k any scalar

---

b. $(kA)C = k(AC) = k(CA) = C(kA)$

Exercise 2.3.19 If A is any matrix, show that both AA^T and A^TA are symmetric.

Exercise 2.3.20 If A and B are symmetric, show that AB is symmetric if and only if $AB = BA$.

We have $A^T = A$ and $B^T = B$, so $(AB)^T = B^TA^T = BA$. Hence AB is symmetric if and only if $AB = BA$.

Exercise 2.3.21 If A is a 2×2 matrix, show that $A^TA = AA^T$ if and only if A is symmetric or $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for some a and b.

Exercise 2.3.22

a. Find all symmetric 2×2 matrices A such that $A^2 = 0$.

b. Repeat (a) if A is 3×3.

c. Repeat (a) if A is $n \times n$.

---

b. $A = 0$

Exercise 2.3.23 Show that there exist no 2×2 matrices A and B such that $AB - BA = I$. [*Hint*: Examine the (1, 1)- and (2, 2)-entries.]

Exercise 2.3.24 Let B be an $n \times n$ matrix. Suppose $AB = 0$ for some nonzero $m \times n$ matrix A. Show that no $n \times n$ matrix C exists such that $BC = I$.

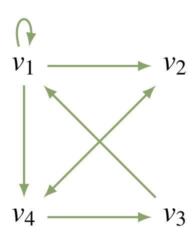
If $BC = I$, then $AB = 0$ gives $0 = 0C = (AB)C = A(BC) = AI = A$, contrary to the assumption that $A \neq 0$.

Exercise 2.3.25 An autoparts manufacturer makes fenders, doors, and hoods. Each requires assembly and packaging carried out at factories: Plant 1, Plant 2, and Plant 3. Matrix A below gives the number of hours for assembly and packaging, and matrix B gives the hourly rates at the three plants. Explain the meaning of the (3, 2)-entry in the matrix AB. Which plant is the most economical to operate? Give reasons.

$$\begin{array}{l} \\ \text{Fenders} \\ \text{Doors} \\ \text{Hoods} \end{array} \begin{array}{c} \begin{array}{cc} \text{Assembly} & \text{Packaging} \end{array} \\ \left[\begin{array}{cc} 12 & 2 \\ 21 & 3 \\ 10 & 2 \end{array}\right] \end{array} = A$$

$$\begin{array}{l} \\ \text{Assembly} \\ \text{Packaging} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} \end{array} \\ \left[\begin{array}{ccc} 21 & 18 & 20 \\ 14 & 10 & 13 \end{array}\right] \end{array} = B$$

Exercise 2.3.26 For the directed graph below, find the adjacency matrix A, compute A^3, and determine the number of paths of length 3 from v_1 to v_4 and from v_2 to v_3.



---

3 paths $v_1 \to v_4$, 0 paths $v_2 \to v_3$

Exercise 2.3.27 In each case either show the statement is true, or give an example showing that it is false.

a. If $A^2 = I$, then $A = I$.

b. If $AJ = A$, then $J = I$.

c. If A is square, then $(A^T)^3 = (A^3)^T$.

d. If A is symmetric, then $I + A$ is symmetric.

e. If $AB = AC$ and $A \neq 0$, then $B = C$.

f. If $A \neq 0$, then $A^2 \neq 0$.

g. If A has a row of zeros, so also does BA for all B.

h. If A commutes with $A + B$, then A commutes with B.

i. If B has a column of zeros, so also does AB.

j. If AB has a column of zeros, so also does B.

k. If A has a row of zeros, so also does AB.

l. If AB has a row of zeros, so also does A.

---

b. False. If $A = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] = J$, then $AJ = A$ but $J \neq I$.

d. True. Since $A^T = A$, we have $(I + AT = I^T + A^T = I + A$.

f. False. If $A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$, then $A \neq 0$ but $A^2 = 0$.

h. True. We have $A(A+B) = (A+B)A$; that is, $A^2 + AB = A^2 + BA$. Subtracting A^2 gives $AB = BA$.

j. False. $A = \left[\begin{array}{cc} 1 & -2 \\ 2 & 4 \end{array}\right]$, $B = \left[\begin{array}{cc} 2 & 4 \\ 1 & 2 \end{array}\right]$

l. False. See **(j)**.

Exercise 2.3.28

a. If A and B are 2×2 matrices whose rows sum to 1, show that the rows of AB also sum to 1.

b. Repeat part (a) for the case where A and B are $n \times n$.

---

b. If $A = [a_{ij}]$ and $B = [b_{ij}]$ and $\sum_j a_{ij} = 1 = \sum_j b_{ij}$, then the $(i,\ j)$-entry of AB is $c_{ij} = \sum_k a_{ik}b_{kj}$, whence $\sum_j c_{ij} = \sum_j \sum_k a_{ik}b_{kj} = \sum_k a_{ik}(\sum_j b_{kj}) = \sum_k a_{ik} = 1$. Alternatively: If $\mathbf{e} = (1,\ 1,\ \dots,\ 1)$, then the rows of A sum to 1 if and only if $A\mathbf{e} = \mathbf{e}$. If also $B\mathbf{e} = \mathbf{e}$ then $(AB)\mathbf{e} = A(B\mathbf{e}) = A\mathbf{e} = \mathbf{e}$.

Exercise 2.3.29 Let A and B be $n \times n$ matrices for which the systems of equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ each have only the trivial solution $\mathbf{x} = \mathbf{0}$. Show that the system $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Exercise 2.3.30 The **trace** of a square matrix A, denoted $\operatorname{tr} A$, is the sum of the elements on the main diagonal of A. Show that, if A and B are $n \times n$ matrices:

a) $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$.

b) $\operatorname{tr}(kA) = k \operatorname{tr}(A)$ for any number k.

c) $\operatorname{tr}(A^T) = \operatorname{tr}(A)$.

d) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

e) $\operatorname{tr}(AA^T)$ is the sum of the squares of all entries of A.

---

b. If $A = [a_{ij}]$, then $\operatorname{tr}(kA) = \operatorname{tr}[ka_{ij}] = \sum_{i=1}^n ka_{ii} = k\sum_{i=1}^n a_{ii} = k \operatorname{tr}(A)$.

e. Write $A^T = \left[a'_{ij}\right]$, where $a'_{ij} = a_{ji}$. Then $AA^T = \left(\sum_{k=1}^n a_{ik}a'_{kj}\right)$, so $\operatorname{tr}(AA^T) = \sum_{i=1}^n \left[\sum_{k=1}^n a_{ik}a'_{ki}\right] = \sum_{i=1}^n \sum_{k=1}^n a_{ik}^2$.

Exercise 2.3.31 Show that $AB - BA = I$ is impossible. [*Hint*: See the preceding exercise.]

Exercise 2.3.32 A square matrix P is called an **idempotent** if $P^2 = P$. Show that:

a. 0 and I are idempotents.

b. $\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right]$, and $\frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$, are idempotents.

c. If P is an idempotent, so is $I - P$. Show further that $P(I-P) = 0$.

d. If P is an idempotent, so is P^T.

e. If P is an idempotent, so is $Q = P + AP - PAP$ for any square matrix A (of the same size as P).

f. If A is $n \times m$ and B is $m \times n$, and if $AB = I_n$, then BA is an idempotent.

---

e. Observe that $PQ = P^2 + PAP - P^2AP = P$, so $Q^2 = PQ + APQ - PAPQ = P + AP - PAP = Q$.

Exercise 2.3.33 Let A and B be $n \times n$ **diagonal matrices** (all entries off the main diagonal are zero).

a. Show that AB is diagonal and $AB = BA$.

b. Formulate a rule for calculating XA if X is $m \times n$.

c. Formulate a rule for calculating AY if Y is $n \times k$.

Exercise 2.3.34 If A and B are $n \times n$ matrices, show that:

a. $AB = BA$ if and only if

$$(A+B)^2 = A^2 + 2AB + B^2$$

b. $AB = BA$ if and only if

$$(A+B)(A-B) = (A-B)(A+B)$$

---

b. $(A+B)(A-B) = A^2 - AB + BA - B^2$, and $(A-B)(A+B) = A^2 + AB - BA - B^2$. These are equal if and only if $-AB + BA = AB - BA$; that is, $2BA = 2AB$; that is, $BA = AB$.

Exercise 2.3.35 In Theorem 2.3.3, prove

a) part 3;

b) part 5.

---

b. $(A+B)(A-B) = A^2 - AB + BA - B^2$ and $(A-B)(A+B) = A^2 - BA + AB - B^2$. These are equal if and only if $-AB + BA = -BA + AB$, that is $2AB = 2BA$, that is $AB = BA$.

Exercise 2.3.36 Show that the product of two reduced row-echelon matrices is also reduced row-echelon.

See V. Camillo, Communications in Algebra 25(6), (1997), 1767–1782; Theorem 2.3.2.

2.4 Matrix Inverses

Three basic operations on matrices, addition, multiplication, and subtraction, are analogs for matrices of the same operations for numbers. In this section we introduce the matrix analog of numerical division.

To begin, consider how a numerical equation $ax = b$ is solved when a and b are known numbers. If $a = 0$, there is no solution (unless $b = 0$). But if $a \neq 0$, we can multiply both sides by the inverse $a^{-1} = \frac{1}{a}$ to obtain the solution $x = a^{-1}b$. Of course multiplying by a^{-1} is just dividing by a, and the property of a^{-1} that makes this work is that $a^{-1}a = 1$. Moreover, we saw in Section 2.2 that the role that 1 plays in arithmetic is played in matrix algebra by the identity matrix I. This suggests the following definition.

Definition 2.11 Matrix Inverses

If A is a square matrix, a matrix B is called an ***inverse*** *of A if and only if*

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an ***invertible matrix****.*[8]

Example 2.4.1

Show that $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution. Compute AB and BA.

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $AB = I = BA$, so B is indeed an inverse of A.

Example 2.4.2

Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ has no inverse.

Solution. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote an arbitrary 2×2 matrix. Then

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a+3c & b+3d \end{bmatrix}$$

so AB has a row of zeros. Hence AB cannot equal I for any B.

[8]Only square matrices have inverses. Even though it is plausible that nonsquare matrices A and B could exist such that $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, we claim that this forces $n = m$. Indeed, if $m < n$ there exists a nonzero column $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ (by Theorem 1.3.1), so $\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B(\mathbf{0}) = \mathbf{0}$, a contradiction. Hence $m \geq n$. Similarly, the condition $AB = I_m$ implies that $n \geq m$. Hence $m = n$ so A is square.

The argument in Example 2.4.2 shows that no zero matrix has an inverse. But Example 2.4.2 also shows that, unlike arithmetic, *it is possible for a nonzero matrix to have no inverse.* However, if a matrix *does* have an inverse, it has only one.

Theorem 2.4.1

If B and C are both inverses of A, then $B = C$.

Proof. Since B and C are both inverses of A, we have $CA = I = AB$. Hence

$$B = IB = (CA)B = C(AB) = CI = C$$

□

If A is an invertible matrix, the (unique) inverse of A is denoted A^{-1}. Hence A^{-1} (when it exists) is a square matrix of the same size as A with the property that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

These equations characterize A^{-1} in the following sense:

> **Inverse Criterion:** *If somehow a matrix B can be found such that $AB = I$ and $BA = I$, then A is invertible and B is the inverse of A; in symbols, $B = A^{-1}$.*

This is a way to verify that the inverse of a matrix exists. Example 2.4.3 and Example 2.4.4 offer illustrations.

Example 2.4.3

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, show that $A^3 = I$ and so find A^{-1}.

Solution. We have $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, and so

$$A^3 = A^2A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $A^3 = I$, as asserted. This can be written as $A^2A = I = AA^2$, so it shows that A^2 is the inverse of A. That is, $A^{-1} = A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

The next example presents a useful formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when it exists. To state it, we define the **determinant** $\det A$ and the **adjugate** $\operatorname{adj} A$ of the matrix A as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \text{and} \quad \operatorname{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.4.4

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that A has an inverse if and only if $\det A \neq 0$, and in this case

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Solution. For convenience, write $e = \det A = ad - bc$ and $B = \operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then $AB = eI = BA$ as the reader can verify. So if $e \neq 0$, scalar multiplication by $\frac{1}{e}$ gives

$$A(\tfrac{1}{e}B) = I = (\tfrac{1}{e}B)A$$

Hence A is invertible and $A^{-1} = \frac{1}{e}B$. Thus it remains only to show that if A^{-1} exists, then $e \neq 0$.

We prove this by showing that assuming $e = 0$ leads to a contradiction. In fact, if $e = 0$, then $AB = eI = 0$, so left multiplication by A^{-1} gives $A^{-1}AB = A^{-1}0$; that is, $IB = 0$, so $B = 0$. But this implies that a, b, c, and d are *all* zero, so $A = 0$, contrary to the assumption that A^{-1} exists.

As an illustration, if $A = \begin{bmatrix} 2 & 4 \\ -3 & 8 \end{bmatrix}$ then $\det A = 2 \cdot 8 - 4 \cdot (-3) = 28 \neq 0$. Hence A is invertible and $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{28} \begin{bmatrix} 8 & -4 \\ 3 & 2 \end{bmatrix}$, as the reader is invited to verify.

The determinant and adjugate will be defined in Chapter 3 for any square matrix, and the conclusions in Example 2.4.4 will be proved in full generality.

Inverses and Linear Systems

Matrix inverses can be used to solve certain systems of linear equations. Recall that a *system* of linear equations can be written as a *single* matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A and $\mathbf{b}$ are known and $\mathbf{x}$ is to be determined. If A is invertible, we multiply each side of the equation on the left by A^{-1} to get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This gives the solution to the system of equations (the reader should verify that $\mathbf{x} = A^{-1}\mathbf{b}$ really does satisfy $A\mathbf{x} = \mathbf{b}$). Furthermore, the argument shows that if $\mathbf{x}$ is *any* solution, then necessarily $\mathbf{x} = A^{-1}\mathbf{b}$, so the solution is unique. Of course the technique works only when the coefficient matrix A has an inverse. This proves Theorem 2.4.2.

Theorem 2.4.2

Suppose a system of n equations in n variables is written in matrix form as

$$A\mathbf{x} = \mathbf{b}$$

If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example 2.4.5

Use Example 2.4.4 to solve the system $\left\{\begin{array}{rl} 5x_1 - 3x_2 = & -4 \\ 7x_1 + 4x_2 = & 8 \end{array}\right.$.

Solution. In matrix form this is $A\mathbf{x} = \mathbf{b}$ where $A = \left[\begin{array}{rr} 5 & -3 \\ 7 & 4 \end{array}\right]$, $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$, and $\mathbf{b} = \left[\begin{array}{r} -4 \\ 8 \end{array}\right]$. Then $\det A = 5 \cdot 4 - (-3) \cdot 7 = 41$, so A is invertible and $A^{-1} = \frac{1}{41}\left[\begin{array}{rr} 4 & 3 \\ -7 & 5 \end{array}\right]$ by Example 2.4.4. Thus Theorem 2.4.2 gives

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{41}\left[\begin{array}{rr} 4 & 3 \\ -7 & 5 \end{array}\right]\left[\begin{array}{r} -4 \\ 8 \end{array}\right] = \frac{1}{41}\left[\begin{array}{r} 8 \\ 68 \end{array}\right]$$

so the solution is $x_1 = \frac{8}{41}$ and $x_2 = \frac{68}{41}$.

An Inversion Method

If a matrix A is $n \times n$ and invertible, it is desirable to have an efficient technique for finding the inverse. The following procedure will be justified in Section 2.5.

Matrix Inversion Algorithm

If A is an invertible (square) matrix, there exists a sequence of elementary row operations that carry A to the identity matrix I of the same size, written $A \to I$. This same series of row operations carries I to A^{-1}; that is, $I \to A^{-1}$. The algorithm can be summarized as follows:

$$\left[\begin{array}{cc} A & I \end{array}\right] \to \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$$

where the row operations on A and I are carried out simultaneously.

Example 2.4.6

Use the inversion algorithm to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution. Apply elementary row operations to the double matrix

$$\left[\begin{array}{cc} A & I \end{array}\right] = \left[\begin{array}{rrr|rrr} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

so as to carry A to I. First interchange rows 1 and 2.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next subtract 2 times row 1 from row 2, and subtract row 1 from row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Continue to reduced row-echelon form.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]$$

Hence $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$, as is readily verified.

Given any $n \times n$ matrix A, Theorem 1.2.1 shows that A can be carried by elementary row operations to a matrix R in reduced row-echelon form. If $R = I$, the matrix A is invertible (this will be proved in the next section), so the algorithm produces A^{-1}. If $R \neq I$, then R has a row of zeros (it is square), so no system of linear equations $A\mathbf{x} = \mathbf{b}$ can have a unique solution. But then A is not invertible by Theorem 2.4.2. Hence, the algorithm is effective in the sense conveyed in Theorem 2.4.3.

Theorem 2.4.3

If A is an $n \times n$ matrix, either A can be reduced to I by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

Properties of Inverses

The following properties of an invertible matrix are used everywhere.

Example 2.4.7: Cancellation Laws

Let A be an invertible matrix. Show that:

1. If $AB = AC$, then $B = C$.
2. If $BA = CA$, then $B = C$.

Solution. Given the equation $AB = AC$, left multiply both sides by A^{-1} to obtain $A^{-1}AB = A^{-1}AC$. Thus $IB = IC$, that is $B = C$. This proves (1) and the proof of (2) is left to the reader.

Properties (1) and (2) in Example 2.4.7 are described by saying that an invertible matrix can be "left cancelled" and "right cancelled", respectively. Note however that "mixed" cancellation does not hold in general: If A is invertible and $AB = CA$, then B and C may *not* be equal, even if both are 2×2. Here is a specific example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Sometimes the inverse of a matrix is given by a formula. Example 2.4.4 is one illustration; Example 2.4.8 and Example 2.4.9 provide two more. The idea is the *Inverse Criterion*: If a matrix B can be found such that $AB = I = BA$, then A is invertible and $A^{-1} = B$.

Example 2.4.8

If A is an invertible matrix, show that the transpose A^T is also invertible. Show further that the inverse of A^T is just the transpose of A^{-1}; in symbols, $(A^T)^{-1} = (A^{-1})^T$.

Solution. A^{-1} exists (by assumption). Its transpose $(A^{-1})^T$ is the candidate proposed for the inverse of A^T. Using the inverse criterion, we test it as follows:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Hence $(A^{-1})^T$ is indeed the inverse of A^T; that is, $(A^T)^{-1} = (A^{-1})^T$.

Example 2.4.9

If A and B are invertible $n \times n$ matrices, show that their product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Solution. We are given a candidate for the inverse of AB, namely $B^{-1}A^{-1}$. We test it as follows:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Hence $B^{-1}A^{-1}$ is the inverse of AB; in symbols, $(AB)^{-1} = B^{-1}A^{-1}$.

We now collect several basic properties of matrix inverses for reference.

Theorem 2.4.4

All the following matrices are square matrices of the same size.

1. *I is invertible and $I^{-1} = I$.*
2. *If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.*
3. *If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *If A_1, A_2, $\dots$, A_k are all invertible, so is their product $A_1A_2\cdots A_k$, and*
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$
5. *If A is invertible, so is A^k for any $k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.*
6. *If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.*
7. *If A is invertible, so is its transpose A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Proof.

1. This is an immediate consequence of the fact that $I^2 = I$.
2. The equations $AA^{-1} = I = A^{-1}A$ show that A is the inverse of A^{-1}; in symbols, $(A^{-1})^{-1} = A$.
3. This is Example 2.4.9.
4. Use induction on k. If $k = 1$, there is nothing to prove, and if $k = 2$, the result is property 3. If $k > 2$, assume inductively that $(A_1A_2\cdots A_{k-1})^{-1} = A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$. We apply this fact together with property 3 as follows:
$$\begin{aligned}[A_1A_2\cdots A_{k-1}A_k]^{-1} &= [(A_1A_2\cdots A_{k-1})A_k]^{-1} \\ &= A_k^{-1}(A_1A_2\cdots A_{k-1})^{-1} \\ &= A_k^{-1}\left(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}\right)\end{aligned}$$

So the proof by induction is complete.

5. This is property 4 with $A_1 = A_2 = \cdots = A_k = A$.

6. This is left as Exercise 2.4.29.

7. This is Example 2.4.8. □

The reversal of the order of the inverses in properties 3 and 4 of Theorem 2.4.4 is a consequence of the fact that matrix multiplication is not commutative. Another manifestation of this comes when matrix equations are dealt with. If a matrix equation $B = C$ is given, it can be *left-multiplied* by a matrix A to yield $AB = AC$. Similarly, *right-multiplication* gives $BA = CA$. However, we cannot mix the two: If $B = C$, it need *not* be the case that $AB = CA$ even if A is invertible, for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = C$.

Part 7 of Theorem 2.4.4 together with the fact that $(A^T)^T = A$ gives

Corollary 2.4.1

A square matrix A is invertible if and only if A^T is invertible.

Example 2.4.10

Find A if $(A^T - 2I)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

Solution. By Theorem 2.4.4(2) and Example 2.4.4, we have

$$(A^T - 2I) = \left[\left(A^T - 2I\right)^{-1}\right]^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

Hence $A^T = 2I + \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, so $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ by Theorem 2.4.4(7).

The following important theorem collects a number of conditions all equivalent[9] to invertibility. It will be referred to frequently below.

Theorem 2.4.5: Inverse Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*

2. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*

[9] If p and q are statements, we say that p **implies** q (written $p \Rightarrow q$) if q is true whenever p is true. The statements are called **equivalent** if both $p \Rightarrow q$ and $q \Rightarrow p$ (written $p \Leftrightarrow q$, spoken "p if and only if q"). See Appendix **??**.

3. *A can be carried to the identity matrix I_n by elementary row operations.*
4. *The system $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x}$ for every choice of column $\mathbf{b}$.*
5. *There exists an $n \times n$ matrix C such that $AC = I_n$.*

Proof. We show that each of these conditions implies the next, and that (5) implies (1).

(1) ⇒ (2). If A^{-1} exists, then $A\mathbf{x} = \mathbf{0}$ gives $\mathbf{x} = I_n\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

(2) ⇒ (3). Assume that (2) is true. Certainly $A \to R$ by row operations where R is a reduced, row-echelon matrix. It suffices to show that $R = I_n$. Suppose that this is not the case. Then R has a row of zeros (being square). Now consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{0}$. Then $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right] \to \left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ is the reduced form, and $\left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ also has a row of zeros. Since R is square there must be at least one nonleading variable, and hence at least one parameter. Hence the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, contrary to (2). So $R = I_n$ after all.

(3) ⇒ (4). Consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{b}$. Using (3), let $A \to I_n$ by a sequence of row operations. Then these same operations carry $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \to \left[\begin{array}{c|c} I_n & \mathbf{c} \end{array}\right]$ for some column $\mathbf{c}$. Hence the system $A\mathbf{x} = \mathbf{b}$ has a solution (in fact unique) by gaussian elimination. This proves (4).

(4) ⇒ (5). Write $I_n = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right]$ where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the columns of I_n. For each $j = 1, 2, \dots, n$, the system $A\mathbf{x} = \mathbf{e}_j$ has a solution $\mathbf{c}_j$ by (4), so $A\mathbf{c}_j = \mathbf{e}_j$. Now let $C = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right]$ be the $n \times n$ matrix with these matrices $\mathbf{c}_j$ as its columns. Then Definition 2.9 gives (5):

$$AC = A\left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{c}_1 & A\mathbf{c}_2 & \cdots & A\mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = I_n$$

(5) ⇒ (1). Assume that (5) is true so that $AC = I_n$ for some matrix C. Then $C\mathbf{x} = 0$ implies $\mathbf{x} = \mathbf{0}$ (because $\mathbf{x} = I_n\mathbf{x} = AC\mathbf{x} = A\mathbf{0} = \mathbf{0}$). Thus condition (2) holds for the matrix C rather than A. Hence the argument above that (2) ⇒ (3) ⇒ (4) ⇒ (5) (with A replaced by C) shows that a matrix C' exists such that $CC' = I_n$. But then

$$A = AI_n = A(CC') = (AC)C' = I_nC' = C'$$

Thus $CA = CC' = I_n$ which, together with $AC = I_n$, shows that C is the inverse of A. This proves (1). □

The proof of (5) ⇒ (1) in Theorem 2.4.5 shows that if $AC = I$ for square matrices, then necessarily $CA = I$, and hence that C and A are inverses of each other. We record this important fact for reference.

Corollary 2.4.1

If A and C are square matrices such that $AC = I$, then also $CA = I$. In particular, both A and C are invertible, $C = A^{-1}$, and $A = C^{-1}$.

Here is a quick way to remember Corollary 2.4.1. If A is a square matrix, then

1. If $AC = I$ then $C = A^{-1}$.

2. If $CA = I$ then $C = A^{-1}$.

Observe that Corollary 2.4.1 is false if A and C are not square matrices. For example, we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = I_2 \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq I_3$$

In fact, it is verified in the footnote on page 91 that if $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, then $m = n$ and A and B are (square) inverses of each other.

An $n \times n$ matrix A has rank n if and only if (3) of Theorem 2.4.5 holds. Hence

Corollary 2.4.2

An $n \times n$ matrix A is invertible if and only if $\operatorname{rank} A = n$.

Here is a useful fact about inverses of block matrices.

Example 2.4.11

Let $P = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ and $Q = \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix}$ be block matrices where A is $m \times m$ and B is $n \times n$ (possibly $m \neq n$).

a. Show that P is invertible if and only if A and B are both invertible. In this case, show that

$$P^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

b. Show that Q is invertible if and only if A and B are both invertible. In this case, show that

$$Q^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}YA^{-1} & B^{-1} \end{bmatrix}$$

Solution. We do (a.) and leave (b.) for the reader.

a. If A^{-1} and B^{-1} both exist, write $R = \begin{bmatrix} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$. Using block multiplication, one verifies that $PR = I_{m+n} = RP$, so P is invertible, and $P^{-1} = R$. Conversely, suppose that P is invertible, and write $P^{-1} = \begin{bmatrix} C & V \\ W & D \end{bmatrix}$ in block form, where C is $m \times m$ and D is $n \times n$.

Then the equation $PP^{-1} = I_{n+m}$ becomes

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \begin{bmatrix} C & V \\ W & D \end{bmatrix} = \begin{bmatrix} AC + XW & AV + XD \\ BW & BD \end{bmatrix} = I_{m+n} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$$

using block notation. Equating corresponding blocks, we find

$$AC + XW = I_m, \quad BW = 0, \quad \text{and } BD = I_n$$

Hence B is invertible because $BD = I_n$ (by Corollary 2.4.1), then $W = 0$ because $BW = 0$, and finally, $AC = I_m$ (so A is invertible, again by Corollary 2.4.1).

Inverses of Matrix Transformations

Let $T = T_A : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by the $n \times n$ matrix A. Since A is square, it may very well be invertible, and this leads to the question:

What does it mean geometrically for T that A is invertible?

To answer this, let $T' = T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$ denote the transformation induced by A^{-1}. Then

$$\begin{aligned} T'[T(\mathbf{x})] &= A^{-1}[A\mathbf{x}] = I\mathbf{x} = \mathbf{x} \\ T[T'(\mathbf{x})] &= A\left[A^{-1}\mathbf{x}\right] = I\mathbf{x} = \mathbf{x} \end{aligned} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2.8}$$

The first of these equations asserts that, if T carries $\mathbf{x}$ to a vector $T(\mathbf{x})$, then T' carries $T(\mathbf{x})$ right back to $\mathbf{x}$; that is T' "reverses" the action of T. Similarly T "reverses" the action of T'. Conditions (2.8) can be stated compactly in terms of composition:

$$T' \circ T = 1_{\mathbb{R}^n} \quad \text{and} \quad T \circ T' = 1_{\mathbb{R}^n} \tag{2.9}$$

When these conditions hold, we say that the matrix transformation T' is an **inverse** of T, and we have shown that if the matrix A of T is invertible, then T has an inverse (induced by A^{-1}).

The converse is also true: If T has an inverse, then its matrix A must be invertible. Indeed, suppose $S : \mathbb{R}^n \to \mathbb{R}^n$ is any inverse of T, so that $S \circ T = 1_{\mathbb{R}_n}$ and $T \circ S = 1_{\mathbb{R}_n}$. It can be shown that S is also a matrix transformation. If B is the matrix of S, we have

$$BA\mathbf{x} = S[T(\mathbf{x})] = (S \circ T)(\mathbf{x}) = 1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} = I_n\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

It follows by Theorem 2.2.6 that $BA = I_n$, and a similar argument shows that $AB = I_n$. Hence A is invertible with $A^{-1} = B$. Furthermore, the inverse transformation S has matrix A^{-1}, so $S = T'$ using the earlier notation. This proves the following important theorem.

Theorem 2.4.6

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by an $n \times n$ matrix A. Then

A is invertible if and only if T has an inverse.

In this case, T has exactly one inverse (which we denote as T^{-1}), and $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the *transformation induced by the matrix* A^{-1}*. In other words*

$$(T_A)^{-1} = T_{A^{-1}}$$

The geometrical relationship between T and T^{-1} is embodied in equations (2.8) above:

$$T^{-1}[T(\mathbf{x})] = \mathbf{x} \quad \text{and} \quad T\left[T^{-1}(\mathbf{x})\right] = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

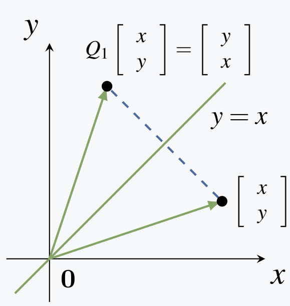
These equations are called the **fundamental identities** relating T and T^{-1}. Loosely speaking, they assert that each of T and T^{-1} "reverses" or "undoes" the action of the other.

This geometric view of the inverse of a linear transformation provides a new way to find the inverse of a matrix A. More precisely, if A is an invertible matrix, we proceed as follows:

1. *Let* T *be the linear transformation induced by* A*.*
2. *Obtain the linear transformation* T^{-1} *which "reverses" the action of* T*.*
3. *Then* A^{-1} *is the matrix of* T^{-1}*.*

Here is an example.

Example 2.4.12



Find the inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by viewing it as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

Solution. If $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ the vector $A\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ is the result of reflecting $\mathbf{x}$ in the line $y = x$ (see the diagram). Hence, if $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection in the line $y = x$, then A is the matrix of Q_1. Now observe that Q_1 reverses itself because reflecting a vector $\mathbf{x}$ twice results in $\mathbf{x}$. Consequently $Q_1^{-1} = Q_1$. Since A^{-1} is the matrix of Q_1^{-1} and A is the matrix of Q, it follows that $A^{-1} = A$. Of course this conclusion is clear by simply observing directly that $A^2 = I$, but the geometric method can often work where these other methods may be less straightforward.

Exercises for 2.4

Exercise 2.4.1 In each case, show that the matrices are inverses of each other.

a. $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 \\ 1 & -4 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 1 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$

Exercise 2.4.2 Find the inverse of each of the following matrices.

a) $\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{bmatrix}$

e) $\begin{bmatrix} 3 & 5 & 0 \\ 3 & 7 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 5 & -1 \end{bmatrix}$

g) $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 3 & 2 \\ 4 & 1 & 4 \end{bmatrix}$ h) $\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

i) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ j) $\begin{bmatrix} -1 & 4 & 5 & 2 \\ 0 & 0 & 0 & -1 \\ 1 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

k) $\begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & 3 & 6 \\ 1 & -1 & 5 & 2 \\ 1 & -1 & 5 & 1 \end{bmatrix}$ l) $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

---

b. $\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

f. $\frac{1}{10}\begin{bmatrix} 1 & 4 & -1 \\ -2 & 2 & 2 \\ -9 & 14 & -1 \end{bmatrix}$

h. $\frac{1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & 2 & 5 \\ -3 & 2 & -1 \end{bmatrix}$

j. $\begin{bmatrix} 0 & 0 & 1 & -2 \\ -1 & -2 & -1 & -3 \\ 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

l. $\begin{bmatrix} 1 & -2 & 6 & -30 & 210 \\ 0 & 1 & -3 & 15 & -105 \\ 0 & 0 & 1 & -5 & 35 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 2.4.3 In each case, solve the systems of equations by finding the inverse of the coefficient matrix.

a) $\begin{aligned} 3x - \;\; y &= 5 \\ 2x + 2y &= 1 \end{aligned}$ b) $\begin{aligned} 2x - 3y &= 0 \\ x - 4y &= 1 \end{aligned}$

c) $\begin{aligned} x + \;\; y + 2z &= \;\;5 \\ x + \;\; y + \;\; z &= \;\;0 \\ x + 2y + 4z &= -2 \end{aligned}$ d) $\begin{aligned} x + 4y + 2z &= \;\;1 \\ 2x + 3y + 3z &= -1 \\ 4x + \;\; y + 4z &= \;\;0 \end{aligned}$

---

b. $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 \\ -2 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 9 & -14 & 6 \\ 4 & -4 & 1 \\ -10 & 15 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 23 \\ 8 \\ -25 \end{bmatrix}$

Exercise 2.4.4 Given $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix}$:

a. Solve the system of equations $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

b. Find a matrix B such that $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

c. Find a matrix C such that $CA = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix}$.

---

b. $B = A^{-1}AB = \begin{bmatrix} 4 & -2 & 1 \\ 7 & -2 & 4 \\ -1 & 2 & -1 \end{bmatrix}$

Exercise 2.4.5 Find A when

a) $(3A)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ b) $(2A)^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1}$

c) $(I+3A)^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

d) $(I-2A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

e) $\left(A\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

f) $\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}A\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

g) $\left(A^T-2I\right)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

h) $\left(A^{-1}-2I\right)^T = -2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

---

b. $\frac{1}{10}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

d. $\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

f. $\frac{1}{2}\begin{bmatrix} 2 & 0 \\ -6 & 1 \end{bmatrix}$

h. $-\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 2.4.6 Find A when:

a) $A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ b) $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

---

b. $A = \frac{1}{2}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & -1 \end{bmatrix}$

Exercise 2.4.7 Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, express the variables x_1, x_2, and x_3 in terms of z_1, z_2, and z_3.

Exercise 2.4.8

a. In the system $\begin{matrix} 3x+4y=7 \\ 4x+5y=1 \end{matrix}$, substitute the new variables x' and y' given by $\begin{matrix} x=-5x'+4y' \\ y=\ \ 4x'-3y' \end{matrix}$. Then find x and y.

b. Explain part (a) by writing the equations as $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x' \\ y' \end{bmatrix}$. What is the relationship between A and B?

---

b. A and B are inverses.

Exercise 2.4.9 In each case either prove the assertion or give an example showing that it is false.

a. If $A \neq 0$ is a square matrix, then A is invertible.

b. If A and B are both invertible, then $A+B$ is invertible.

c. If A and B are both invertible, then $(A^{-1}B)^T$ is invertible.

d. If $A^4 = 3I$, then A is invertible.

e. If $A^2 = A$ and $A \neq 0$, then A is invertible.

f. If $AB = B$ for some $B \neq 0$, then A is invertible.

g. If A is invertible and skew symmetric ($A^T = -A$), the same is true of A^{-1}.

h. If A^2 is invertible, then A is invertible.

i. If $AB = I$, then A and B commute.

---

b. False. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

d. True. $A^{-1} = \frac{1}{3}A^3$

f. False. $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

h. True. If $(A^2)B = I$, then $A(AB) = I$; use Theorem 2.4.5.

Exercise 2.4.10

a. If A, B, and C are square matrices and $AB = I$, $I = CA$, show that A is invertible and $B = C = A^{-1}$.

b. If $C^{-1} = A$, find the inverse of C^T in terms of A.

---

b. $(C^T)^{-1} = (C^{-1})^T = A^T$ because $C^{-1} = (A^{-1})^{-1} = A$.

Exercise 2.4.11 Suppose $CA = I_m$, where C is $m \times n$ and A is $n \times m$. Consider the system $A\mathbf{x} = \mathbf{b}$ of n equations in m variables.

a. Show that this system has a unique solution $C\mathbf{B}$ if it is consistent.

b. If $C = \begin{bmatrix} 0 & -5 & 1 \\ 3 & 0 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \\ 6 & -10 \end{bmatrix}$, find $\mathbf{x}$ (if it exists) when (i) $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$; and (ii) $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ 22 \end{bmatrix}$.

---

b. (i) Inconsistent. (ii) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Exercise 2.4.12 Verify that $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ satisfies $A^2 - 3A + 2I = 0$, and use this fact to show that $A^{-1} = \frac{1}{2}(3I - A)$.

Exercise 2.4.13 Let $Q = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$. Compute QQ^T and so find Q^{-1} if $Q \neq 0$.

Exercise 2.4.14 Let $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that each of U, $-U$, and $-I_2$ is its own inverse and that the product of any two of these is the third.

Exercise 2.4.15 Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{bmatrix}$. Find the inverses by computing (a) A^6; (b) B^4; and (c) C^3.

---

b. $B^4 = I$, so $B^{-1} = B^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Exercise 2.4.16 Find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ c & 1 & c \\ 3 & c & 2 \end{bmatrix}$ in terms of c.

$\begin{bmatrix} c^2-2 & -c & 1 \\ -c & 1 & 0 \\ 3-c^2 & c & -1 \end{bmatrix}$

Exercise 2.4.17 If $c \neq 0$, find the inverse of $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 2 & c \end{bmatrix}$ in terms of c.

Exercise 2.4.18 Show that A has no inverse when:

a. A has a row of zeros.

b. A has a column of zeros.

c. each row of A sums to 0. [*Hint*: Theorem 2.4.5(2).]

d. each column of A sums to 0. [*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

---

b. If column j of A is zero, $A\mathbf{y} = \mathbf{0}$ where $\mathbf{y}$ is column j of the identity matrix. Use Theorem 2.4.5.

d. If each column of A sums to 0, $XA = 0$ where X is the row of 1s. Hence $A^T X^T = 0$ so A has no inverse by Theorem 2.4.5 ($X^T \neq 0$).

Exercise 2.4.19 Let A denote a square matrix.

a. Let $YA = 0$ for some matrix $Y \neq 0$. Show that A has no inverse. [*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

b. Use part (a) to show that (i) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$; and (ii) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ have no inverse. [*Hint*: For part (ii) compare row 3 with the difference between row 1 and row 2.]

---

b. (ii) $(-1,\ 1,\ 1)A = 0$

Exercise 2.4.20 If A is invertible, show that

a) $A^2 \neq 0$.

b) $A^k \neq 0$ for all $k = 1,\ 2,\ \dots$.

---

b. Each power A^k is invertible by Theorem 2.4.4 (because A is invertible). Hence A^k cannot be 0.

Exercise 2.4.21 Suppose $AB = 0$, where A and B are square matrices. Show that:

a. If one of A and B has an inverse, the other is zero.

b. It is impossible for both A and B to have inverses.

c. $(BA)^2 = 0$.

---

b. By (a), if one has an inverse the other is zero and so has no inverse.

Exercise 2.4.22 Find the inverse of the x-expansion in Example 2.2.16 and describe it geometrically.

If $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, $a > 1$, then $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}$ is an x-compression because $\frac{1}{a} < 1$.

Exercise 2.4.23 Find the inverse of the shear transformation in Example 2.2.17 and describe it geometrically.

Exercise 2.4.24 In each case assume that A is a square matrix that satisfies the given condition. Show that A is invertible and find a formula for A^{-1} in terms of A.

a. $A^3 - 3A + 2I = 0$.

b. $A^4 + 2A^3 - A - 4I = 0$.

---

b. $A^{-1} = \frac{1}{4}(A^3 + 2A^2 - 1)$

Exercise 2.4.25 Let A and B denote $n \times n$ matrices.

a. If A and AB are invertible, show that B is invertible using only (2) and (3) of Theorem 2.4.4.

b. If AB is invertible, show that both A and B are invertible using Theorem 2.4.5.

---

b. If $B\mathbf{x} = \mathbf{0}$, then $(AB)\mathbf{x} = (A)B\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ because AB is invertible. Hence B is invertible by Theorem 2.4.5. But then $A = (AB)B^{-1}$ is invertible by Theorem 2.4.4.

Exercise 2.4.26 In each case find the inverse of the matrix A using Example 2.4.11.

a) $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 1 & 0 \\ 5 & 2 & 0 \\ 1 & 3 & -1 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 1 & 4 \end{bmatrix}$

d) $A = \begin{bmatrix} 2 & 1 & 5 & 2 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

---

b. $\left[\begin{array}{rr|r} 2 & -1 & 0 \\ -5 & 3 & 0 \\ \hline -13 & 8 & -1 \end{array}\right]$

d. $\left[\begin{array}{rr|rr} 1 & -1 & -14 & 8 \\ -1 & 2 & 16 & -9 \\ \hline 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Exercise 2.4.27 If A and B are invertible symmetric matrices such that $AB = BA$, show that A^{-1}, AB, AB^{-1}, and $A^{-1}B^{-1}$ are also invertible and symmetric.

Exercise 2.4.28 Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix.

a. If $A^2 = 0$, verify that $(I - A)^{-1} = I + A$.

b. If $A^3 = 0$, verify that $(I - A)^{-1} = I + A + A^2$.

c. Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

d. If $A^n = 0$, find the formula for $(I - A)^{-1}$.

---

d. If $A^n = 0$, $(I - A)^{-1} = I + A + \cdots + A^{n-1}$.

Exercise 2.4.29 Prove property 6 of Theorem 2.4.4: If A is invertible and $a \neq 0$, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$

Exercise 2.4.30 Let A, B, and C denote $n \times n$ matrices. Using only Theorem 2.4.4, show that:

a. If A, C, and ABC are all invertible, B is invertible.

b. If AB and BA are both invertible, A and B are both invertible.

---

b. $A[B(AB)^{-1}] = I = [(BA)^{-1}B]A$, so A is invertible by Exercise 2.4.10.

Exercise 2.4.31 Let A and B denote invertible $n \times n$ matrices.

a. If $A^{-1} = B^{-1}$, does it mean that $A = B$? Explain.

b. Show that $A = B$ if and only if $A^{-1}B = I$.

Exercise 2.4.32 Let A, B, and C be $n \times n$ matrices, with A and B invertible. Show that

a. If A commutes with C, then A^{-1} commutes with C.

b. If A commutes with B, then A^{-1} commutes with B^{-1}.

---

a. Have $AC = CA$. Left-multiply by A^{-1} to get $C = A^{-1}CA$. Then right-multiply by A^{-1} to get $CA^{-1} = A^{-1}C$.

Exercise 2.4.33 Let A and B be square matrices of the same size.

a. Show that $(AB)^2 = A^2B^2$ if $AB = BA$.

b. If A and B are invertible and $(AB)^2 = A^2B^2$, show that $AB = BA$.

c. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(AB)^2 = A^2B^2$ but $AB \neq BA$.

---

b. Given $ABAB = AABB$. Left multiply by A^{-1}, then right multiply by B^{-1}.

Exercise 2.4.34 Let A and B be $n \times n$ matrices for which AB is invertible. Show that A and B are both invertible. ______

If $B\mathbf{x}=\mathbf{0}$ where $\mathbf{x}$ is $n\times 1$, then $AB\mathbf{x}=\mathbf{0}$ so $\mathbf{x}=\mathbf{0}$ as AB is invertible. Hence B is invertible by Theorem 2.4.5, so $A=(AB)B^{-1}$ is invertible.

Exercise 2.4.35 Consider $A=\begin{bmatrix}1 & 3 & -1\\ 2 & 1 & 5\\ 1 & -7 & 13\end{bmatrix}$, $B=\begin{bmatrix}1 & 1 & 2\\ 3 & 0 & -3\\ -2 & 5 & 17\end{bmatrix}$.

a. Show that A is not invertible by finding a nonzero 1×3 matrix Y such that $YA=0$. [*Hint*: Row 3 of A equals 2(row 2) -3(row 1).]

b. Show that B is not invertible. [*Hint*: Column 3 = 3(column 2) − column 1.]

---

b. $B\begin{bmatrix}-1\\ 3\\ -1\end{bmatrix}=0$ so B is not invertible by Theorem 2.4.5.

Exercise 2.4.36 Show that a square matrix A is invertible if and only if it can be left-cancelled: $AB=AC$ implies $B=C$.

Exercise 2.4.37 If $U^2=I$, show that $I+U$ is not invertible unless $U=I$.

Exercise 2.4.38

a. If J is the 4×4 matrix with every entry 1, show that $I-\frac{1}{2}J$ is self-inverse and symmetric.

b. If X is $n\times m$ and satisfies $X^TX=I_m$, show that I_n-2XX^T is self-inverse and symmetric.

---

b. Write $U=I_n-2XX^T$. Then $U^T=I_n^T-2X^{TT}X^T=U$, and $U^2=I_n^2-(2XX^T)I_n-I_n(2XX^T)+4(XX^T)(XX^T)=I_n-4XX^T+4XX^T=I_n$.

Exercise 2.4.39 An $n\times n$ matrix P is called an idempotent if $P^2=P$. Show that:

a. I is the only invertible idempotent.

b. P is an idempotent if and only if $I-2P$ is self-inverse.

c. U is self-inverse if and only if $U=I-2P$ for some idempotent P.

d. $I-aP$ is invertible for any $a\neq 1$, and that $(I-aP)^{-1}=I+\left(\frac{a}{1-a}\right)^P$.

---

b. $(I-2P)^2=I-4P+4P^2$, and this equals I if and only if $P^2=P$.

Exercise 2.4.40 If $A^2=kA$, where $k\neq 0$, show that A is invertible if and only if $A=kI$.

Exercise 2.4.41 Let A and B denote $n\times n$ invertible matrices.

a. Show that $A^{-1}+B^{-1}=A^{-1}(A+B)B^{-1}$.

b. If $A+B$ is also invertible, show that $A^{-1}+B^{-1}$ is invertible and find a formula for $(A^{-1}+B^{-1})^{-1}$.

---

b. $(A^{-1}+B^{-1})^{-1}=B(A+B)^{-1}A$

Exercise 2.4.42 Let A and B be $n\times n$ matrices, and let I be the $n\times n$ identity matrix.

a. Verify that $A(I+BA)=(I+AB)A$ and that $(I+BA)B=B(I+AB)$.

b. If $I+AB$ is invertible, verify that $I+BA$ is also invertible and that $(I+BA)^{-1}=I-B(I+AB)^{-1}A$.

2.5 Elementary Matrices

It is now clear that elementary row operations are important in linear algebra: They are essential in solving linear systems (using the gaussian algorithm) and in inverting a matrix (using the matrix inversion algorithm). It turns out that they can be performed by left multiplying by certain invertible matrices. These matrices are the subject of this section.

Definition 2.12 Elementary Matrices

An $n \times n$ matrix E is called an ***elementary matrix*** *if it can be obtained from the identity matrix I_n by a single elementary row operation (called the operation* ***corresponding*** *to E). We say that E is of type I, II, or III if the operation is of that type (see Definition 1.2).*

Hence

$$E_1 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \quad E_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 9 \end{array} \right], \quad \text{and} \quad E_3 = \left[\begin{array}{cc} 1 & 5 \\ 0 & 1 \end{array} \right]$$

are elementary of types I, II, and III, respectively, obtained from the 2×2 identity matrix by interchanging rows 1 and 2, multiplying row 2 by 9, and adding 5 times row 2 to row 1.

Suppose now that the matrix $A = \left[\begin{array}{ccc} a & b & c \\ p & q & r \end{array} \right]$ is left multiplied by the above elementary matrices E_1, E_2, and E_3. The results are:

$$\begin{aligned}
E_1A &= \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{ccc} a & b & c \\ p & q & r \end{array} \right] = \left[\begin{array}{ccc} p & q & r \\ a & b & c \end{array} \right] \\
E_2A &= \left[\begin{array}{cc} 1 & 0 \\ 0 & 9 \end{array} \right] \left[\begin{array}{ccc} a & b & c \\ p & q & r \end{array} \right] = \left[\begin{array}{ccc} a & b & c \\ 9p & 9q & 9r \end{array} \right] \\
E_3A &= \left[\begin{array}{cc} 1 & 5 \\ 0 & 1 \end{array} \right] \left[\begin{array}{ccc} a & b & c \\ p & q & r \end{array} \right] = \left[\begin{array}{ccc} a+5p & b+5q & c+5r \\ p & q & r \end{array} \right]
\end{aligned}$$

In each case, left multiplying A by the elementary matrix has the *same* effect as doing the corresponding row operation to A. This works in general.

Lemma 2.5.1: [10]

If an elementary row operation is performed on an $m \times n$ matrix A, the result is EA where E is the elementary matrix obtained by performing the same operation on the $m \times m$ identity matrix.

Proof. We prove it for operations of type III; the proofs for types I and II are left as exercises. Let E be the elementary matrix corresponding to the operation that adds k times row p to row $q \neq p$. The proof depends on the fact that each row of EA is equal to the corresponding row of E times

[10] A *lemma* is an auxiliary theorem used in the proof of other theorems.

A. Let K_1, K_2, $\dots$, K_m denote the rows of I_m. Then row i of E is K_i if $i \neq q$, while row q of E is $K_q + kK_p$. Hence:

$$\begin{aligned}
&\text{If } i \neq q \text{ then row } i \text{ of } EA = K_iA = (\text{row } i \text{ of } A).\\
&\text{Row } q \text{ of } EA = (K_q + kK_p)A = K_qA + k(K_pA)\\
&\qquad\qquad\qquad\qquad\qquad\quad = (\text{row } q \text{ of } A) \text{ plus } k \text{ (row } p \text{ of } A).
\end{aligned}$$

Thus EA is the result of adding k times row p of A to row q, as required. □

The effect of an elementary row operation can be reversed by another such operation (called its inverse) which is also elementary of the same type (see the discussion following (Example 1.1.3). It follows that each elementary matrix E is invertible. In fact, if a row operation on I produces E, then the inverse operation carries E back to I. If F is the elementary matrix corresponding to the inverse operation, this means $FE = I$ (by Lemma 2.5.1). Thus $F = E^{-1}$ and we have proved

Lemma 2.5.2

Every elementary matrix E is invertible, and E^{-1} is also a elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E.

The following table gives the inverse of each type of elementary row operation:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$, $k \neq 0$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q, $q \neq p$

Note that elementary matrices of type I are self-inverse.

Example 2.5.1

Find the inverse of each of the elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. E_1, E_2, and E_3 are of type I, II, and III respectively, so the table gives

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}, \quad \text{and} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Inverses and Elementary Matrices

Suppose that an $m \times n$ matrix A is carried to a matrix B (written $A \to B$) by a series of k elementary row operations. Let $E_1, E_2, \dots, E_k$ denote the corresponding elementary matrices. By Lemma 2.5.1, the reduction becomes

$$A \to E_1 A \to E_2 E_1 A \to E_3 E_2 E_1 A \to \cdots \to E_k E_{k-1} \cdots E_2 E_1 A = B$$

In other words,

$$A \to UA = B \quad \text{where } U = E_k E_{k-1} \cdots E_2 E_1$$

The matrix $U = E_k E_{k-1} \cdots E_2 E_1$ is invertible, being a product of invertible matrices by Lemma 2.5.2. Moreover, U can be computed without finding the E_i as follows: If the above series of operations carrying $A \to B$ is performed on I_m in place of A, the result is $I_m \to UI_m = U$. Hence this series of operations carries the block matrix $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} B & U \end{array}\right]$. This, together with the above discussion, proves

Theorem 2.5.1

Suppose A is $m \times n$ and $A \to B$ by elementary row operations.

1. *$B = UA$ where U is an $m \times m$ invertible matrix.*
2. *U can be computed by $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} B & U \end{array}\right]$ using the operations carrying $A \to B$.*
3. *$U = E_k E_{k-1} \cdots E_2 E_1$ where $E_1, E_2, \dots, E_k$ are the elementary matrices corresponding (in order) to the elementary row operations carrying A to B.*

Example 2.5.2

If $A = \left[\begin{array}{ccc} 2 & 3 & 1 \\ 1 & 2 & 1 \end{array}\right]$, express the reduced row-echelon form R of A as $R = UA$ where U is invertible.

Solution. Reduce the double matrix $\left[\begin{array}{cc} A & I \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$ as follows:

$$\left[\begin{array}{cc} A & I \end{array}\right] = \left[\begin{array}{ccc|cc} 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

Hence $R = \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$ and $U = \left[\begin{array}{cc} 2 & -3 \\ -1 & 2 \end{array}\right]$.

Now suppose that A is invertible. We know that $A \to I$ by Theorem 2.4.5, so taking $B = I$ in Theorem 2.5.1 gives $\left[\begin{array}{cc} A & I \end{array}\right] \to \left[\begin{array}{cc} I & U \end{array}\right]$ where $I = UA$. Thus $U = A^{-1}$, so we have $\left[\begin{array}{cc} A & I \end{array}\right] \to \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$.

This is the matrix inversion algorithm in Section 2.4. However, more is true: Theorem 2.5.1 gives $A^{-1} = U = E_k E_{k-1} \cdots E_2 E_1$ where $E_1, E_2, \dots, E_k$ are the elementary matrices corresponding (in order) to the row operations carrying $A \to I$. Hence

$$A = \left(A^{-1}\right)^{-1} = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} \tag{2.10}$$

By Lemma 2.5.2, this shows that every invertible matrix A is a product of elementary matrices. Since elementary matrices are invertible (again by Lemma 2.5.2), this proves the following important characterization of invertible matrices.

Theorem 2.5.2

A square matrix is invertible if and only if it is a product of elementary matrices.

It follows from Theorem 2.5.1 that $A \to B$ by row operations if and only if $B = UA$ for some invertible matrix B. In this case we say that A and B are **row-equivalent**. (See Exercise 2.5.17.)

Example 2.5.3

Express $A = \left[\begin{array}{rr} -2 & 3 \\ 1 & 0 \end{array} \right]$ as a product of elementary matrices.

Solution. Using Lemma 2.5.1, the reduction of $A \to I$ is as follows:

$$A = \left[\begin{array}{rr} -2 & 3 \\ 1 & 0 \end{array} \right] \to E_1 A = \left[\begin{array}{rr} 1 & 0 \\ -2 & 3 \end{array} \right] \to E_2 E_1 A = \left[\begin{array}{rr} 1 & 0 \\ 0 & 3 \end{array} \right] \to E_3 E_2 E_1 A = \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right]$$

where the corresponding elementary matrices are

$$E_1 = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right], \quad E_2 = \left[\begin{array}{rr} 1 & 0 \\ 2 & 1 \end{array} \right], \quad E_3 = \left[\begin{array}{rr} 1 & 0 \\ 0 & \frac{1}{3} \end{array} \right]$$

Hence $(E_3 \ E_2 \ E_1)A = I$, so:

$$A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ -2 & 1 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ 0 & 3 \end{array} \right]$$

Smith Normal Form

Let A be an $m \times n$ matrix of $\operatorname{rank} r$, and let R be the reduced row-echelon form of A. Theorem 2.5.1 shows that $R = UA$ where U is invertible, and that U can be found from $\left[\begin{array}{cc} A & I_m \end{array} \right] \to \left[\begin{array}{cc} R & U \end{array} \right]$.

The matrix R has r leading ones (since $\operatorname{rank} A = r$) so, as R is reduced, the $n \times m$ matrix R^T contains each row of I_r in the first r columns. Thus row operations will carry $R^T \to \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right]_{n \times m}$.

Hence Theorem 2.5.1 (again) shows that $\left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right]_{n \times m} = U_1 R^T$ where U_1 is an $n \times n$ invertible matrix.

Writing $V = U_1^T$, we obtain

$$UAV = RV = RU_1^T = \left(U_1R^T\right)^T = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times m}\right)^T = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$$

Moreover, the matrix $U_1 = V^T$ can be computed by $\left[\begin{array}{cc} R^T & I_n \end{array}\right] \to \left[\begin{array}{cc} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times m} & V^T \end{array}\right]$. This proves

Theorem 2.5.3

Let A be an $m\times n$ matrix of rank *r. There exist invertible matrices U and V of size $m\times m$ and $n\times n$, respectively, such that*

$$UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$$

Moreover, if R is the reduced row-echelon form of A, then:

1. *U can be computed by $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$;*
2. *V can be computed by $\left[\begin{array}{cc} R^T & I_n \end{array}\right] \to \left[\begin{array}{cc} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times m} & V^T \end{array}\right]$.*

If A is an $m\times n$ matrix of rank r, the matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called the **Smith normal form**[11] of A. Whereas the reduced row-echelon form of A is the "nicest" matrix to which A can be carried by row operations, the Smith canonical form is the "nicest" matrix to which A can be carried by *row and column* operations. This is because doing row operations to R^T amounts to doing *column* operations to R and then transposing.

Example 2.5.4

Given $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$, find invertible matrices U and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r = \text{rank } A$.

Solution. The matrix U and the reduced row-echelon form R of A are computed by the row reduction $\left[\begin{array}{cc} A & I_3 \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$:

$$\left[\begin{array}{rrrr|rrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & -2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrr} 1 & -1 & 0 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

[11] Named after Henry John Stephen Smith (1826–83).

Hence

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

In particular, $r = \text{rank } R = 2$. Now row-reduce $\left[\begin{array}{cc} R^T & I_4 \end{array}\right] \to \left[\begin{array}{cc} \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right] & V^T \end{array}\right]$:

$$\left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -3 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -5 & 1 \end{array}\right]$$

whence

$$V^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & -5 & -1 \end{bmatrix} \quad \text{so} \quad V = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $UAV = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ as is easily verified.

Uniqueness of the Reduced Row-echelon Form

In this short subsection, Theorem 2.5.1 is used to prove the following important theorem.

Theorem 2.5.4

If a matrix A is carried to reduced row-echelon matrices R and S by row operations, then $R = S$.

Proof. Observe first that $UR = S$ for some invertible matrix U (by Theorem 2.5.1 there exist invertible matrices P and Q such that $R = PA$ and $S = QA$; take $U = QP^{-1}$). We show that $R = S$ by induction on the number m of rows of R and S. The case $m = 1$ is left to the reader. If R_j and S_j denote column j in R and S respectively, the fact that $UR = S$ gives

$$UR_j = S_j \quad \text{for each } j \tag{2.11}$$

Since U is invertible, this shows that R and S have the same zero columns. Hence, by passing to the matrices obtained by deleting the zero columns from R and S, we may assume that R and S have no zero columns.

But then the first column of R and S is the first column of I_m because R and S are row-echelon, so (2.11) shows that the first column of U is column 1 of I_m. Now write U, R, and S in block form as follows.

$$U = \begin{bmatrix} 1 & X \\ 0 & V \end{bmatrix}, \quad R = \begin{bmatrix} 1 & X \\ 0 & R' \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} 1 & Z \\ 0 & S' \end{bmatrix}$$

Since $UR = S$, block multiplication gives $VR' = S'$ so, since V is invertible (U is invertible) and both R' and S' are reduced row-echelon, we obtain $R' = S'$ by induction. Hence R and S have the same number (say r) of leading 1s, and so both have m–r zero rows.

In fact, R and S have leading ones in the same columns, say r of them. Applying (2.11) to these columns shows that the first r columns of U are the first r columns of I_m. Hence we can write U, R, and S in block form as follows:

$$U = \begin{bmatrix} I_r & M \\ 0 & W \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$$

where R_1 and S_1 are $r \times r$. Then block multiplication gives $UR = R$; that is, $S = R$. This completes the proof. □

Exercises for 2.5

Exercise 2.5.1 For each of the following elementary matrices, describe the corresponding elementary row operation and write the inverse.

a) $E = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d) $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ f) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

---

b. Interchange rows 1 and 3 of I. $E^{-1} = E$.

d. Add (-2) times row 1 of I to row 2. $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. Multiply row 3 of I by 5. $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$

Exercise 2.5.2 In each case find an elementary matrix E such that $B = EA$.

a. $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

b. $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$

d. $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

e. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

f. $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

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b. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 2.5.3 Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

a. Find elementary matrices E_1 and E_2 such that $C = E_2E_1A$.

b. Show that there is *no* elementary matrix E such that $C = EA$.

---

b. The only possibilities for E are $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$. In each case, EA has a row different from C.

Exercise 2.5.4 If E is elementary, show that A and EA differ in at most two rows.

Exercise 2.5.5

a. Is I an elementary matrix? Explain.

b. Is 0 an elementary matrix? Explain.

---

b. No, 0 is not invertible.

Exercise 2.5.6 In each case find an invertible matrix U such that $UA = R$ is in reduced row-echelon form, and express U as a product of elementary matrices.

a) $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 12 & -1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & -3 & 3 & 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 \end{bmatrix}$

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b. $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \end{bmatrix}$. Alternatively, $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \end{bmatrix}$.

d. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A =$ $\begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{7}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Exercise 2.5.7 In each case find an invertible matrix U such that $UA = B$, and express U as a product of elementary matrices.

a. $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

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b. $U = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 2.5.8 In each case factor A as a product of elementary matrices.

a) $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ b) $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ d) $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ -2 & 2 & 15 \end{bmatrix}$

---

b. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 2.5.9 Let E be an elementary matrix.

a. Show that E^T is also elementary of the same type.

b. Show that $E^T = E$ if E is of type I or II.

Exercise 2.5.10 Show that every matrix A can be factored as $A = UR$ where U is invertible and R is in reduced row-echelon form.

$UA = R$ by Theorem 2.5.1, so $A = U^{-1}R$.

Exercise 2.5.11 If $A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ -5 & -3 \end{bmatrix}$ find an elementary matrix F such that $AF = B$. [*Hint*: See Exercise 2.5.9.]

Exercise 2.5.12 In each case find invertible U and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r = \text{rank } A$.

a) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -2 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 0 & 3 \\ 0 & 1 & -4 & 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$

---

b. $U = A^{-1}$, $V = I^2$; rank $A = 2$

d. $U = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$,

$V = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; rank $A = 2$

Exercise 2.5.13 Prove Lemma 2.5.1 for elementary matrices of:

a) type I;

b) type II.

Exercise 2.5.14 While trying to invert A, $\begin{bmatrix} A & I \end{bmatrix}$ is carried to $\begin{bmatrix} P & Q \end{bmatrix}$ by row operations. Show that $P = QA$.

Exercise 2.5.15 If A and B are $n \times n$ matrices and AB is a product of elementary matrices, show that the same is true of A.

Exercise 2.5.16 If U is invertible, show that the reduced row-echelon form of a matrix $\begin{bmatrix} U & A \end{bmatrix}$ is $\begin{bmatrix} I & U^{-1}A \end{bmatrix}$.

Write $U^{-1} = E_kE_{k-1} \cdots E_2E_1$, E_i elementary. Then $\begin{bmatrix} I & U^{-1}A \end{bmatrix} = \begin{bmatrix} U^{-1}U & U^{-1}A \end{bmatrix}$ $= U^{-1}\begin{bmatrix} U & A \end{bmatrix} = E_kE_{k-1} \cdots E_2E_1 \begin{bmatrix} U & A \end{bmatrix}$. So $\begin{bmatrix} U & A \end{bmatrix} \to \begin{bmatrix} I & U^{-1}A \end{bmatrix}$ by row operations (Lemma 2.5.1).

Exercise 2.5.17 Two matrices A and B are called **row-equivalent** (written $A \overset{r}{\sim} B$) if there is a sequence of elementary row operations carrying A to B.

a. Show that $A \overset{r}{\sim} B$ if and only if $A = UB$ for some invertible matrix U.

b. Show that:

   i. $A \overset{r}{\sim} A$ for all matrices A.

   ii. If $A \overset{r}{\sim} B$, then $B \overset{r}{\sim} A$

   iii. If $A \overset{r}{\sim} B$ and $B \overset{r}{\sim} C$, then $A \overset{r}{\sim} C$.

c. Show that, if A and B are both row-equivalent to some third matrix, then $A \overset{r}{\sim} B$.

d. Show that $\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 8 & 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 4 & 5 \\ -2 & 1 & -11 & -8 \\ -1 & 2 & 2 & 2 \end{bmatrix}$ are row-equivalent. [*Hint*: Consider (c) and Theorem 1.2.1.]

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b. (i) $A \overset{r}{\sim} A$ because $A = IA$. (ii) If $A \overset{r}{\sim} B$, then $A = UB$, U invertible, so $B = U^{-1}A$. Thus $B \overset{r}{\sim} A$. (iii) If $A \overset{r}{\sim} B$ and $B \overset{r}{\sim} C$, then $A = UB$ and $B = VC$, U and V invertible. Hence $A = U(VC) = (UV)C$, so $A \overset{r}{\sim} C$.

Exercise 2.5.18 If U and V are invertible $n \times n$ matrices, show that $U \overset{r}{\sim} V$. (See Exercise 2.5.17.)

Exercise 2.5.19 (See Exercise 2.5.17.) Find all matrices that are row-equivalent to:

a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

---

b. If $B \overset{r}{\sim} A$, let $B = UA$, U invertible. If $U = \begin{bmatrix} d & b \\ -b & d \end{bmatrix}$, $B = UA = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & d \end{bmatrix}$ where b and d are not both zero (as U is invertible). Every such matrix B arises in this way: Use $U = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$–it is invertible by Example 2.3.5.

Exercise 2.5.20 Let A and B be $m \times n$ and $n \times m$ matrices, respectively. If $m > n$, show that AB is not invertible. [*Hint*: Use Theorem 1.3.1 to find $\mathbf{x} \neq \mathbf{0}$ with $B\mathbf{x} = \mathbf{0}$.]

Exercise 2.5.21 Define an *elementary column operation* on a matrix to be one of the following: (I) Interchange two columns. (II) Multiply a column by a nonzero scalar. (III) Add a multiple of a column to another column. Show that:

a. If an elementary column operation is done to an $m \times n$ matrix A, the result is AF, where F is an $n \times n$ elementary matrix.

b. Given any $m \times n$ matrix A, there exist $m \times m$ elementary matrices $E_1, \dots, E_k$ and $n \times n$ elementary matrices $F_1, \dots, F_p$ such that, in block form,

$$E_k \cdots E_1 A F_1 \cdots F_p = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise 2.5.22 Suppose B is obtained from A by:

a. interchanging rows i and j;

b. multiplying row i by $k \neq 0$;

c. adding k times row i to row j $(i \neq j)$.

In each case describe how to obtain B^{-1} from A^{-1}. [*Hint*: See part (a) of the preceding exercise.]

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b. Multiply column i by $1/k$.

Exercise 2.5.23 Two $m \times n$ matrices A and B are called **equivalent** (written $A \overset{e}{\sim} B$) if there exist invertible matrices U and V (sizes $m \times m$ and $n \times n$) such that $A = UBV$.

a. Prove the following the properties of equivalence.

   i. $A \overset{e}{\sim} A$ for all $m \times n$ matrices A.

   ii. If $A \overset{e}{\sim} B$, then $B \overset{e}{\sim} A$.

   iii. If $A \overset{e}{\sim} B$ and $B \overset{e}{\sim} C$, then $A \overset{e}{\sim} C$.

b. Prove that two $m \times n$ matrices are equivalent if they have the same rank. [*Hint*: Use part (a) and Theorem 2.5.3.]

2.6 Linear Transformations

If A is an $m \times n$ matrix, recall that the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

is called the *matrix transformation induced* by A. In Section 2.2, we saw that many important geometric transformations were in fact matrix transformations. These transformations can be characterized in a different way. The new idea is that of a linear transformation, one of the basic notions in linear algebra. We define these transformations in this section, and show that they are really just the matrix transformations looked at in another way. Having these two ways to view them turns out to be useful because, in a given situation, one perspective or the other may be preferable.

Linear Transformations

Definition 2.13 Linear Transformations $\mathbb{R}^n \to \mathbb{R}^m$

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a ***linear transformation*** *if it satisfies the following two conditions for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a:*

T1 $\quad T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

T2 $\quad T(a\mathbf{x}) = aT(\mathbf{x})$

Of course, $\mathbf{x}+\mathbf{y}$ and $a\mathbf{x}$ here are computed in $\mathbb{R}^n$, while $T(\mathbf{x}) + T(\mathbf{y})$ and $aT(\mathbf{x})$ are in $\mathbb{R}^m$. We say that T *preserves addition* if T1 holds, and that T *preserves scalar multiplication* if T2 holds. Moreover, taking $a = 0$ and $a = -1$ in T2 gives

$$T(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad T(-\mathbf{x}) = -T(\mathbf{x}) \quad \text{for all } \mathbf{x}$$

Hence T preserves the zero vector and the negative of a vector. Even more is true.

Recall that a vector $\mathbf{y}$ in $\mathbb{R}^n$ is called a **linear combination** of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ if $\mathbf{y}$ has the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$$

for some scalars $a_1, a_2, \dots, a_k$. Conditions T1 and T2 combine to show that every linear transformation T *preserves linear combinations* in the sense of the following theorem. This result is used repeatedly in linear algebra.

Theorem 2.6.1: Linearity Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for each $k = 1, 2, \dots$

$$T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) = a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2) + \cdots + a_kT(\mathbf{x}_k)$$

for all scalars a_i and all vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

Proof. If $k=1$, it reads $T(a_1\mathbf{x}_1)=a_1T(\mathbf{x}_1)$ which is Condition T1. If $k=2$, we have

$$\begin{aligned} T(a_1\mathbf{x}_1+a_2\mathbf{x}_2) &= T(a_1\mathbf{x}_1)+T(a_2\mathbf{x}_2) && \text{by Condition T1} \\ &= a_1T(\mathbf{x}_1)+a_2T(\mathbf{x}_2) && \text{by Condition T2} \end{aligned}$$

If $k=3$, we use the case $k=2$ to obtain

$$\begin{aligned} T(a_1\mathbf{x}_1+a_2\mathbf{x}_2+a_3\mathbf{x}_3) &= T\left[(a_1\mathbf{x}_1+a_2\mathbf{x}_2)+a_3\mathbf{x}_3\right] && \text{collect terms} \\ &= T(a_1\mathbf{x}_1+a_2\mathbf{x}_2)+T(a_3\mathbf{x}_3) && \text{by Condition T1} \\ &= \left[a_1T(\mathbf{x}_1)+a_2T(\mathbf{x}_2)\right]+T(a_3\mathbf{x}_3) && \text{by the case } k=2 \\ &= \left[a_1T(\mathbf{x}_1)+a_2T(\mathbf{x}_2)\right]+a_3T(\mathbf{x}_3) && \text{by Condition T2} \end{aligned}$$

The proof for any k is similar, using the previous case $k-1$ and Conditions T1 and T2. □

The method of proof in Theorem 2.6.1 is called *mathematical induction* (Appendix **??**).

Theorem 2.6.1 shows that if T is a linear transformation and $T(\mathbf{x}_1)$, $T(\mathbf{x}_2)$, ..., $T(\mathbf{x}_k)$ are all known, then $T(\mathbf{y})$ can be easily computed for any linear combination $\mathbf{y}$ of $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$. This is a very useful property of linear transformations, and is illustrated in the next example.

Example 2.6.1

If $T:\mathbb{R}^2\to\mathbb{R}^2$ is a linear transformation, $T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix}$ and $T\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$, find $T\begin{bmatrix}4\\3\end{bmatrix}$.

Solution. Write $\mathbf{z}=\begin{bmatrix}4\\3\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}1\\1\end{bmatrix}$, and $\mathbf{y}=\begin{bmatrix}1\\-2\end{bmatrix}$ for convenience. Then we know $T(\mathbf{x})$ and $T(\mathbf{y})$ and we want $T(\mathbf{z})$, so it is enough by Theorem 2.6.1 to express $\mathbf{z}$ as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. That is, we want to find numbers a and b such that $\mathbf{z}=a\mathbf{x}+b\mathbf{y}$. Equating entries gives two equations $4=a+b$ and $3=a-2b$. The solution is, $a=\frac{11}{3}$ and $b=\frac{1}{3}$, so $\mathbf{z}=\frac{11}{3}\mathbf{x}+\frac{1}{3}\mathbf{y}$. Thus Theorem 2.6.1 gives

$$T(\mathbf{z})=\tfrac{11}{3}T(\mathbf{x})+\tfrac{1}{3}T(\mathbf{y})=\tfrac{11}{3}\begin{bmatrix}2\\-3\end{bmatrix}+\tfrac{1}{3}\begin{bmatrix}5\\1\end{bmatrix}=\tfrac{1}{3}\begin{bmatrix}27\\-32\end{bmatrix}$$

This is what we wanted.

Example 2.6.2

If A is $m\times n$, the matrix transformation $T_A:\mathbb{R}^n\to\mathbb{R}^m$, is a linear transformation.

Solution. We have $T_A(\mathbf{x})=A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, so Theorem 2.2.2 gives

$$T_A(\mathbf{x}+\mathbf{y})=A(\mathbf{x}+\mathbf{y})=A\mathbf{x}+A\mathbf{y}=T_A(\mathbf{x})+T_A(\mathbf{y})$$

and

$$T_A(a\mathbf{x})=A(a\mathbf{x})=a(A\mathbf{x})=aT_A(\mathbf{x})$$

hold for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. Hence T_A satisfies T1 and T2, and so is linear.

The remarkable thing is that the *converse* of Example 2.6.2 is true: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is actually a matrix transformation. To see why, we define the **standard basis** of $\mathbb{R}^n$ to be the set of columns

$$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$$

of the identity matrix I_n. Then each $\mathbf{e}_i$ is in $\mathbb{R}^n$ and every vector $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ in $\mathbb{R}^n$ is a linear combination of the $\mathbf{e}_i$. In fact:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

as the reader can verify. Hence Theorem 2.6.1 shows that

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n)$$

Now observe that each $T(\mathbf{e}_i)$ is a column in $\mathbb{R}^m$, so

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$$

is an $m \times n$ matrix. Hence we can apply Definition 2.5 to get

$$T(\mathbf{x}) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n) = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] = A\mathbf{x}$$

Since this holds for every $\mathbf{x}$ in $\mathbb{R}^n$, it shows that T is the matrix transformation induced by A, and so proves most of the following theorem.

Theorem 2.6.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. *T is linear if and only if it is a matrix transformation.*
2. *In this case $T = T_A$ is the matrix transformation induced by a unique $m \times n$ matrix A, given in terms of its columns by*

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Proof. It remains to verify that the matrix A is unique. Suppose that T is induced by another matrix B. Then $T(\mathbf{x}) = B\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. But $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$, so $B\mathbf{x} = A\mathbf{x}$ for every $\mathbf{x}$. Hence $A = B$ by Theorem 2.2.6. □

Hence we can speak of *the* matrix of a linear transformation. Because of Theorem 2.6.2 we may (and shall) use the phrases "linear transformation" and "matrix transformation" interchangeably.

Example 2.6.3

Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$. Show that T is a linear transformation and use Theorem 2.6.2 to find its matrix.

Solution. Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, so that $\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$. Hence

$$T(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\mathbf{x}) + T(\mathbf{y})$$

Similarly, the reader can verify that $T(a\mathbf{x}) = aT(\mathbf{x})$ for all a in $\mathbb{R}$, so T is a linear transformation. Now the standard basis of $\mathbb{R}^3$ is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so, by Theorem 2.6.2, the matrix of T is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Of course, the fact that $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ shows directly that T is a matrix transformation (hence linear) and reveals the matrix.

To illustrate how Theorem 2.6.2 is used, we rederive the matrices of the transformations in Examples 2.2.13 and 2.2.15.

Example 2.6.4

Let $Q_0 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the x axis (as in Example 2.2.13) and let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation through $\frac{\pi}{2}$ about the origin (as in Example 2.2.15). Use Theorem 2.6.2 to find the matrices of Q_0 and $R_{\frac{\pi}{2}}$.

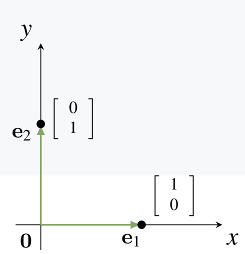


Figure 2.6.1

Solution. Observe that Q_0 and $R_{\frac{\pi}{2}}$ are linear by Example 2.6.2 (they are matrix transformations), so Theorem 2.6.2 applies to them. The standard basis of $\mathbb{R}^2$ is

$\{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ points along the positive x axis, and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ points along the positive y axis (see Figure 2.6.1).

The reflection of $\mathbf{e}_1$ in the x axis is $\mathbf{e}_1$ itself because $\mathbf{e}_1$ points along the x axis, and the reflection of $\mathbf{e}_2$ in the x axis is $-\mathbf{e}_2$ because $\mathbf{e}_2$ is perpendicular to the x axis. In other words, $Q_0(\mathbf{e}_1) = \mathbf{e}_1$ and $Q_0(\mathbf{e}_2) = -\mathbf{e}_2$. Hence Theorem 2.6.2 shows that the matrix of Q_0 is

$$\begin{bmatrix} Q_0(\mathbf{e}_1) & Q_0(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & -\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which agrees with Example 2.2.13.

Similarly, rotating $\mathbf{e}_1$ through $\frac{\pi}{2}$ counterclockwise about the origin produces $\mathbf{e}_2$, and rotating $\mathbf{e}_2$ through $\frac{\pi}{2}$ counterclockwise about the origin gives $-\mathbf{e}_1$. That is, $R_{\frac{\pi}{2}}(\mathbf{e}_1) = \mathbf{e}_2$ and $R_{\frac{\pi}{2}}(\mathbf{e}_2) = -\mathbf{e}_2$. Hence, again by Theorem 2.6.2, the matrix of $R_{\frac{\pi}{2}}$ is

$$\begin{bmatrix} R_{\frac{\pi}{2}}(\mathbf{e}_1) & R_{\frac{\pi}{2}}(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_2 & -\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

agreeing with Example 2.2.15.

Example 2.6.5

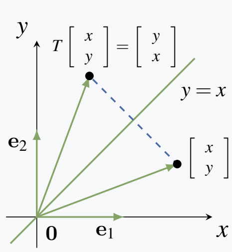


Figure 2.6.2

Let $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = x$. Show that Q_1 is a matrix transformation, find its matrix, and use it to illustrate Theorem 2.6.2.

Solution. Figure 2.6.2 shows that $Q_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. Hence $Q_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}$, so Q_1 is the matrix transformation induced by the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence Q_1 is linear (by Example 2.6.2) and so Theorem 2.6.2 applies. If $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the standard basis of $\mathbb{R}^2$, then it is clear geometrically that $Q_1(\mathbf{e}_1) = \mathbf{e}_2$ and $Q_1(\mathbf{e}_2) = \mathbf{e}_1$. Thus (by Theorem 2.6.2) the matrix of Q_1 is $\begin{bmatrix} Q_1(\mathbf{e}_1) & Q_1(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_2 & \mathbf{e}_1 \end{bmatrix} = A$ as before.

Recall that, given two "linked" transformations

$$\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$$

we can apply T first and then apply S, and so obtain a new transformation

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m$$

called the **composite** of S and T, defined by

$$(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^k$$

If S and T are linear, the action of $S \circ T$ can be computed by multiplying their matrices.

Theorem 2.6.3

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and let A and B be the matrices of S and T respectively. Then $S \circ T$ is linear with matrix AB.

Proof. $(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] = A[B\mathbf{x}] = (AB)\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^k$. □

Theorem 2.6.3 shows that the action of the composite $S \circ T$ is determined by the matrices of S and T. But it also provides a very useful interpretation of matrix multiplication. If A and B are matrices, the product matrix AB induces the transformation resulting from first applying B and then applying A. Thus the study of matrices can cast light on geometrical transformations and vice-versa. Here is an example.

Example 2.6.6

Show that reflection in the x axis followed by rotation through $\frac{\pi}{2}$ is reflection in the line $y = x$.

Solution. The composite in question is $R_{\frac{\pi}{2}} \circ Q_0$ where Q_0 is reflection in the x axis and $R_{\frac{\pi}{2}}$ is rotation through $\frac{\pi}{2}$. By Example 2.6.4, $R_{\frac{\pi}{2}}$ has matrix $A = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$ and Q_0 has matrix $B = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$. Hence Theorem 2.6.3 shows that the matrix of $R_{\frac{\pi}{2}} \circ Q_0$ is $AB = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right]$, which is the matrix of reflection in the line $y = x$ by Example 2.6.3.

This conclusion can also be seen geometrically. Let $\mathbf{x}$ be a typical point in $\mathbb{R}^2$, and assume that $\mathbf{x}$ makes an angle α with the positive x axis. The effect of first applying Q_0 and then applying $R_{\frac{\pi}{2}}$ is shown in Figure 2.6.3. The fact that $R_{\frac{\pi}{2}}[Q_0(\mathbf{x})]$ makes the angle α with the positive y axis shows that $R_{\frac{\pi}{2}}[Q_0(\mathbf{x})]$ is the reflection of $\mathbf{x}$ in the line $y = x$.

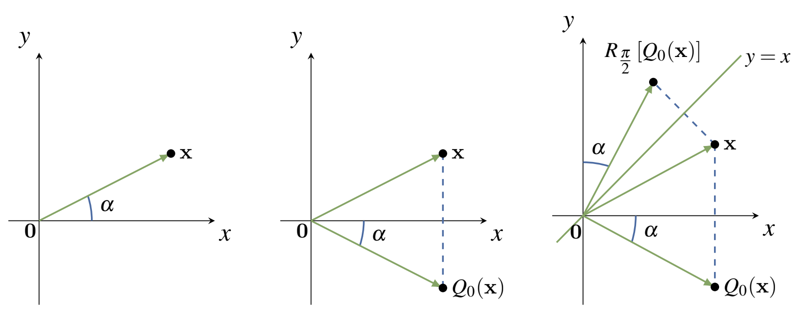


Figure 2.6.3

In Theorem 2.6.3, we saw that the matrix of the composite of two linear transformations is the product of their matrices (in fact, matrix products were defined so that this is the case). We are going to apply this fact to rotations, reflections, and projections in the plane. Before proceeding, we pause to present useful geometrical descriptions of vector addition and scalar multiplication in the plane, and to give a short review of angles and the trigonometric functions.

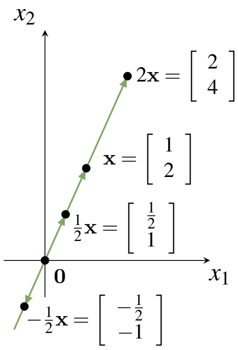


Figure 2.6.4

Some Geometry

As we have seen, it is convenient to view a vector $\mathbf{x}$ in $\mathbb{R}^2$ as an arrow from the origin to the point $\mathbf{x}$ (see Section 2.2). This enables us to visualize what sums and scalar multiples mean geometrically. For example consider $\mathbf{x} = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$ in $\mathbb{R}^2$. Then $2\mathbf{x} = \left[\begin{array}{c} 2 \\ 4 \end{array}\right]$, $\frac{1}{2}\mathbf{x} = \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]$ and $-\frac{1}{2}\mathbf{x} = \left[\begin{array}{c} -\frac{1}{2} \\ -1 \end{array}\right]$, and these are shown as arrows in Figure 2.6.4.

Observe that the arrow for $2\mathbf{x}$ is twice as long as the arrow for $\mathbf{x}$ and in the same direction, and that the arrows for $\frac{1}{2}\mathbf{x}$ is also in the same direction as the arrow for $\mathbf{x}$, but only half as long. On the other hand, the arrow for $-\frac{1}{2}\mathbf{x}$ is half as long as the arrow for $\mathbf{x}$, but in the *opposite* direction. More generally, we have the following geometrical description of scalar multiplication in $\mathbb{R}^2$:

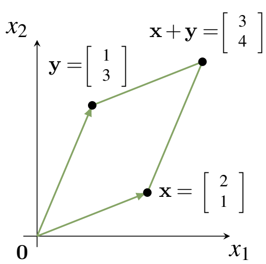


Figure 2.6.5

Scalar Multiple Law

Let $\mathbf{x}$ be a vector in $\mathbb{R}^2$. The arrow for $k\mathbf{x}$ is $|k|$ times[12] as long as the arrow for $\mathbf{x}$, and is in the same direction as the arrow for $\mathbf{x}$ if $k > 0$, and in the opposite direction if $k < 0$.

Now consider two vectors $\mathbf{x} = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$ and $\mathbf{y} = \left[\begin{array}{c} 1 \\ 3 \end{array}\right]$ in $\mathbb{R}^2$. They are plotted in Figure 2.6.5 along with their sum $\mathbf{x}+\mathbf{y} = \left[\begin{array}{c} 3 \\ 4 \end{array}\right]$. It is a routine matter to verify that the four points $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}+\mathbf{y}$ form the vertices of a **parallelogram**–that is opposite sides are parallel and of the same length. (The reader should verify that the side from $\mathbf{0}$ to $\mathbf{x}$ has slope of $\frac{1}{2}$, as does the side from $\mathbf{y}$ to $\mathbf{x}+\mathbf{y}$, so these sides are parallel.) We state this as follows:

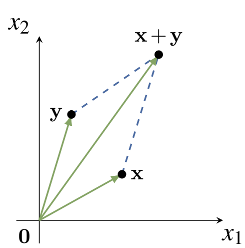


Figure 2.6.6

Parallelogram Law

Consider vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$. If the arrows for $\mathbf{x}$ and $\mathbf{y}$ are drawn (see Figure 2.6.6), the arrow for $\mathbf{x}+\mathbf{y}$ corresponds to the fourth vertex of the parallelogram determined by the

[12]If k is a real number, $|k|$ denotes the **absolute value** of k; that is, $|k| = k$ if $k \geq 0$ and $|k| = -k$ if $k < 0$.

Figure 2.6.7

points $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{0}$.

We will have more to say about this in Chapter 4.

Before proceeding we turn to a brief review of angles and the trigonometric functions. Recall that an angle θ is said to be in **standard position** if it is measured counterclockwise from the positive x axis (as in Figure 2.6.7). Then θ uniquely determines a point $\mathbf{p}$ on the **unit circle** (radius 1, centre at the origin). The **radian** measure of θ is the length of the arc on the unit circle from the positive x axis to $\mathbf{p}$. Thus $360^\circ = 2\pi$ radians, $180^\circ = \pi$, $90^\circ = \frac{\pi}{2}$, and so on.

The point $\mathbf{p}$ in Figure 2.6.7 is also closely linked to the trigonometric functions **cosine** and **sine**, written $\cos\theta$ and $\sin\theta$ respectively. In fact these functions are *defined* to be the x and y coordinates of $\mathbf{p}$; that is $\mathbf{p} = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right]$. This defines $\cos\theta$ and $\sin\theta$ for the arbitrary angle θ (possibly negative), and agrees with the usual values when θ is an acute angle $\left(0 \le \theta \le \frac{\pi}{2}\right)$ as the reader should verify. For more discussion of this, see Appendix **??**.

Rotations

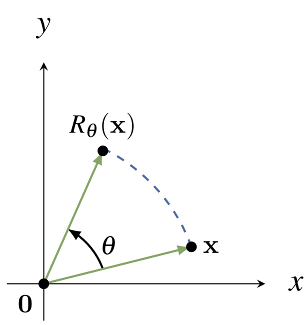


Figure 2.6.8

We can now describe rotations in the plane. Given an angle θ, let

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denote counterclockwise rotation of $\mathbb{R}^2$ about the origin through the angle θ. The action of R_θ is depicted in Figure 2.6.8. We have already looked at $R_{\frac{\pi}{2}}$ (in Example 2.2.15) and found it to be a matrix transformation. It turns out that R_θ is a matrix transformation for *every* angle θ (with a simple formula for the matrix), but it is not clear how to find the matrix. Our approach is to first establish the (somewhat surprising) fact that R_θ is *linear*, and then obtain the matrix from Theorem 2.6.2.

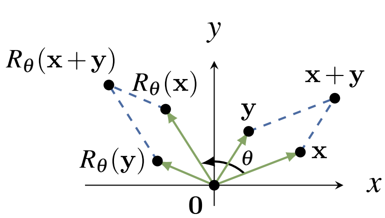


Figure 2.6.9

Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^2$. Then $\mathbf{x}+\mathbf{y}$ is the diagonal of the parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$ as in Figure 2.6.9.

The effect of R_θ is to rotate the *entire* parallelogram to obtain the new parallelogram determined by $R_\theta(\mathbf{x})$ and $R_\theta(\mathbf{y})$, with diagonal $R_\theta(\mathbf{x}+\mathbf{y})$. But this diagonal is $R_\theta(\mathbf{x}) + R_\theta(\mathbf{y})$ by the parallelogram law (applied to the new parallelogram). It follows that

$$R_\theta(\mathbf{x}+\mathbf{y}) = R_\theta(\mathbf{x}) + R_\theta(\mathbf{y})$$

A similar argument shows that $R_\theta(a\mathbf{x}) = aR_\theta(\mathbf{x})$ for any scalar a, so $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is indeed a linear transformation.

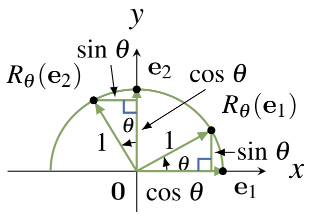


Figure 2.6.10

With linearity established we can find the matrix of R_θ. Let $\mathbf{e}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$ and $\mathbf{e}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$ denote the standard basis of $\mathbb{R}^2$. By Figure 2.6.10 we see that

$$R_\theta(\mathbf{e}_1) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array} \right] \quad \text{and} \quad R_\theta(\mathbf{e}_2) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array} \right]$$

Hence Theorem 2.6.2 shows that R_θ is induced by the matrix

$$\left[\begin{array}{cc} R_\theta(\mathbf{e}_1) & R_\theta(\mathbf{e}_2) \end{array} \right] = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]$$

We record this as

Theorem 2.6.4

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ *is the linear transformation with matrix* $\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]$.

For example, $R_{\frac{\pi}{2}}$ and R_π have matrices $\left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array} \right]$ and $\left[\begin{array}{rr} -1 & 0 \\ 0 & -1 \end{array} \right]$, respectively, by Theorem 2.6.4. The first of these confirms the result in Example 2.2.15. The second shows that rotating a vector $\mathbf{x} = \left[\begin{array}{c} x \\ y \end{array} \right]$ through the angle π results in $R_\pi(\mathbf{x}) = \left[\begin{array}{rr} -1 & 0 \\ 0 & -1 \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} -x \\ -y \end{array} \right] = -\mathbf{x}$. Thus applying R_π is the same as negating $\mathbf{x}$, a fact that is evident without Theorem 2.6.4.

Example 2.6.7

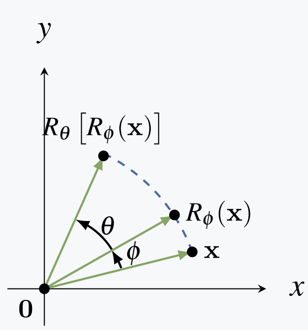


Figure 2.6.11

Let θ and ϕ be angles. By finding the matrix of the composite $R_\theta \circ R_\phi$, obtain expressions for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$.

Solution. Consider the transformations $\mathbb{R}^2 \xrightarrow{R_\phi} \mathbb{R}^2 \xrightarrow{R_\theta} \mathbb{R}^2$. Their composite $R_\theta \circ R_\phi$ is the transformation that first rotates the plane through ϕ and then rotates it through θ, and so is the rotation through the angle $\theta + \phi$ (see Figure 2.6.11). In other words

$$R_{\theta+\phi} = R_\theta \circ R_\phi$$

Theorem 2.6.3 shows that the corresponding equation holds for the matrices of these transformations, so Theorem 2.6.4 gives:

$$\left[\begin{array}{cc} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{array} \right] = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right] \left[\begin{array}{cc} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{array} \right]$$

If we perform the matrix multiplication on the right, and then compare first column entries, we obtain

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

These are the two basic identities from which most of trigonometry can be derived.

Reflections

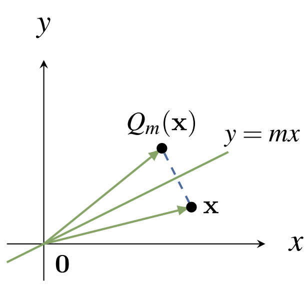


Figure 2.6.12

The line through the origin with slope m has equation $y = mx$, and we let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$.

This transformation is described geometrically in Figure 2.6.12. In words, $Q_m(\mathbf{x})$ is the "mirror image" of $\mathbf{x}$ in the line $y = mx$. If $m = 0$ then Q_0 is reflection in the x axis, so we already know Q_0 is linear. While we could show directly that Q_m is linear (with an argument like that for R_θ), we prefer to do it another way that is instructive and derives the matrix of Q_m directly without using Theorem 2.6.2.

Let θ denote the angle between the positive x axis and the line $y = mx$. The key observation is that the transformation Q_m can be accomplished in three steps: First rotate through $-\theta$ (so our line coincides with the x axis), then reflect in the x axis, and finally rotate back through θ. In other words:

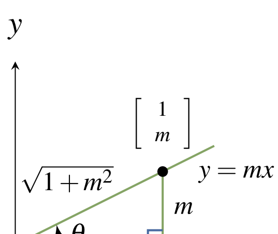
$$Q_m = R_\theta \circ Q_0 \circ R_{-\theta}$$

Since $R_{-\theta}$, Q_0, and R_θ are all linear, this (with Theorem 2.6.3) shows that Q_m is linear and that its matrix is the product of the matrices of R_θ, Q_0, and $R_{-\theta}$. If we write $c = \cos\theta$ and $s = \sin\theta$ for simplicity, then the matrices of R_θ, $R_{-\theta}$, and Q_0 are

$$\left[\begin{array}{rr} c & -s \\ s & c \end{array}\right], \quad \left[\begin{array}{rr} c & s \\ -s & c \end{array}\right], \quad \text{and} \quad \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] \text{ respectively.}$$[13]

Hence, by Theorem 2.6.3, the matrix of $Q_m = R_\theta \circ Q_0 \circ R_{-\theta}$ is

$$\left[\begin{array}{rr} c & -s \\ s & c \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{rr} c & s \\ -s & c \end{array}\right] = \left[\begin{array}{cc} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{array}\right]$$



We can obtain this matrix in terms of m alone. Figure 2.6.13 shows that

$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \text{ and } \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

so the matrix $\left[\begin{array}{cc} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{array}\right]$ of Q_m becomes $\frac{1}{1+m^2}\left[\begin{array}{cc} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{array}\right]$.

Theorem 2.6.5

Let Q_m denote reflection in the line $y = mx$. Then Q_m is a

[13] The matrix of $R_{-\theta}$ comes from the matrix of R_θ using the fact that, for all angles θ, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin(\theta)$.

linear transformation with matrix $\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$.

Note that if $m = 0$, the matrix in Theorem 2.6.5 becomes $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, as expected. Of course this analysis fails for reflection in the y axis because vertical lines have no slope. However it is an easy exercise to verify directly that reflection in the y axis is indeed linear with matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.[14]

Example 2.6.8

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through $-\frac{\pi}{2}$ followed by reflection in the y axis. Show that T is a reflection in a line through the origin and find the line.

Solution. The matrix of $R_{-\frac{\pi}{2}}$ is $\begin{bmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and the matrix of reflection in the y axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence the matrix of T is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and this is reflection in the line $y = -x$ (take $m = -1$ in Theorem 2.6.5).

Projections

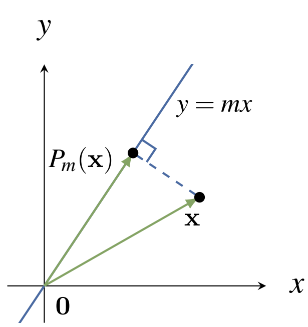


Figure 2.6.14

The method in the proof of Theorem 2.6.5 works more generally. Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote projection on the line $y = mx$. This transformation is described geometrically in Figure 2.6.14.

If $m = 0$, then $P_0\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, so P_0 is linear with matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence the argument above for Q_m goes through for P_m. First observe that

$$P_m = R_\theta \circ P_0 \circ R_{-\theta}$$

as before. So, P_m is linear with matrix

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$

where $c = \cos\theta = \frac{1}{\sqrt{1+m^2}}$ and $s = \sin\theta = \frac{m}{\sqrt{1+m^2}}$.

[14]Note that $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \lim_{m\to\infty} \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$.

This gives:

Theorem 2.6.6

Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ be projection on the line $y = mx$. Then P_m is a linear transformation with matrix $\frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$.

Again, if $m = 0$, then the matrix in Theorem 2.6.6 reduces to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as expected. As the y axis has no slope, the analysis fails for projection on the y axis, but this transformation is indeed linear with matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ as is easily verified directly.

Note that the formula for the matrix of Q_m in Theorem 2.6.5 can be derived from the above formula for the matrix of P_m. Using Figure 2.6.12, observe that $Q_m(\mathbf{x}) = \mathbf{x} + 2[P_m(\mathbf{x}) - \mathbf{x}]$ so $Q_m(x) = 2P_m(\mathbf{x}) - \mathbf{x}$. Substituting the matrices for $P_m(\mathbf{x})$ and $1_{\mathbb{R}^2}(\mathbf{x})$ gives the desired formula.

Example 2.6.9

Given $\mathbf{x}$ in $\mathbb{R}^2$, write $\mathbf{y} = P_m(\mathbf{x})$. The fact that $\mathbf{y}$ lies on the line $y = mx$ means that $P_m(\mathbf{y}) = \mathbf{y}$. But then

$$(P_m \circ P_m)(\mathbf{x}) = P_m(\mathbf{y}) = \mathbf{y} = P_m(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^2, \quad \text{that is, } P_m \circ P_m = P_m.$$

In particular, if we write the matrix of P_m as $A = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$, then $A^2 = A$. The reader should verify this directly.

Exercises for 2.6

Exercise 2.6.1 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation.

a. Find $T\begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}$ if $T\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

b. Find $T\begin{bmatrix} 5 \\ 6 \\ -13 \end{bmatrix}$ if $T\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $T\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

---

b. $\begin{bmatrix} 5 \\ 6 \\ -13 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$, so $T\begin{bmatrix} 5 \\ 6 \\ -13 \end{bmatrix} = 3T\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - 2T\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix}$

Exercise 2.6.2 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation.

a. Find $T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -3 \end{bmatrix}$ if $T\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.

b. Find $T\begin{bmatrix} 5 \\ -1 \\ 2 \\ -4 \end{bmatrix}$ if $T\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$ and $T\begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

---

b. As in 1(b), $T\begin{bmatrix} 5 \\ -1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}$.

Exercise 2.6.3 In each case assume that the transformation T is linear, and use Theorem 2.6.2 to obtain the matrix A of T.

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = -x$.

b. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\mathbf{x}) = -\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$.

c. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is clockwise rotation through $\frac{\pi}{4}$.

d. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation through $\frac{\pi}{4}$.

---

b. $T(\mathbf{e}_1) = -\mathbf{e}_2$ and $T(\mathbf{e}_2) = -\mathbf{e}_1$. So $A\left[\begin{array}{cc} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} -\mathbf{e}_2 & -\mathbf{e}_1 \end{array}\right] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

d. $T(\mathbf{e}_1) = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ So $A = \left[\begin{array}{cc} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{array}\right] = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Exercise 2.6.4 In each case use Theorem 2.6.2 to obtain the matrix A of the transformation T. You may assume that T is linear in each case.

a. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the $x-z$ plane.

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the $y-z$ plane.

---

b. $T(\mathbf{e}_1) = -\mathbf{e}_1$, $T(\mathbf{e}_2) = \mathbf{e}_2$ and $T(\mathbf{e}_3) = \mathbf{e}_3$. Hence Theorem 2.6.2 gives $A\left[\begin{array}{ccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{array}\right] = \left[\begin{array}{ccc} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{array}\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercise 2.6.5 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a. If $\mathbf{x}$ is in $\mathbb{R}^n$, we say that $\mathbf{x}$ is in the *kernel* of T if $T(\mathbf{x}) = \mathbf{0}$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are both in the kernel of T, show that $a\mathbf{x}_1 + b\mathbf{x}_2$ is also in the kernel of T for all scalars a and b.

b. If $\mathbf{y}$ is in $\mathbb{R}^n$, we say that $\mathbf{y}$ is in the *image* of T if $\mathbf{y} = T(\mathbf{x})$ for some $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{y}_1$ and $\mathbf{y}_2$ are both in the image of T, show that $a\mathbf{y}_1 + b\mathbf{y}_2$ is also in the image of T for all scalars a and b.

---

b. We have $\mathbf{y}_1 = T(\mathbf{x}_1)$ for some $\mathbf{x}_1$ in $\mathbb{R}^n$, and $\mathbf{y}_2 = T(\mathbf{x}_2)$ for some $\mathbf{x}_2$ in $\mathbb{R}^n$. So $a\mathbf{y}_1 + b\mathbf{y}_2 = aT(\mathbf{x}_1) + bT(\mathbf{x}_2) = T(a\mathbf{x}_1 + b\mathbf{x}_2)$. Hence $a\mathbf{y}_1 + b\mathbf{y}_2$ is also in the image of T.

Exercise 2.6.6 Use Theorem 2.6.2 to find the matrix of the **identity transformation** $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 2.6.7 In each case show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is not a linear transformation.

a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ 0 \end{bmatrix}$ b) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y^2 \end{bmatrix}$

---

b. $T\left(2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq 2\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Exercise 2.6.8 In each case show that T is either reflection in a line or rotation through an angle, and find the line or angle.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x+y \\ -x+y \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} 8x+6y \\ 6x-8y \end{bmatrix}$

---

b. $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, rotation through $\theta = -\frac{\pi}{4}$.

d. $A = \frac{1}{10}\begin{bmatrix} -8 & -6 \\ -6 & 8 \end{bmatrix}$, reflection in the line $y = -3x$.

Exercise 2.6.9 Express reflection in the line $y = -x$ as the composition of a rotation followed by reflection in the line $y = x$.

Exercise 2.6.10 Find the matrix of $T : \mathbb{R}^3 \to \mathbb{R}^3$ in each case:

a. T is rotation through θ about the x axis (from the y axis to the z axis).

b. T is rotation through θ about the y axis (from the x axis to the z axis).

---

b. $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

Exercise 2.6.11 Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line making an angle θ with the positive x axis.

a. Show that the matrix of T_θ is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ for all θ.

b. Show that $T_\theta \circ R_{2\phi} = T_{\theta-\phi}$ for all θ and ϕ.

Exercise 2.6.12 In each case find a rotation or reflection that equals the given transformation.

a. Reflection in the y axis followed by rotation through $\frac{\pi}{2}$.

b. Rotation through π followed by reflection in the x axis.

c. Rotation through $\frac{\pi}{2}$ followed by reflection in the line $y = x$.

d. Reflection in the x axis followed by rotation through $\frac{\pi}{2}$.

e. Reflection in the line $y = x$ followed by reflection in the x axis.

f. Reflection in the x axis followed by reflection in the line $y = x$.

---

b. Reflection in the y axis

d. Reflection in $y = x$

f. Rotation through $\frac{\pi}{2}$

Exercise 2.6.13 Let R and S be matrix transformations $\mathbb{R}^n \to \mathbb{R}^m$ induced by matrices A and B respectively. In each case, show that T is a matrix transformation and describe its matrix in terms of A and B.

a. $T(\mathbf{x}) = R(\mathbf{x}) + S(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

b. $T(\mathbf{x}) = aR(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$ (where a is a fixed real number).

---

b. $T(\mathbf{x}) = aR(\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}$. Hence T is induced by aA.

Exercise 2.6.14 Show that the following hold for all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$:

a) $T(\mathbf{0}) = \mathbf{0}$

b) $T(-\mathbf{x}) = -T(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$

---

b. If $\mathbf{x}$ is in $\mathbb{R}^n$, then $T(-\mathbf{x}) = T[(-1)\mathbf{x}] = (-1)T(\mathbf{x}) = -T(\mathbf{x})$.

Exercise 2.6.15 The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ is called the **zero transformation**.

a. Show that the zero transformation is linear and find its matrix.

b. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $T(\mathbf{e}_i) = \mathbf{0}$ for each i, show that T is the zero transformation. [*Hint*: Theorem 2.6.1.]

Exercise 2.6.16 Write the elements of $\mathbb{R}^n$ and $\mathbb{R}^m$ as rows. If A is an $m \times n$ matrix, define $T : \mathbb{R}^m \to \mathbb{R}^n$ by $T(\mathbf{y}) = \mathbf{y}A$ for all rows $\mathbf{y}$ in $\mathbb{R}^m$. Show that:

a. T is a linear transformation.

b. the rows of A are $T(\mathbf{f}_1), T(\mathbf{f}_2), \dots, T(\mathbf{f}_m)$ where $\mathbf{f}_i$ denotes row i of I_m. [*Hint*: Show that $\mathbf{f}_iA$ is row i of A.]

Exercise 2.6.17 Let $S : \mathbb{R}^n \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations with matrices A and B respectively.

a. Show that $B^2 = B$ if and only if $T^2 = T$ (where T^2 means $T \circ T$).

b. Show that $B^2 = I$ if and only if $T^2 = 1_{\mathbb{R}^n}$.

c. Show that $AB = BA$ if and only if $S \circ T = T \circ S$.

[*Hint*: Theorem 2.6.3.]

---

b. If $B^2 = I$ then $T^2(\mathbf{x}) = T[T(\mathbf{x})] = B(B\mathbf{x}) = B^2\mathbf{x} = I\mathbf{x} = \mathbf{x} = 1_{\mathbb{R}^2}(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Hence $T^2 = 1_{\mathbb{R}^2}$. If $T^2 = 1_{\mathbb{R}^2}$, then $B^2\mathbf{x} = T^2(\mathbf{x}) = 1_{\mathbb{R}^2}(\mathbf{x}) = \mathbf{x} = I\mathbf{x}$ for all $\mathbf{x}$, so $B^2 = I$ by Theorem 2.2.6.

Exercise 2.6.18 Let $Q_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the x axis, let $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$, let $Q_{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = -x$, and let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation through $\frac{\pi}{2}$.

a. Show that $Q_1 \circ R_{\frac{\pi}{2}} = Q_0$.

b. Show that $Q_1 \circ Q_0 = R_{\frac{\pi}{2}}$.

c. Show that $R_{\frac{\pi}{2}} \circ Q_0 = Q_1$.

d. Show that $Q_0 \circ R_{\frac{\pi}{2}} = Q_{-1}$.

---

b. The matrix of $Q_1 \circ Q_0$ is $\left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$, which is the matrix of $R_{\frac{\pi}{2}}$.

d. The matrix of $Q_0 \circ R_{\frac{\pi}{2}}$ is $\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ -1 & 0 \end{array}\right]$, which is the matrix of Q_{-1}.

Exercise 2.6.19 For any slope m, show that:

a) $Q_m \circ P_m = P_m$

b) $P_m \circ Q_m = P_m$

Exercise 2.6.20 Define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$. Show that T is a linear transformation and find its matrix.

We have $T(\mathbf{x}) = x_1 + x_2 + \cdots + x_n = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$, so T is the matrix transformation induced by the matrix $A = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right]$. In particular, T is linear. On the other hand, we can use Theorem 2.6.2 to get A, but to do this we must first show directly that T is linear. If we write $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ and $\mathbf{y} = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right]$.

Then

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= T\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} \\ &= (x_1+y_1)+(x_2+y_2)+\cdots+(x_n+y_n) \\ &= (x_1+x_2+\cdots+x_n)+(y_1+y_2+\cdots+y_n) \\ &= T(\mathbf{x})+T(\mathbf{y}) \end{aligned}$$

Similarly, $T(a\mathbf{x}) = aT(\mathbf{x})$ for any scalar a, so T is linear. By Theorem 2.6.2, T has matrix $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$, as before.

Exercise 2.6.21 Given c in $\mathbb{R}$, define $T_c : \mathbb{R}^n \to \mathbb{R}$ by $T_c(\mathbf{x}) = c\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that T_c is a linear transformation and find its matrix.

Exercise 2.6.22 Given vectors $\mathbf{w}$ and $\mathbf{x}$ in $\mathbb{R}^n$, denote their dot product by $\mathbf{w}\cdot\mathbf{x}$.

a. Given $\mathbf{w}$ in $\mathbb{R}^n$, define $T_{\mathbf{w}} : \mathbb{R}^n \to \mathbb{R}$ by $T_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}\cdot\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that $T_{\mathbf{w}}$ is a linear transformation.

b. Show that *every* linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ is given as in (a); that is $T = T_{\mathbf{w}}$ for some $\mathbf{w}$ in $\mathbb{R}^n$.

---

b. If $T : \mathbb{R}^n \to \mathbb{R}$ is linear, write $T(\mathbf{e}_j) = w_j$ for each $j = 1, 2, \dots, n$ where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. Since $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$, Theorem 2.6.1 gives

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n) \\ &= x_1w_1 + x_2w_2 + \cdots + x_nw_n \\ &= \mathbf{w}\cdot\mathbf{x} = T_{\mathbf{w}}(\mathbf{x}) \end{aligned}$$

where $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$. Since this holds for all $\mathbf{x}$ in $\mathbb{R}^n$, it shows that $T = T_{\mathbf{W}}$. This also follows from Theorem 2.6.2, but we have first to verify that T is linear. (This comes to showing that $\mathbf{w}\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{w}\cdot\mathbf{s} + \mathbf{w}\cdot\mathbf{y}$ and $\mathbf{w}\cdot(a\mathbf{x}) = a(\mathbf{w}\cdot\mathbf{x})$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all a in $\mathbb{R}$.) Then T has matrix $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$ by Theorem 2.6.2. Hence if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}$, then $T(\mathbf{x}) = A\mathbf{x} = \mathbf{w}\cdot\mathbf{x}$, as required.

Exercise 2.6.23 If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, show that there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$. [*Hint*: By Definition 2.5, find a matrix A such that $A\mathbf{x} = \mathbf{y}$.]

---

b. Given $\mathbf{x}$ in $\mathbb{R}$ and a in $\mathbb{R}$, we have

$(S\circ T)(a\mathbf{x})$	$=$	$S[T(a\mathbf{x})]$	Definition of $S\circ T$
	$=$	$S[aT(\mathbf{x})]$	Because T is linear.
	$=$	$a[S[T(\mathbf{x})]]$	Because S is linear.
	$=$	$a[S\circ T(\mathbf{x})]$	Definition of $S\circ T$

Exercise 2.6.24 Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^k$ be two linear transformations. Show directly that $S\circ T$ is linear. That is:

a. Show that $(S\circ T)(\mathbf{x}+\mathbf{y}) = (S\circ T)\mathbf{x} + (S\circ T)\mathbf{y}$ for all $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$.

b. Show that $(S\circ T)(a\mathbf{x}) = a[(S\circ T)\mathbf{x}]$ for all $\mathbf{x}$ in $\mathbb{R}^n$ and all a in $\mathbb{R}$.

Exercise 2.6.25 Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^k \xrightarrow{R} \mathbb{R}^k$ be linear. Show that $R\circ(S\circ T) = (R\circ S)\circ T$ by showing directly that $[R\circ(S\circ T)](\mathbf{x}) = [(R\circ S)\circ T)](\mathbf{x})$ holds for each vector $\mathbf{x}$ in $\mathbb{R}^n$.

2.7 LU-Factorization[15]

The solution to a system $A\mathbf{x} = \mathbf{b}$ of linear equations can be solved quickly if A can be factored as $A = LU$ where L and U are of a particularly nice form. In this section we show that gaussian elimination can be used to find such factorizations.

Triangular Matrices

As for square matrices, if $A = \left[a_{ij}\right]$ is an $m \times n$ matrix, the elements a_{11}, a_{22}, a_{33}, ... form the **main diagonal** of A. Then A is called **upper triangular** if every entry below and to the left of the main diagonal is zero. Every row-echelon matrix is upper triangular, as are the matrices

$$\left[\begin{array}{rrrr} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -3 & 0 \end{array} \right] \quad \left[\begin{array}{rrrrr} 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array} \right] \quad \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

By analogy, a matrix A is called **lower triangular** if its transpose is upper triangular, that is if each entry above and to the right of the main diagonal is zero. A matrix is called **triangular** if it is upper or lower triangular.

Example 2.7.1

Solve the system

$$\begin{array}{rcl} x_1 + 2x_2 - 3x_3 - x_4 + 5x_5 &=& 3 \\ 5x_3 + x_4 + x_5 &=& 8 \\ 2x_5 &=& 6 \end{array}$$

where the coefficient matrix is upper triangular.

Solution. As in gaussian elimination, let the "non-leading" variables be parameters: $x_2 = s$ and $x_4 = t$. Then solve for x_5, x_3, and x_1 in that order as follows. The last equation gives

$$x_5 = \frac{6}{2} = 3$$

Substitution into the second last equation gives

$$x_3 = 1 - \frac{1}{5}t$$

Finally, substitution of both x_5 and x_3 into the first equation gives

$$x_1 = -9 - 2s + \frac{2}{5}t$$

The method used in Example 2.7.1 is called **back substitution** because later variables are substituted into earlier equations. It works because the coefficient matrix is upper triangular.

[15]This section is not used later and so may be omitted with no loss of continuity.

Similarly, if the coefficient matrix is lower triangular the system can be solved by **forward substitution** where earlier variables are substituted into later equations. As observed in Section 1.2, these procedures are more numerically efficient than gaussian elimination.

Now consider a system $A\mathbf{x} = \mathbf{b}$ where A can be factored as $A = LU$ where L is lower triangular and U is upper triangular. Then the system $A\mathbf{x} = \mathbf{b}$ can be solved in two stages as follows:

1. *First solve* $L\mathbf{y} = \mathbf{b}$ *for* $\mathbf{y}$ *by forward substitution.*
2. *Then solve* $U\mathbf{x} = \mathbf{y}$ *for* $\mathbf{x}$ *by back substitution.*

Then $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$ because $A\mathbf{x} = LU\mathbf{x} = L\mathbf{y} = \mathbf{b}$. Moreover, every solution $\mathbf{x}$ arises this way (take $\mathbf{y} = U\mathbf{x}$). Furthermore the method adapts easily for use in a computer.

This focuses attention on efficiently obtaining such factorizations $A = LU$. The following result will be needed; the proof is straightforward and is left as Exercises 2.7.7 and 2.7.8.

Lemma 2.7.1

Let A and B denote matrices.

1. *If A and B are both lower (upper) triangular, the same is true of AB.*
2. *If A is $n \times n$ and lower (upper) triangular, then A is invertible if and only if every main diagonal entry is nonzero. In this case A^{-1} is also lower (upper) triangular.*

LU-Factorization

Let A be an $m \times n$ matrix. Then A can be carried to a row-echelon matrix U (that is, upper triangular). As in Section 2.5, the reduction is

$$A \to E_1A \to E_2E_1A \to E_3E_2E_1A \to \cdots \to E_kE_{k-1}\cdots E_2E_1A = U$$

where $E_1, E_2, \dots, E_k$ are elementary matrices corresponding to the row operations used. Hence

$$A = LU$$

where $L = (E_kE_{k-1}\cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$. If we do not insist that U is reduced then, except for row interchanges, none of these row operations involve adding a row to a row *above* it. Thus, if no row interchanges are used, all the E_i are *lower* triangular, and so L is lower triangular (and invertible) by Lemma 2.7.1. This proves the following theorem. For convenience, let us say that A can be **lower reduced** if it can be carried to row-echelon form using no row interchanges.

Theorem 2.7.1

If A can be lower reduced to a row-echelon matrix U, then

$$A = LU$$

where L is lower triangular and invertible and U is upper triangular and row-echelon.

Definition 2.14 LU-factorization

A factorization $A = LU$ as in Theorem 2.7.1 is called an ***LU-factorization*** *of A.*

Such a factorization may not exist (Exercise 2.7.4) because A cannot be carried to row-echelon form using no row interchange. A procedure for dealing with this situation will be outlined later. However, if an LU-factorization $A = LU$ does exist, then the gaussian algorithm gives U and also leads to a procedure for finding L. Example 2.7.2 provides an illustration. For convenience, the first nonzero column from the left in a matrix A is called the **leading column** of A.

Example 2.7.2

Find an LU-factorization of $A = \left[\begin{array}{rrrrr} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{array} \right]$.

Solution. We lower reduce A to row-echelon form as follows:

$$A = \left[\begin{array}{rrrrr} 0 & \boxed{2} & -6 & -2 & 4 \\ 0 & \boxed{-1} & 3 & 3 & 2 \\ 0 & \boxed{-1} & 3 & 7 & 10 \end{array} \right] \rightarrow \left[\begin{array}{rrrrr} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & \boxed{6} & 12 \end{array} \right] \rightarrow \left[\begin{array}{rrrrr} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] = U$$

The circled columns are determined as follows: The first is the leading column of A, and is used (by lower reduction) to create the first leading 1 and create zeros below it. This completes the work on row 1, and we repeat the procedure on the matrix consisting of the remaining rows. Thus the second circled column is the leading column of this smaller matrix, which we use to create the second leading 1 and the zeros below it. As the remaining row is zero here, we are finished. Then $A = LU$ where

$$L = \left[\begin{array}{rrr} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 6 & 1 \end{array} \right]$$

This matrix L is obtained from I_3 by replacing the bottom of the first two columns by the circled columns in the reduction. Note that the rank of A is 2 here, and this is the number of circled columns.

The calculation in Example 2.7.2 works in general. There is no need to calculate the elementary matrices E_i, and the method is suitable for use in a computer because the circled columns can be stored in memory as they are created. The procedure can be formally stated as follows:

LU-Algorithm

Let A be an $m \times n$ matrix of rank r, *and suppose that A can be lower reduced to a row-echelon matrix U. Then $A = LU$ where the lower triangular, invertible matrix L is constructed as follows:*

1. *If $A = 0$, take $L = I_m$ and $U = 0$.*
2. *If $A \neq 0$, write $A_1 = A$ and let $\mathbf{c}_1$ be the leading column of A_1. Use $\mathbf{c}_1$ to create the first leading 1 and create zeros below it (using lower reduction). When this is completed, let A_2 denote the matrix consisting of rows 2 to m of the matrix just created.*
3. *If $A_2 \neq 0$, let $\mathbf{c}_2$ be the leading column of A_2 and repeat Step 2 on A_2 to create A_3.*
4. *Continue in this way until U is reached, where all rows below the last leading 1 consist of zeros. This will happen after r steps.*
5. *Create L by placing $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ at the bottom of the first r columns of I_m.*

A proof of the LU-algorithm is given at the end of this section.

LU-factorization is particularly important if, as often happens in business and industry, a series of equations $A\mathbf{x} = B_1, A\mathbf{x} = B_2, \dots, A\mathbf{x} = B_k$, must be solved, each with the same coefficient matrix A. It is very efficient to solve the first system by gaussian elimination, simultaneously creating an LU-factorization of A, and then using the factorization to solve the remaining systems by forward and back substitution.

Example 2.7.3

Find an LU-factorization for $A = \left[\begin{array}{rrrrr} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{array} \right]$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 8 & 2 & 4 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 8 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

If U denotes this row-echelon matrix, then $A = LU$, where

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$$

The next example deals with a case where no row of zeros is present in U (in fact, A is invertible).

Example 2.7.4

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Hence $A = LU$ where $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$.

There are matrices (for example $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) that have no LU-factorization and so require at least one row interchange when being carried to row-echelon form via the gaussian algorithm. However, it turns out that, if all the row interchanges encountered in the algorithm are carried out first, the resulting matrix requires no interchanges and so has an LU-factorization. Here is the precise result.

Theorem 2.7.2

Suppose an $m \times n$ matrix A is carried to a row-echelon matrix U via the gaussian algorithm. Let $P_1, P_2, \dots, P_s$ be the elementary matrices corresponding (in order) to the row interchanges used, and write $P = P_s \cdots P_2 P_1$. (If no interchanges are used take $P = I_m$.) Then:

1. *PA is the matrix obtained from A by doing these interchanges (in order) to A.*
2. *PA has an LU-factorization.*

The proof is given at the end of this section.

A matrix P that is the product of elementary matrices corresponding to row interchanges is called a **permutation matrix**. Such a matrix is obtained from the identity matrix by arranging the rows in a different order, so it has exactly one 1 in each row and each column, and has zeros elsewhere. We regard the identity matrix as a permutation matrix. The elementary permutation matrices are those obtained from I by a single row interchange, and every permutation matrix is a product of elementary ones.

Example 2.7.5

If $A = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix}$, find a permutation matrix P such that PA has an LU-factorization, and then find the factorization.

Solution. Apply the gaussian algorithm to A:

$$A \xrightarrow{*} \begin{bmatrix} -1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 10 \\ 0 & 1 & -1 & 4 \end{bmatrix} \xrightarrow{*} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Two row interchanges were needed (marked with $*$), first rows 1 and 2 and then rows 2 and

3. Hence, as in Theorem 2.7.2,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we do these interchanges (in order) to A, the result is PA. Now apply the LU-algorithm to PA:

$$PA = \begin{bmatrix} -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

Hence, $PA = LU$, where $L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 10 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Theorem 2.7.2 provides an important general factorization theorem for matrices. If A is any $m \times n$ matrix, it asserts that there exists a permutation matrix P and an LU-factorization $PA = LU$. Moreover, it shows that either $P = I$ or $P = P_s \cdots P_2 P_1$, where $P_1, P_2, \dots, P_s$ are the elementary permutation matrices arising in the reduction of A to row-echelon form. Now observe that $P_i^{-1} = P_i$ for each i (they are elementary row interchanges). Thus, $P^{-1} = P_1 P_2 \cdots P_s$, so the matrix A can be factored as

$$A = P^{-1}LU$$

where P^{-1} is a permutation matrix, L is lower triangular and invertible, and U is a row-echelon matrix. This is called a **PLU-factorization** of A.

The LU-factorization in Theorem 2.7.1 is not unique. For example,

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

However, it is necessary here that the row-echelon matrix has a row of zeros. Recall that the rank of a matrix A is the number of nonzero rows in any row-echelon matrix U to which A can be carried by row operations. Thus, if A is $m \times n$, the matrix U has no row of zeros if and only if A has rank m.

Theorem 2.7.3

Let A be an $m \times n$ matrix that has an LU-factorization

$$A = LU$$

If A has rank m *(that is, U has no row of zeros), then L and U are uniquely determined by A.*

Proof. Suppose $A = MV$ is another LU-factorization of A, so M is lower triangular and invertible and V is row-echelon. Hence $LU = MV$, and we must show that $L = M$ and $U = V$. We write $N = M^{-1}L$. Then N is lower triangular and invertible (Lemma 2.7.1) and $NU = V$, so it suffices to prove that $N = I$. If N is $m \times m$, we use induction on m. The case $m = 1$ is left to the reader. If $m > 1$, observe first that column 1 of V is N times column 1 of U. Thus if either column is zero, so is the other (N is invertible). Hence, we can assume (by deleting zero columns) that the $(1, 1)$-entry is 1 in both U and V.

Now we write $N = \left[\begin{array}{cc} a & 0 \\ X & N_1 \end{array} \right]$, $U = \left[\begin{array}{cc} 1 & Y \\ 0 & U_1 \end{array} \right]$, and $V = \left[\begin{array}{cc} 1 & Z \\ 0 & V_1 \end{array} \right]$ in block form. Then $NU = V$ becomes $\left[\begin{array}{cc} a & aY \\ X & XY + N_1U_1 \end{array} \right] = \left[\begin{array}{cc} 1 & Z \\ 0 & V_1 \end{array} \right]$. Hence $a = 1$, $Y = Z$, $X = 0$, and $N_1U_1 = V_1$. But $N_1U_1 = V_1$ implies $N_1 = I$ by induction, whence $N = I$. □

If A is an $m \times m$ invertible matrix, then A has rank m by Theorem 2.4.5. Hence, we get the following important special case of Theorem 2.7.3.

Corollary 2.7.1

If an invertible matrix A has an LU-factorization $A = LU$, then L and U are uniquely determined by A.

Of course, in this case U is an upper triangular matrix with 1s along the main diagonal.

Proofs of Theorems

Proof of the LU-Algorithm. If $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ are columns of lengths $m, m-1, \dots, m-r+1$, respectively, write $L^{(m)}(\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r)$ for the lower triangular $m \times m$ matrix obtained from I_m by placing $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ at the bottom of the first r columns of I_m.

Proceed by induction on n. If $A = 0$ or $n = 1$, it is left to the reader. If $n > 1$, let $\mathbf{c}_1$ denote the leading column of A and let $\mathbf{k}_1$ denote the first column of the $m \times m$ identity matrix. There exist elementary matrices $E_1, \dots, E_k$ such that, in block form,

$$(E_k \cdots E_2E_1)A = \left[\begin{array}{c|c|c} 0 & \mathbf{k}_1 & \begin{array}{c} X_1 \\ \hline A_1 \end{array} \end{array} \right] \quad \text{where } (E_k \cdots E_2E_1)\mathbf{c}_1 = \mathbf{k}_1$$

Moreover, each E_j can be taken to be lower triangular (by assumption). Write

$$G = (E_k \cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$$

Then G is lower triangular, and $G\mathbf{k}_1 = \mathbf{c}_1$. Also, each E_j (and so each E_j^{-1}) is the result of either multiplying row 1 of I_m by a constant or adding a multiple of row 1 to another row. Hence,

$$G = (E_1^{-1}E_2^{-1}\cdots E_k^{-1})I_m = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array}\right]$$

in block form. Now, by induction, let $A_1 = L_1U_1$ be an LU-factorization of A_1, where $L_1 = L^{(m-1)}[\mathbf{c}_2, \dots, \mathbf{c}_r]$ and U_1 is row-echelon. Then block multiplication gives

$$G^{-1}A = \left[\begin{array}{c|c|c} 0 & \mathbf{k}_1 & \begin{array}{c} X_1 \\ \hline L_1U_1 \end{array} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array}\right]\left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array}\right]$$

Hence $A = LU$, where $U = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array}\right]$ is row-echelon and

$$L = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline L \end{array} \end{array}\right] = L^{(m)}[\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r]$$

This completes the proof. □

Proof of Theorem 2.7.2. Let A be a nonzero $m \times n$ matrix and let $\mathbf{k}_j$ denote column j of I_m. There is a permutation matrix P_1 (where either P_1 is elementary or $P_1 = I_m$) such that the first nonzero column $\mathbf{c}_1$ of P_1A has a nonzero entry on top. Hence, as in the LU-algorithm,

$$L^{(m)}[\mathbf{c}_1]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1 \end{array}\right]$$

in block form. Then let P_2 be a permutation matrix (either elementary or I_m) such that

$$P_2 \cdot L^{(m)}[\mathbf{c}_1]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1' \end{array}\right]$$

and the first nonzero column $\mathbf{c}_2$ of A_1' has a nonzero entry on top. Thus,

$$L^{(m)}[\mathbf{k}_1, \mathbf{c}_2]^{-1} \cdot P_2 \cdot L^{(m)}[\mathbf{c}_1]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & \begin{array}{c|c|c} 0 & 1 & X_2 \\ \hline 0 & 0 & A_2 \end{array} \end{array}\right]$$

in block form. Continue to obtain elementary permutation matrices $P_1, P_2, \dots, P_r$ and columns $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ of lengths $m, m-1, \dots$, such that

$$(L_rP_rL_{r-1}P_{r-1}\cdots L_2P_2L_1P_1)A = U$$

where U is a row-echelon matrix and $L_j = L^{(m)}\left[\mathbf{k}_1, \dots, \mathbf{k}_{j-1}, \mathbf{c}_j\right]^{-1}$ for each j, where the notation means the first $j-1$ columns are those of I_m. It is not hard to verify that each L_j has the form $L_j = L^{(m)}\left[\mathbf{k}_1, \dots, \mathbf{k}_{j-1}, \mathbf{c}_j'\right]$ where $\mathbf{c}_j'$ is a column of length $m-j+1$. We now claim that each permutation matrix P_k can be "moved past" each matrix L_j to the right of it, in the sense that

$$P_kL_j = L_j'P_k$$

where $L'_j = L^{(m)}\left[\mathbf{k}_1,\ \ldots,\ \mathbf{k}_{j-1},\ \mathbf{c}''_j\right]$ for some column $\mathbf{c}''_j$ of length $m-j+1$. Given that this is true, we obtain a factorization of the form

$$(L_r L'_{r-1} \cdots L'_2 L'_1)(P_r P_{r-1} \cdots P_2 P_1)A = U$$

If we write $P = P_r P_{r-1} \cdots P_2 P_1$, this shows that PA has an LU-factorization because $L_r L'_{r-1} \cdots L'_2 L'_1$ is lower triangular and invertible. All that remains is to prove the following rather technical result. □

Lemma 2.7.2

Let P_k result from interchanging row k of I_m with a row below it. If $j < k$, let c_j be a column of length $m-j+1$. Then there is another column $\mathbf{c}'_j$ of length $m-j+1$ such that

$$P_k \cdot L^{(m)}\left[\mathbf{k}_1,\ \ldots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j\right] = L^{(m)}\left[\mathbf{k}_1,\ \ldots,\ \mathbf{k}_{j-1},\ \mathbf{c}'_j\right] \cdot P_k$$

The proof is left as Exercise 2.7.11.

Exercises for 2.7

Exercise 2.7.1 Find an LU-factorization of the following matrices.

a. $\left[\begin{array}{rrrrr} 2 & 6 & -2 & 0 & 2 \\ 3 & 9 & -3 & 3 & 1 \\ -1 & -3 & 1 & -3 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr} 2 & 4 & 2 \\ 1 & -1 & 3 \\ -1 & 7 & -7 \end{array}\right]$

c. $\left[\begin{array}{rrrrr} 2 & 6 & -2 & 0 & 2 \\ 1 & 5 & -1 & 2 & 5 \\ 3 & 7 & -3 & -2 & 5 \\ -1 & -1 & 1 & 2 & 3 \end{array}\right]$

d. $\left[\begin{array}{rrrrr} -1 & -3 & 1 & 0 & -1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 & 1 \\ 0 & -2 & -4 & -2 & 0 \end{array}\right]$

e. $\left[\begin{array}{rrrrrr} 2 & 2 & 4 & 6 & 0 & 2 \\ 1 & -1 & 2 & 1 & 3 & 1 \\ -2 & 2 & -4 & -1 & 1 & 6 \\ 0 & 2 & 0 & 3 & 4 & 8 \\ -2 & 4 & -4 & 1 & -2 & 6 \end{array}\right]$

f. $\left[\begin{array}{rrrrr} 2 & 2 & -2 & 4 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & -2 & 6 & 3 \\ 1 & 3 & -2 & 2 & 1 \end{array}\right]$

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b. $\left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & -3 & 0 \\ -1 & 9 & 1 \end{array}\right]\left[\begin{array}{rrr} 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{array}\right]$

d. $\left[\begin{array}{rrrr} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{array}\right]\left[\begin{array}{rrrrr} 1 & 3 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

f. $\left[\begin{array}{rrrr} 2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]\left[\begin{array}{rrrrr} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Exercise 2.7.2 Find a permutation matrix P and an LU-factorization of PA if A is:

a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 4 \\ 3 & 5 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & -1 & 2 & 1 & 3 \\ -1 & 1 & 3 & 1 & 4 \\ 1 & -1 & -3 & 6 & 2 \\ 2 & -2 & -4 & 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \end{bmatrix}$

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b. $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$PA = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

d. $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$PA = \begin{bmatrix} -1 & -2 & 3 & 0 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \\ 2 & 4 & -6 & 5 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & -2 & 0 \\ 2 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 2.7.3 In each case use the given LU-decomposition of A to solve the system $A\mathbf{x} = \mathbf{b}$ by finding $\mathbf{y}$ such that $L\mathbf{y} = \mathbf{b}$, and then $\mathbf{x}$ such that $U\mathbf{x} = \mathbf{y}$:

a. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

c. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 4 \\ -6 \\ 4 \\ 5 \end{bmatrix}$

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b. $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} -1+2t \\ -t \\ s \\ t \end{bmatrix}$ s and t arbitrary

d. $\mathbf{y} = \begin{bmatrix} 2 \\ 8 \\ -1 \\ 0 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} 8-2t \\ 6-t \\ -1-t \\ t \end{bmatrix}$ t arbitrary

Exercise 2.7.4 Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = LU$ is impossible where L is lower triangular and U is upper triangular.

Exercise 2.7.5 Show that we can accomplish any row interchange by using only row operations of other types.

$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1 + R_2 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1 + R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ R_1 \end{bmatrix}$

Exercise 2.7.6

a. Let L and L_1 be invertible lower triangular matrices, and let U and U_1 be invertible upper triangular matrices. Show that $LU = L_1U_1$ if and only if there exists an invertible diagonal matrix D such that $L_1 = LD$ and $U_1 = D^{-1}U$. [*Hint*: Scrutinize $L^{-1}L_1 = UU_1^{-1}$.]

b. Use part (a) to prove Theorem 2.7.3 in the case that A is invertible.

---

b. Let $A = LU = L_1U_1$ be LU-factorizations of the invertible matrix A. Then U and U_1 have no row of zeros and so (being row-echelon) are upper triangular with 1's on the main diagonal. Thus, using **(a.)**, the diagonal matrix $D = UU_1^{-1}$ has 1's on the main diagonal. Thus $D = I$, $U = U_1$, and $L = L_1$.

Exercise 2.7.7 Prove Lemma 2.7.1(1). [*Hint*: Use block multiplication and induction.]

If $A = \begin{bmatrix} a & 0 \\ X & A_1 \end{bmatrix}$ and $B = \begin{bmatrix} b & 0 \\ Y & B_1 \end{bmatrix}$ in block form, then $AB = \begin{bmatrix} ab & 0 \\ Xb + A_1Y & A_1B_1 \end{bmatrix}$, and A_1B_1 is lower triangular by induction.

Exercise 2.7.8 Prove Lemma 2.7.1(2). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.9 A triangular matrix is called **unit triangular** if it is square and every main diagonal element is a 1.

a. If A can be carried by the gaussian algorithm to row-echelon form using no row interchanges, show that $A = LU$ where L is unit lower triangular and U is upper triangular.

b. Show that the factorization in (a.) is unique.

---

b. Let $A = LU = L_1U_1$ be two such factorizations. Then $UU_1^{-1} = L^{-1}L_1$; write this matrix as $D = UU_1^{-1} = L^{-1}L_1$. Then D is lower triangular (apply Lemma 2.7.1 to $D = L^{-1}L_1$); and D is also upper triangular (consider UU_1^{-1}). Hence D is diagonal, and so $D = I$ because L^{-1} and L_1 are unit triangular. Since $A = LU$; this completes the proof.

Exercise 2.7.10 Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ be columns of lengths $m, m-1, \dots, m-r+1$. If $\mathbf{k}_j$ denotes column j of I_m, show that $L^{(m)}[\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r] = L^{(m)}[\mathbf{c}_1]L^{(m)}[\mathbf{k}_1, \mathbf{c}_2]L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_3]\cdots$
$L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{r-1}, \mathbf{c}_r]$. The notation is as in the proof of Theorem 2.7.2. [*Hint*: Use induction on m and block multiplication.]

Exercise 2.7.11 Prove Lemma 2.7.2. [*Hint*: $P_k^{-1} = P_k$. Write $P_k = \begin{bmatrix} I_k & 0 \\ 0 & P_0 \end{bmatrix}$ in block form where P_0 is an $(m-k) \times (m-k)$ permutation matrix.]

3. Determinants and Diagonalization

Contents

With each square matrix we can calculate a number, called the determinant of the matrix, which tells us whether or not the matrix is invertible. In fact, determinants can be used to give a formula for the inverse of a matrix. They also arise in calculating certain numbers (called eigenvalues) associated with the matrix. These eigenvalues are essential to a technique called diagonalization that is used in many applications where it is desired to predict the future behaviour of a system. For example, we use it to predict whether a species will become extinct.

Determinants were first studied by Leibnitz in 1696, and the term "determinant" was first used in 1801 by Gauss is his *Disquisitiones Arithmeticae*. Determinants are much older than matrices (which were introduced by Cayley in 1878) and were used extensively in the eighteenth and nineteenth centuries, primarily because of their significance in geometry (see Section 4.4). Although they are somewhat less important today, determinants still play a role in the theory and application of matrix algebra.

3.1 The Cofactor Expansion

In Section 2.4 we defined the determinant of a 2×2 matrix $A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$ as follows:[1]

$$\det A = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = ad - bc$$

and showed (in Example 2.4.4) that A has an inverse if and only if $\det A \neq 0$. One objective of this chapter is to do this for *any* square matrix A. There is no difficulty for 1×1 matrices: If $A = [a]$, we define $\det A = \det [a] = a$ and note that A is invertible if and only if $a \neq 0$.

If A is 3×3 and invertible, we look for a suitable definition of $\det A$ by trying to carry A to the identity matrix by row operations. The first column is not zero (A is invertible); suppose the (1, 1)-entry a is not zero. Then row operations give

$$A = \left[\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right] \rightarrow \left[\begin{array}{ccc} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{array} \right] \rightarrow \left[\begin{array}{ccc} a & b & c \\ 0 & ae - bd & af - cd \\ 0 & ah - bg & ai - cg \end{array} \right] = \left[\begin{array}{ccc} a & b & c \\ 0 & u & af - cd \\ 0 & v & ai - cg \end{array} \right]$$

where $u = ae - bd$ and $v = ah - bg$. Since A is invertible, one of u and v is nonzero (by Example 2.4.11); suppose that $u \neq 0$. Then the reduction proceeds

$$A \rightarrow \left[\begin{array}{ccc} a & b & c \\ 0 & u & af - cd \\ 0 & v & ai - cg \end{array} \right] \rightarrow \left[\begin{array}{ccc} a & b & c \\ 0 & u & af - cd \\ 0 & uv & u(ai - cg) \end{array} \right] \rightarrow \left[\begin{array}{ccc} a & b & c \\ 0 & u & af - cd \\ 0 & 0 & w \end{array} \right]$$

where $w = u(ai - cg) - v(af - cd) = a(aei + bfg + cdh - ceg - afh - bdi)$. We define

$$\det A = aei + bfg + cdh - ceg - afh - bdi \tag{3.1}$$

and observe that $\det A \neq 0$ because $a \det A = w \neq 0$ (is invertible).

To motivate the definition below, collect the terms in Equation 3.1 involving the entries a, b, and c in row 1 of A:

$$\begin{aligned} \det A = \left| \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right| &= aei + bfg + cdh - ceg - afh - bdi \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= a \left| \begin{array}{cc} e & f \\ h & i \end{array} \right| - b \left| \begin{array}{cc} d & f \\ g & i \end{array} \right| + c \left| \begin{array}{cc} d & e \\ g & h \end{array} \right| \end{aligned}$$

This last expression can be described as follows: To compute the determinant of a 3×3 matrix A, multiply each entry in row 1 by a sign times the determinant of the 2×2 matrix obtained by deleting the row and column of that entry, and add the results. The signs alternate down row 1, starting with $+$. It is this observation that we generalize below.

[1] Determinants are commonly written $|A| = \det A$ using vertical bars. We will use both notations.

Example 3.1.1

$$\det\begin{bmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{bmatrix} = 2\begin{vmatrix} 0 & 6 \\ 5 & 0 \end{vmatrix} - 3\begin{vmatrix} -4 & 6 \\ 1 & 0 \end{vmatrix} + 7\begin{vmatrix} -4 & 0 \\ 1 & 5 \end{vmatrix}$$
$$= 2(-30) - 3(-6) + 7(-20)$$
$$= -182$$

This suggests an inductive method of defining the determinant of any square matrix in terms of determinants of matrices one size smaller. The idea is to define determinants of 3×3 matrices in terms of determinants of 2×2 matrices, then we do 4×4 matrices in terms of 3×3 matrices, and so on.

To describe this, we need some terminology.

Definition 3.1 Cofactors of a Matrix

Assume that determinants of $(n-1)\times(n-1)$ matrices have been defined. Given the $n\times n$ matrix A, let

A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j.

*Then the $(i,\ j)$-**cofactor** $c_{ij}(A)$ is the scalar defined by*

$$c_{ij}(A) = (-1)^{i+j}\det(A_{ij})$$

*Here $(-1)^{i+j}$ is called the **sign** of the $(i,\ j)$-position.*

The sign of a position is clearly 1 or -1, and the following diagram is useful for remembering it:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Note that the signs alternate along each row and column with $+$ in the upper left corner.

Example 3.1.2

Find the cofactors of positions $(1,\ 2)$, $(3,\ 1)$, and $(2,\ 3)$ in the following matrix.

$$A = \begin{bmatrix} 3 & -1 & 6 \\ 5 & 2 & 7 \\ 8 & 9 & 4 \end{bmatrix}$$

Solution. Here A_{12} is the matrix $\left[\begin{array}{cc} 5 & 7 \\ 8 & 4 \end{array}\right]$ that remains when row 1 and column 2 are deleted. The sign of position $(1, 2)$ is $(-1)^{1+2} = -1$ (this is also the $(1, 2)$-entry in the sign diagram), so the $(1, 2)$-cofactor is

$$c_{12}(A) = (-1)^{1+2}\left|\begin{array}{cc} 5 & 7 \\ 8 & 4 \end{array}\right| = (-1)(5\cdot 4 - 7\cdot 8) = (-1)(-36) = 36$$

Turning to position $(3, 1)$, we find

$$c_{31}(A) = (-1)^{3+1}A_{31} = (-1)^{3+1}\left|\begin{array}{cc} -1 & 6 \\ 2 & 7 \end{array}\right| = (+1)(-7-12) = -19$$

Finally, the $(2, 3)$-cofactor is

$$c_{23}(A) = (-1)^{2+3}A_{23} = (-1)^{2+3}\left|\begin{array}{cc} 3 & -1 \\ 8 & 9 \end{array}\right| = (-1)(27+8) = -35$$

Clearly other cofactors can be found—there are nine in all, one for each position in the matrix.

We can now define $\det A$ for any square matrix A

Definition 3.2 Cofactor expansion of a Matrix

Assume that determinants of $(n-1)\times(n-1)$ matrices have been defined. If $A = \left[a_{ij}\right]$ is $n\times n$ define

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

*This is called the **cofactor expansion** of $\det A$ along row 1.*

It asserts that $\det A$ can be computed by multiplying the entries of row 1 by the corresponding cofactors, and adding the results. The astonishing thing is that $\det A$ can be computed by taking the cofactor expansion along *any row or column*: Simply multiply each entry of that row or column by the corresponding cofactor and add.

Theorem 3.1.1: Cofactor Expansion Theorem[2]

The determinant of an $n\times n$ matrix A can be computed by using the cofactor expansion along any row or column of A. That is $\det A$ can be computed by multiplying each entry of the row or column by the corresponding cofactor and adding the results.

The proof will be given in Section **??**.

[2]The cofactor expansion is due to Pierre Simon de Laplace (1749–1827), who discovered it in 1772 as part of a study of linear differential equations. Laplace is primarily remembered for his work in astronomy and applied mathematics.

Example 3.1.3

Compute the determinant of $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 7 & 2 \\ 9 & 8 & -6 \end{bmatrix}$.

Solution. The cofactor expansion along the first row is as follows:

$$\begin{aligned} \det A &= 3c_{11}(A) + 4c_{12}(A) + 5c_{13}(A) \\ &= 3\begin{vmatrix} 7 & 2 \\ 8 & -6 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 \\ 9 & -6 \end{vmatrix} + 3\begin{vmatrix} 1 & 7 \\ 9 & 8 \end{vmatrix} \\ &= 3(-58) - 4(-24) + 5(-55) \\ &= -353 \end{aligned}$$

Note that the signs alternate along the row (indeed along *any* row or column). Now we compute $\det A$ by expanding along the first column.

$$\begin{aligned} \det A &= 3c_{11}(A) + 1c_{21}(A) + 9c_{31}(A) \\ &= 3\begin{vmatrix} 7 & 2 \\ 8 & -6 \end{vmatrix} - \begin{vmatrix} 4 & 5 \\ 8 & -6 \end{vmatrix} + 9\begin{vmatrix} 4 & 5 \\ 7 & 2 \end{vmatrix} \\ &= 3(-58) - (-64) + 9(-27) \\ &= -353 \end{aligned}$$

The reader is invited to verify that $\det A$ can be computed by expanding along any other row or column.

The fact that the cofactor expansion along *any row or column* of a matrix A always gives the same result (the determinant of A) is remarkable, to say the least. The choice of a particular row or column can simplify the calculation.

Example 3.1.4

Compute $\det A$ where $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ 2 & 6 & 0 & -1 \\ -6 & 3 & 1 & 0 \end{bmatrix}$.

Solution. The first choice we must make is which row or column to use in the cofactor expansion. The expansion involves multiplying entries by cofactors, so the work is minimized when the row or column contains as many zero entries as possible. Row 1 is a best choice in this matrix (column 4 would do as well), and the expansion is

$$\begin{aligned} \det A &= 3c_{11}(A) + 0c_{12}(A) + 0c_{13}(A) + 0c_{14}(A) \\ &= 3\begin{vmatrix} 1 & 2 & 0 \\ 6 & 0 & -1 \\ 3 & 1 & 0 \end{vmatrix} \end{aligned}$$

This is the first stage of the calculation, and we have succeeded in expressing the determinant of the 4×4 matrix A in terms of the determinant of a 3×3 matrix. The next stage involves this 3×3 matrix. Again, we can use any row or column for the cofactor expansion. The third column is preferred (with two zeros), so

$$\begin{aligned}\det A &= 3\left(0\begin{vmatrix} 6 & 0 \\ 3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix}\right) \\ &= 3[0 + 1(-5) + 0] \\ &= -15\end{aligned}$$

This completes the calculation.

Computing the determinant of a matrix A can be tedious. For example, if A is a 4×4 matrix, the cofactor expansion along any row or column involves calculating four cofactors, each of which involves the determinant of a 3×3 matrix. And if A is 5×5, the expansion involves five determinants of 4×4 matrices! There is a clear need for some techniques to cut down the work.[3]

The motivation for the method is the observation (see Example 3.1.4) that calculating a determinant is simplified a great deal when a row or column consists mostly of zeros. (In fact, when a row or column consists *entirely* of zeros, the determinant is zero—simply expand along that row or column.)

Recall next that one method of *creating* zeros in a matrix is to apply elementary row operations to it. Hence, a natural question to ask is what effect such a row operation has on the determinant of the matrix. It turns out that the effect is easy to determine and that elementary *column* operations can be used in the same way. These observations lead to a technique for evaluating determinants that greatly reduces the labour involved. The necessary information is given in Theorem 3.1.2.

Theorem 3.1.2

Let A denote an $n \times n$ matrix.

1. *If A has a row or column of zeros,* $\det A = 0$.
2. *If two distinct rows (or columns) of A are interchanged, the determinant of the resulting matrix is* $-\det A$.
3. *If a row (or column) of A is multiplied by a constant u, the determinant of the resulting matrix is* $u(\det A)$.
4. *If two distinct rows (or columns) of A are identical,* $\det A = 0$.

[3] If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ we can calculate $\det A$ by considering $\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$ obtained from A by adjoining columns 1 and 2 on the right. Then $\det A = aei + bfg + cdh - ceg - afh - bdi$, where the positive terms aei, bfg, and cdh are the products down and to the right starting at a, b, and c, and the negative terms ceg, afh, and bdi are the products down and to the left starting at c, a, and b. **Warning**: This rule does **not** apply to $n \times n$ matrices where $n > 3$ or $n = 2$.

5. *If a multiple of one row of A is added to a different row (or if a multiple of a column is added to a different column), the determinant of the resulting matrix is* $\det A$.

Proof. We prove properties 2, 4, and 5 and leave the rest as exercises.

Property 2. If A is $n \times n$, this follows by induction on n. If $n = 2$, the verification is left to the reader. If $n > 2$ and two rows are interchanged, let B denote the resulting matrix. Expand $\det A$ and $\det B$ along a row *other than* the two that were interchanged. The entries in this row are the same for both A and B, but the cofactors in B are the negatives of those in A (by induction) because the corresponding $(n-1) \times (n-1)$ matrices have two rows interchanged. Hence, $\det B = -\det A$, as required. A similar argument works if two columns are interchanged.

Property 4. If two rows of A are equal, let B be the matrix obtained by interchanging them. Then $B = A$, so $\det B = detA$. But $\det B = -\det A$ by property 2, so $\det A = \det B = 0$. Again, the same argument works for columns.

Property 5. Let B be obtained from $A = \left[a_{ij}\right]$ by adding u times row p to row q. Then row q of B is

$$(a_{q1} + ua_{p1},\ a_{q2} + ua_{p2},\ \dots,\ a_{qn} + ua_{pn})$$

The cofactors of these elements in B are the same as in A (they do not involve row q): in symbols, $c_{qj}(B) = c_{qj}(A)$ for each j. Hence, expanding B along row q gives

$$\begin{aligned}\det A &= (a_{q1} + ua_{p1})c_{q1}(A) + (a_{q2} + ua_{p2})c_{q2}(A) + \cdots + (a_{qn} + ua_{pn})c_{qn}(A) \\ &= [a_{q1}c_{q1}(A) + a_{q2}c_{q2}(A) + \cdots + a_{qn}c_{qn}(A)] + u[a_{p1}c_{q1}(A) + a_{p2}c_{q2}(A) + \cdots + a_{pn}c_{qn}(A)] \\ &= \det A + u \det C\end{aligned}$$

where C is the matrix obtained from A by replacing row q by row p (and both expansions are along row q). Because rows p and q of C are equal, $\det C = 0$ by property 4. Hence, $\det B = detA$, as required. As before, a similar proof holds for columns. □

To illustrate Theorem 3.1.2, consider the following determinants.

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$ (because the last row consists of zeros)

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 8 & 7 \\ 1 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} 5 & -1 & 3 \\ 7 & 8 & 2 \\ -1 & 2 & 1 \end{vmatrix}$$ (because two columns are interchanged)

$$\begin{vmatrix} 8 & 1 & 2 \\ 3 & 0 & 9 \\ 1 & 2 & -1 \end{vmatrix} = 3\begin{vmatrix} 8 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & -1 \end{vmatrix}$$ (because the second row of the matrix on the left is 3 times the second row of the matrix on the right)

$$\begin{vmatrix} 2 & 1 & 2 \\ 4 & 0 & 4 \\ 1 & 3 & 1 \end{vmatrix} = 0$$ (because two columns are identical)

$$\begin{vmatrix} 2 & 5 & 2 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 9 & 20 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix}$$ (because twice the second row of the matrix on the left was added to the first row)

The following four examples illustrate how Theorem 3.1.2 is used to evaluate determinants.

Example 3.1.5

Evaluate $\det A$ when $A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 6 \end{bmatrix}$.

Solution. The matrix does have zero entries, so expansion along (say) the second row would involve somewhat less work. However, a column operation can be used to get a zero in position (2, 3)—namely, add column 1 to column 3. Because this does not change the value of the determinant, we obtain

$$\det A = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 4 \\ 1 & 0 & 0 \\ 2 & 1 & 8 \end{vmatrix} = -\begin{vmatrix} -1 & 4 \\ 1 & 8 \end{vmatrix} = 12$$

where we expanded the second 3×3 matrix along row 2.

Example 3.1.6

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 6$, evaluate $\det A$ where $A = \begin{bmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{bmatrix}$.

Solution. First take common factors out of rows 2 and 3.

$$\det A = 3(-1)\det \begin{bmatrix} a+x & b+y & c+z \\ x & y & z \\ p & q & r \end{bmatrix}$$

Now subtract the second row from the first and interchange the last two rows.

$$\det A = -3\det \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix} = 3\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 3 \cdot 6 = 18$$

The determinant of a matrix is a sum of products of its entries. In particular, if these entries are polynomials in x, then the determinant itself is a polynomial in x. It is often of interest to determine which values of x make the determinant zero, so it is very useful if the determinant is given in factored form. Theorem 3.1.2 can help.

Example 3.1.7

Find the values of x for which $\det A = 0$, where $A = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$.

Solution. To evaluate $\det A$, first subtract x times row 1 from rows 2 and 3.

$$\det A = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x \\ 0 & 1-x^2 & x-x^2 \\ 0 & x-x^2 & 1-x^2 \end{vmatrix} = \begin{vmatrix} 1-x^2 & x-x^2 \\ x-x^2 & 1-x^2 \end{vmatrix}$$

At this stage we could simply evaluate the determinant (the result is $2x^3 - 3x^2 + 1$). But then we would have to factor this polynomial to find the values of x that make it zero. However, this factorization can be obtained directly by first factoring each entry in the determinant and taking a common factor of $(1-x)$ from each row.

$$\begin{aligned} \det A = \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ x(1-x) & (1-x)(1+x) \end{vmatrix} &= (1-x)^2 \begin{vmatrix} 1+x & x \\ x & 1+x \end{vmatrix} \\ &= (1-x)^2(2x+1) \end{aligned}$$

Hence, $\det A = 0$ means $(1-x)^2(2x+1) = 0$, that is $x = 1$ or $x = -\frac{1}{2}$.

Example 3.1.8

If a_1, a_2, and a_3 are given show that

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = (a_3 - a_1)(a_3 - a_2)(a_2 - a_1)$$

Solution. Begin by subtracting row 1 from rows 2 and 3, and then expand along column 1:

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 \\ 0 & a_3 - a_1 & a_3^2 - a_1^2 \end{bmatrix} = \begin{bmatrix} a_2 - a_1 & a_2^2 - a_1^2 \\ a_3 - a_1 & a_3^2 - a_1^2 \end{bmatrix}$$

Now $(a_2 - a_1)$ and $(a_3 - a_1)$ are common factors in rows 1 and 2, respectively, so

$$\begin{aligned} \det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} &= (a_2 - a_1)(a_3 - a_1) \det \begin{bmatrix} 1 & a_2 + a_1 \\ 1 & a_3 + a_1 \end{bmatrix} \\ &= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2) \end{aligned}$$

The matrix in Example 3.1.8 is called a Vandermonde matrix, and the formula for its determinant can be generalized to the $n \times n$ case (see Theorem 3.2.7).

If A is an $n \times n$ matrix, forming uA means multiplying *every* row of A by u. Applying property 3 of Theorem 3.1.2, we can take the common factor u out of each row and so obtain the following useful result.

Theorem 3.1.3

If A is an $n \times n$ matrix, then $\det(uA) = u^n \det A$ for any number u.

The next example displays a type of matrix whose determinant is easy to compute.

Example 3.1.9

Evaluate $\det A$ if $A = \begin{bmatrix} a & 0 & 0 & 0 \\ u & b & 0 & 0 \\ v & w & c & 0 \\ x & y & z & d \end{bmatrix}$.

Solution. Expand along row 1 to get $\det A = a \begin{vmatrix} b & 0 & 0 \\ w & c & 0 \\ y & z & d \end{vmatrix}$. Now expand this along the top row to get $\det A = ab \begin{vmatrix} c & 0 \\ z & d \end{vmatrix} = abcd$, the product of the main diagonal entries.

A square matrix is called a **lower triangular matrix** if all entries above the main diagonal are zero (as in Example 3.1.9). Similarly, an **upper triangular matrix** is one for which all entries below the main diagonal are zero. A **triangular matrix** is one that is either upper or lower triangular. Theorem 3.1.4 gives an easy rule for calculating the determinant of any triangular matrix. The proof is like the solution to Example 3.1.9.

Theorem 3.1.4

If A is a square triangular matrix, then det A is the product of the entries on the main diagonal.

Theorem 3.1.4 is useful in computer calculations because it is a routine matter to carry a matrix to triangular form using row operations.

Block matrices such as those in the next theorem arise frequently in practice, and the theorem gives an easy method for computing their determinants. This dovetails with Example 2.4.11.

Theorem 3.1.5

Consider matrices $\left[\begin{array}{cc} A & X \\ 0 & B \end{array} \right]$ and $\left[\begin{array}{cc} A & 0 \\ Y & B \end{array} \right]$ in block form, where A and B are square matrices. Then

$$\det \left[\begin{array}{cc} A & X \\ 0 & B \end{array} \right] = \det A \det B \text{ and } \det \left[\begin{array}{cc} A & 0 \\ Y & B \end{array} \right] = \det A \det B$$

Proof. Write $T = \det \left[\begin{array}{cc} A & X \\ 0 & B \end{array} \right]$ and proceed by induction on k where A is $k \times k$. If $k = 1$, it is the cofactor expansion along column 1. In general let $S_i(T)$ denote the matrix obtained from T by deleting row i and column 1. Then the cofactor expansion of $\det T$ along the first column is

$$\det T = a_{11} \det(S_1(T)) - a_{21} \det(S_2(T)) + \cdots \pm a_{k1} \det(S_k(T)) \tag{3.2}$$

where $a_{11}, a_{21}, \cdots, a_{k1}$ are the entries in the first column of A. But $S_i(T) = \left[\begin{array}{cc} S_i(A) & X_i \\ 0 & B \end{array} \right]$ for each $i = 1, 2, \cdots, k$, so $\det(S_i(T)) = \det(S_i(A)) \cdot \det B$ by induction. Hence, Equation 3.2 becomes

$$\begin{aligned} \det T &= \{a_{11} \det(S_1(T)) - a_{21} \det(S_2(T)) + \cdots \pm a_{k1} \det(S_k(T))\} \det B \\ &= \{\det A\} \det B \end{aligned}$$

as required. The lower triangular case is similar. □

Example 3.1.10

$$\det\begin{bmatrix} 2 & 3 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 1 \end{bmatrix} = -\begin{vmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -(-3)(-3) = -9$$

The next result shows that $\det A$ is a linear transformation when regarded as a function of a fixed column of A. The proof is Exercise 3.1.21.

Theorem 3.1.6

Given columns $\mathbf{c}_1, \cdots, \mathbf{c}_{j-1}, \mathbf{c}_{j+1}, \cdots, \mathbf{c}_n$ in $\mathbb{R}^n$, define $T : \mathbb{R}^n \to \mathbb{R}$ by

$$T(\mathbf{x}) = \det\left[\begin{array}{ccccccc} \mathbf{c}_1 & \cdots & \mathbf{c}_{j-1} & \mathbf{x} & \mathbf{c}_{j+1} & \cdots & \mathbf{c}_n \end{array}\right] \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Then, for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all a in $\mathbb{R}$,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \quad \text{and} \quad T(a\mathbf{x}) = aT(\mathbf{x})$$

Exercises for 3.1

Exercise 3.1.1 Compute the determinants of the following matrices.

a) $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 6 & 9 \\ 8 & 12 \end{bmatrix}$

c) $\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$

d) $\begin{bmatrix} a+1 & a \\ a & a-1 \end{bmatrix}$

e) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

f) $\begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 5 \\ 0 & 3 & 0 \end{bmatrix}$

g) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

h) $\begin{bmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{bmatrix}$

i) $\begin{bmatrix} 1 & b & c \\ b & c & 1 \\ c & 1 & b \end{bmatrix}$

j) $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$

k) $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \end{bmatrix}$

l) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 2 & 6 & 0 \\ -1 & 0 & -3 & 1 \\ 4 & 1 & 12 & 0 \end{bmatrix}$

m) $\begin{bmatrix} 3 & 1 & -5 & 2 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 5 & 2 \\ 1 & 1 & 2 & -1 \end{bmatrix}$

n) $\begin{bmatrix} 4 & -1 & 3 & -1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & -1 & 1 \end{bmatrix}$

o) $\begin{bmatrix} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix}$

p) $\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & p \\ 0 & c & q & k \\ d & s & t & u \end{bmatrix}$

---

b. 0

d. -1

f. -39

h. 0

j. $2abc$

l. 0

n. -56

p. $abcd$

Exercise 3.1.2 Show that $\det A = 0$ if A has a row or column consisting of zeros.

Exercise 3.1.3 Show that the sign of the position in the last row and the last column of A is always $+1$.

Exercise 3.1.4 Show that $\det I = 1$ for any identity matrix I.

Exercise 3.1.5 Evaluate the determinant of each matrix by reducing it to upper triangular form.

a) $\left[\begin{array}{rrr} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -1 & 3 \end{array}\right]$ b) $\left[\begin{array}{rrr} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{array}\right]$

c) $\left[\begin{array}{rrrr} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{array}\right]$ d) $\left[\begin{array}{rrrr} 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{array}\right]$

---

b. -17

d. 106

Exercise 3.1.6 Evaluate by cursory inspection:

a. $\det \left[\begin{array}{ccc} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{array}\right]$

b. $\det \left[\begin{array}{ccc} a & b & c \\ a+b & 2b & c+b \\ 2 & 2 & 2 \end{array}\right]$

---

b. 0

Exercise 3.1.7 If $\det \left[\begin{array}{ccc} a & b & c \\ p & q & r \\ x & y & z \end{array}\right] = -1$ compute:

a. $\det \left[\begin{array}{ccc} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{array}\right]$

b. $\det \left[\begin{array}{ccc} -2a & -2b & -2c \\ 2p+x & 2q+y & 2r+z \\ 3x & 3y & 3z \end{array}\right]$

---

b. 12

Exercise 3.1.8 Show that:

a. $\det \left[\begin{array}{ccc} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{array}\right] = 2 \det \left[\begin{array}{ccc} a & b & c \\ p & q & r \\ x & y & z \end{array}\right]$

b. $\det \left[\begin{array}{ccc} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{array}\right] = 9 \det \left[\begin{array}{ccc} a & b & c \\ p & q & r \\ x & y & z \end{array}\right]$

---

b.
$$\begin{aligned} \det \left[\begin{array}{ccc} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{array}\right] &= 3 \det \left[\begin{array}{ccc} a+p+x & b+q+y & c+r+z \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{array}\right] \\ &= 3 \det \left[\begin{array}{ccc} a+p+x & b+q+y & c+r+z \\ p-a & q-b & r-c \\ x-p & y-q & z-r \end{array}\right] \\ &= 3 \det \left[\begin{array}{ccc} 3x & 3y & 3z \\ p-a & q-b & r-c \\ x-p & y-q & z-r \end{array}\right] \cdots \end{aligned}$$

Exercise 3.1.9 In each case either prove the statement or give an example showing that it is false:

a. $\det(A+B) = \det A + \det B$.

b. If $\det A = 0$, then A has two equal rows.

c. If A is 2×2, then $\det(A^T) = \det A$.

d. If R is the reduced row-echelon form of A, then $\det A = \det R$.

e. If A is 2×2, then $\det(7A) = 49 \det A$.

f. $\det(A^T) = -\det A$.

g. $\det(-A) = -\det A$.

h. If $\det A = \det B$ where A and B are the same size, then $A = B$.

---

b. False. $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

d. False. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

f. False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

h. False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Exercise 3.1.10 Compute the determinant of each matrix, using Theorem 3.1.5.

a. $\begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ -1 & 3 & 1 & 4 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 1 \end{bmatrix}$

---

b. 35

Exercise 3.1.11 If $\det A = 2$, $\det B = -1$, and $\det C = 3$, find:

a) $\det \begin{bmatrix} A & X & Y \\ 0 & B & Z \\ 0 & 0 & C \end{bmatrix}$

b) $\det \begin{bmatrix} A & 0 & 0 \\ X & B & 0 \\ Y & Z & C \end{bmatrix}$

c) $\det \begin{bmatrix} A & X & Y \\ 0 & B & 0 \\ 0 & Z & C \end{bmatrix}$

d) $\det \begin{bmatrix} A & X & 0 \\ 0 & B & 0 \\ Y & Z & C \end{bmatrix}$

---

b. -6

d. -6

Exercise 3.1.12 If A has three columns with only the top two entries nonzero, show that $\det A = 0$.

Exercise 3.1.13

a. Find $\det A$ if A is 3×3 and $\det(2A) = 6$.

b. Under what conditions is $\det(-A) = \det A$?

Exercise 3.1.14 Evaluate by first adding all other rows to the first row.

a. $\det \begin{bmatrix} x-1 & 2 & 3 \\ 2 & -3 & x-2 \\ -2 & x & -2 \end{bmatrix}$

b. $\det \begin{bmatrix} x-1 & -3 & 1 \\ 2 & -1 & x-1 \\ -3 & x+2 & -2 \end{bmatrix}$

---

b. $-(x-2)(x^2+2x-12)$

Exercise 3.1.15

a. Find b if $\det \begin{bmatrix} 5 & -1 & x \\ 2 & 6 & y \\ -5 & 4 & z \end{bmatrix} = ax+by+cz$.

b. Find c if $\det \begin{bmatrix} 2 & x & -1 \\ 1 & y & 3 \\ -3 & z & 4 \end{bmatrix} = ax+by+cz$.

---

b. -7

Exercise 3.1.16 Find the real numbers x and y such that $\det A = 0$ if:

a) $A = \begin{bmatrix} 0 & x & y \\ y & 0 & x \\ x & y & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & x & x \\ -x & -2 & x \\ -x & -x & -3 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{bmatrix}$

d) $A = \begin{bmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ y & 0 & 0 & x \end{bmatrix}$

---

b. $\pm\frac{\sqrt{6}}{2}$

d. $x = \pm y$

Exercise 3.1.17 Show that

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{bmatrix} = -3x^2$$

Exercise 3.1.18 Show that

$$\det \begin{bmatrix} 1 & x & x^2 & x^3 \\ a & 1 & x & x^2 \\ p & b & 1 & x \\ q & r & c & 1 \end{bmatrix} = (1-ax)(1-bx)(1-cx).$$

Exercise 3.1.19
Given the polynomial $p(x) = a + bx + cx^2 + dx^3 + x^4$, the matrix $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{bmatrix}$ is called the **companion matrix** of $p(x)$. Show that $\det(xI - C) = p(x)$.

Exercise 3.1.20 Show that

$$\det \begin{bmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{bmatrix}$$
$$= (a+b+c+3x)[(ab+ac+bc) - (a^2+b^2+c^2)]$$

Exercise 3.1.21 . Prove Theorem 3.1.6. [*Hint*: Expand the determinant along column j.]

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{x}+\mathbf{y} & \cdots & \mathbf{c}_n \end{bmatrix}$ where $\mathbf{x}+\mathbf{y}$ is in column j. Expanding $\det A$ along column j (the one containing $\mathbf{x}+\mathbf{y}$):

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) = \det A &= \sum_{i=1}^{n} (x_i + y_i) c_{ij}(A) \\ &= \sum_{i=1}^{n} x_i c_{ij}(A) + \sum_{i=1}^{n} y_i c_{ij}(A) \\ &= T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

Similarly for $T(a\mathbf{x}) = aT(\mathbf{x})$.

Exercise 3.1.22 Show that

$$\det \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_2 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & * & * \\ a_n & * & \cdots & * & * \end{bmatrix} = (-1)^k a_1 a_2 \cdots a_n$$

where either $n = 2k$ or $n = 2k+1$, and $*$-entries are arbitrary.

Exercise 3.1.23 By expanding along the first column, show that:

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = 1 + (-1)^{n+1}$$

if the matrix is $n \times n$, $n \geq 2$.

Exercise 3.1.24 Form matrix B from a matrix A by writing the columns of A in reverse order. Express $\det B$ in terms of $\det A$.

If A is $n \times n$, then $\det B = (-1)^k \det A$ where $n = 2k$ or $n = 2k+1$.

Exercise 3.1.25 Prove property 3 of Theorem 3.1.2 by expanding along the row (or column) in question.

Exercise 3.1.26 Show that the line through two distinct points (x_1, y_1) and (x_2, y_2) in the plane has equation

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

Exercise 3.1.27 Let A be an $n \times n$ matrix. Given a polynomial $p(x) = a_0 + a_1 x + \cdots + a_m x^m$, we write

$p(A) = a_0 I + a_1 A + \cdots + a_m A^m$. For example, if $p(x) = 2 - 3x + 5x^2$, then
$p(A) = 2I - 3A + 5A^2$. The *characteristic polynomial* of A is defined to be $c_A(x) = \det[xI - A]$, and the Cayley-Hamilton theorem asserts that $c_A(A) = 0$ for any matrix A.

a. Verify the theorem for

i. $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ ii. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 8 & 2 & 2 \end{bmatrix}$

b. Prove the theorem for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

3.2 Determinants and Matrix Inverses

In this section, several theorems about determinants are derived. One consequence of these theorems is that a square matrix A is invertible if and only if $\det A \neq 0$. Moreover, determinants are used to give a formula for A^{-1} which, in turn, yields a formula (called Cramer's rule) for the solution of any system of linear equations with an invertible coefficient matrix.

We begin with a remarkable theorem (due to Cauchy in 1812) about the determinant of a product of matrices. The proof is given at the end of this section.

Theorem 3.2.1: Product Theorem

If A and B are $n \times n$ matrices, then $\det(AB) = \det A \det B$.

The complexity of matrix multiplication makes the product theorem quite unexpected. Here is an example where it reveals an important numerical identity.

Example 3.2.1

If $A = \left[\begin{array}{rr} a & b \\ -b & a \end{array} \right]$ and $B = \left[\begin{array}{rr} c & d \\ -d & c \end{array} \right]$ then $AB = \left[\begin{array}{cc} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{array} \right]$.

Hence $\det A \det B = \det(AB)$ gives the identity

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

Theorem 3.2.1 extends easily to $\det(ABC) = \det A \det B \det C$. In fact, induction gives

$$\det(A_1 A_2 \cdots A_{k-1} A_k) = \det A_1 \det A_2 \cdots \det A_{k-1} \det A_k$$

for any square matrices $A_1, \dots, A_k$ of the same size. In particular, if each $A_i = A$, we obtain

$$\det(A^k) = (detA)^k, \quad \text{for any } k \geq 1$$

We can now give the invertibility condition.

Theorem 3.2.2

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. When this is the case, $\det(A^{-1}) = \frac{1}{\det A}$

Proof. If A is invertible, then $AA^{-1} = I$; so the product theorem gives

$$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}$$

Hence, $\det A \neq 0$ and also $\det A^{-1} = \frac{1}{\det A}$.

Conversely, if $\det A \neq 0$, we show that A can be carried to I by elementary row operations (and invoke Theorem 2.4.5). Certainly, A can be carried to its reduced row-echelon form R, so $R = E_k \cdots E_2 E_1 A$ where the E_i are elementary matrices (Theorem 2.5.1). Hence the product theorem gives

$$\det R = \det E_k \cdots \det E_2 \det E_1 \det A$$

Since $\det E \neq 0$ for all elementary matrices E, this shows $\det R \neq 0$. In particular, R has no row of zeros, so $R = I$ because R is square and reduced row-echelon. This is what we wanted. □

Example 3.2.2

For which values of c does $A = \left[\begin{array}{rrr} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{array} \right]$ have an inverse?

Solution. Compute $\det A$ by first adding c times column 1 to column 3 and then expanding along row 1.

$$\det A = \det \left[\begin{array}{rrr} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{array} \right] = \det \left[\begin{array}{rrc} 1 & 0 & 0 \\ -1 & 3 & 1-c \\ 0 & 2c & -4 \end{array} \right] = 2(c+2)(c-3)$$

Hence, $\det A = 0$ if $c = -2$ or $c = 3$, and A has an inverse if $c \neq -2$ and $c \neq 3$.

Example 3.2.3

If a product $A_1 A_2 \cdots A_k$ of square matrices is invertible, show that each A_i is invertible.

Solution. We have $\det A_1 \det A_2 \cdots \det A_k = \det (A_1 A_2 \cdots A_k)$ by the product theorem, and $\det (A_1 A_2 \cdots A_k) \neq 0$ by Theorem 3.2.2 because $A_1 A_2 \cdots A_k$ is invertible. Hence

$$\det A_1 \det A_2 \cdots \det A_k \neq 0$$

so $\det A_i \neq 0$ for each i. This shows that each A_i is invertible, again by Theorem 3.2.2.

Theorem 3.2.3

If A is any square matrix, $\det A^T = \det A$.

Proof. Consider first the case of an elementary matrix E. If E is of type I or II, then $E^T = E$; so certainly $\det E^T = \det E$. If E is of type III, then E^T is also of type III; so $\det E^T = 1 = \det E$ by Theorem 3.1.2. Hence, $\det E^T = \det E$ for every elementary matrix E.

Now let A be any square matrix. If A is not invertible, then neither is A^T; so $\det A^T = 0 = \det A$ by Theorem 3.2.2. On the other hand, if A is invertible, then $A = E_k \cdots E_2 E_1$, where the E_i are elementary matrices (Theorem 2.5.2). Hence, $A^T = E_1^T E_2^T \cdots E_k^T$ so the product theorem gives

$$\begin{aligned}\det A^T &= \det E_1^T \det E_2^T \cdots \det E_k^T = \det E_1 \det E_2 \cdots \det E_k \\ &= \det E_k \cdots \det E_2 \det E_1 \\ &= \det A\end{aligned}$$

This completes the proof. □

Example 3.2.4

If $\det A = 2$ and $\det B = 5$, calculate $\det(A^3B^{-1}A^TB^2)$.

Solution. We use several of the facts just derived.

$$\begin{aligned}\det(A^3B^{-1}A^TB^2) &= \det(A^3)\det(B^{-1})\det(A^T)\det(B^2) \\ &= (\det A)^3\frac{1}{\det B}\det A(\det B)^2 \\ &= 2^3 \cdot \frac{1}{5} \cdot 2 \cdot 5^2 \\ &= 80\end{aligned}$$

Example 3.2.5

A square matrix is called **orthogonal** if $A^{-1} = A^T$. What are the possible values of $\det A$ if A is orthogonal?

Solution. If A is orthogonal, we have $I = AA^T$. Take determinants to obtain

$$1 = \det I = \det(AA^T) = \det A \det A^T = (\det A)^2$$

Since $\det A$ is a number, this means $\det A = \pm 1$.

Hence Theorems 2.6.4 and 2.6.5 imply that rotation about the origin and reflection about a line through the origin in $\mathbb{R}^2$ have orthogonal matrices with determinants 1 and -1 respectively. In fact they are the *only* such transformations of $\mathbb{R}^2$. We have more to say about this in Section 8.2.

Adjugates

In Section 2.4 we defined the adjugate of a 2 × 2 matrix $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ to be $\text{adj}(A) = \left[\begin{array}{rr} d & -b \\ -c & a \end{array}\right]$. Then we verified that $A(\text{adj } A) = (\det A)I = (\text{adj } A)A$ and hence that, if $\det A \neq 0$, $A^{-1} = \frac{1}{\det A}\text{adj } A$. We are now able to define the adjugate of an arbitrary square matrix and to show that this formula for the inverse remains valid (when the inverse exists).

Recall that the (i, j)-cofactor $c_{ij}(A)$ of a square matrix A is a number defined for each position (i, j) in the matrix. If A is a square matrix, the **cofactor matrix of** A is defined to be the matrix $\left[c_{ij}(A)\right]$ whose (i, j)-entry is the (i, j)-cofactor of A.

Definition 3.3 Adjugate of a Matrix

The ***adjugate***[4] *of* A, *denoted* $\text{adj}\,(A)$, *is the transpose of this cofactor matrix; in symbols,*

$$\text{adj}\,(A) = \left[c_{ij}(A)\right]^T$$

This agrees with the earlier definition for a 2×2 matrix A as the reader can verify.

Example 3.2.6

Compute the adjugate of $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix}$ and calculate $A(\text{adj}\,A)$ and $(\text{adj}\,A)A$.

Solution. We first find the cofactor matrix.

$$\begin{bmatrix} c_{11}(A) & c_{12}(A) & c_{13}(A) \\ c_{21}(A) & c_{22}(A) & c_{23}(A) \\ c_{31}(A) & c_{32}(A) & c_{33}(A) \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 1 & 5 \\ -6 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ -6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}$$

Then the adjugate of A is the transpose of this cofactor matrix.

$$\text{adj}\,A = \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}^T = \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

The computation of $A(\text{adj}\,A)$ gives

$$A(\text{adj}\,A) = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix} \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I$$

and the reader can verify that also $(\text{adj}\,A)A = 3I$. Hence, analogy with the 2×2 case would indicate that $\det A = 3$; this is, in fact, the case.

The relationship $A(\text{adj}\,A) = (\det A)I$ holds for any square matrix A. To see why this is so,

[4] This is also called the classical adjoint of A, but the term “adjoint” has another meaning.

consider the general 3×3 case. Writing $c_{ij}(A) = c_{ij}$ for short, we have

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

If $A = \left[a_{ij} \right]$ in the usual notation, we are to verify that $A(\text{adj } A) = (\det A)I$. That is,

$$A(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

Consider the $(1,\ 1)$-entry in the product. It is given by $a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$, and this is just the cofactor expansion of $\det A$ along the first row of A. Similarly, the $(2,\ 2)$-entry and the $(3,\ 3)$-entry are the cofactor expansions of $\det A$ along rows 2 and 3, respectively.

So it remains to be seen why the off-diagonal elements in the matrix product $A(\text{adj } A)$ are all zero. Consider the $(1,\ 2)$-entry of the product. It is given by $a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23}$. This *looks* like the cofactor expansion of the determinant of *some* matrix. To see which, observe that c_{21}, c_{22}, and c_{23} are all computed by *deleting* row 2 of A (and one of the columns), so they remain the same if row 2 of A is changed. In particular, if row 2 of A is replaced by row 1, we obtain

$$a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0$$

where the expansion is along row 2 and where the determinant is zero because two rows are identical. A similar argument shows that the other off-diagonal entries are zero.

This argument works in general and yields the first part of Theorem 3.2.4. The second assertion follows from the first by multiplying through by the scalar $\frac{1}{\det A}$.

Theorem 3.2.4: Adjugate Formula

If A is any square matrix, then

$$A(\text{adj } A) = (\det A)I = (\text{adj } A)A$$

In particular, if det A ≠ 0, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$

It is important to note that this theorem is *not* an efficient way to find the inverse of the matrix A. For example, if A were 10×10, the calculation of $\text{adj } A$ would require computing $10^2 = 100$ determinants of 9×9 matrices! On the other hand, the matrix inversion algorithm would find A^{-1} with about the same effort as finding $\det A$. Clearly, Theorem 3.2.4 is not a *practical* result: its virtue is that it gives a formula for A^{-1} that is useful for *theoretical* purposes.

Example 3.2.7

Find the $(2, 3)$-entry of A^{-1} if $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Solution. First compute

$$\det A = \begin{vmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 7 \\ 5 & -7 & 11 \\ 3 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 7 \\ -7 & 11 \end{vmatrix} = 180$$

Since $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{180} \left[c_{ij}(A)\right]^T$, the $(2, 3)$-entry of A^{-1} is the $(3, 2)$-entry of the matrix $\frac{1}{180} \left[c_{ij}(A)\right]$; that is, it equals $\frac{1}{180} c_{32}(A) = \frac{1}{180} \left(- \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} \right) = \frac{13}{180}$.

Example 3.2.8

If A is $n \times n$, $n \geq 2$, show that $\det(\operatorname{adj} A) = (\det A)^{n-1}$.

Solution. Write $d = \det A$; we must show that $\det(\operatorname{adj} A) = d^{n-1}$. We have $A(\operatorname{adj} A) = dI$ by Theorem 3.2.4, so taking determinants gives $d \det(\operatorname{adj} A) = d^n$. Hence we are done if $d \neq 0$. Assume $d = 0$; we must show that $\det(\operatorname{adj} A) = 0$, that is, $\operatorname{adj} A$ is not invertible. If $A \neq 0$, this follows from $A(\operatorname{adj} A) = dI = 0$; if $A = 0$, it follows because then $\operatorname{adj} A = 0$.

Cramer's Rule

Theorem 3.2.4 has a nice application to linear equations. Suppose

$$A\mathbf{x} = \mathbf{b}$$

is a system of n equations in n variables $x_1, x_2, \dots, x_n$. Here A is the $n \times n$ coefficient matrix, and $\mathbf{x}$ and $\mathbf{b}$ are the columns

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

of variables and constants, respectively. If $\det A \neq 0$, we left multiply by A^{-1} to obtain the solution $\mathbf{x} = A^{-1}\mathbf{b}$. When we use the adjugate formula, this becomes

$$\begin{aligned}
\left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] &= \frac{1}{\det A}(\operatorname{adj} A)\mathbf{b} \\
&= \frac{1}{\det A}\left[\begin{array}{cccc} c_{11}(A) & c_{21}(A) & \cdots & c_{n1}(A) \\ c_{12}(A) & c_{22}(A) & \cdots & c_{n2}(A) \\ \vdots & \vdots & & \vdots \\ c_{1n}(A) & c_{2n}(A) & \cdots & c_{nn}(A) \end{array}\right]\left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right]
\end{aligned}$$

Hence, the variables $x_1, x_2, \dots, x_n$ are given by

$$\begin{aligned}
x_1 &= \frac{1}{\det A}\left[b_1c_{11}(A) + b_2c_{21}(A) + \cdots + b_nc_{n1}(A)\right] \\
x_2 &= \frac{1}{\det A}\left[b_1c_{12}(A) + b_2c_{22}(A) + \cdots + b_nc_{n2}(A)\right] \\
\vdots & \qquad\qquad \vdots \\
x_n &= \frac{1}{\det A}\left[b_1c_{1n}(A) + b_2c_{2n}(A) + \cdots + b_nc_{nn}(A)\right]
\end{aligned}$$

Now the quantity $b_1c_{11}(A) + b_2c_{21}(A) + \cdots + b_nc_{n1}(A)$ occurring in the formula for x_1 looks like the cofactor expansion of the determinant of a matrix. The cofactors involved are $c_{11}(A), c_{21}(A), \dots, c_{n1}(A)$, corresponding to the first column of A. If A_1 is obtained from A by replacing the first column of A by $\mathbf{b}$, then $c_{i1}(A_1) = c_{i1}(A)$ for each i because column 1 is deleted when computing them. Hence, expanding $\det(A_1)$ by the first column gives

$$\begin{aligned}
\det A_1 &= b_1c_{11}(A_1) + b_2c_{21}(A_1) + \cdots + b_nc_{n1}(A_1) \\
&= b_1c_{11}(A) + b_2c_{21}(A) + \cdots + b_nc_{n1}(A) \\
&= (\det A)x_1
\end{aligned}$$

Hence, $x_1 = \frac{\det A_1}{\det A}$ and similar results hold for the other variables.

Theorem 3.2.5: Cramer's Rule[5]

If A is an invertible $n \times n$ matrix, the solution to the system

$$A\mathbf{x} = \mathbf{b}$$

of n equations in the variables $x_1, x_2, \dots, x_n$ is given by

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots, \quad x_n = \frac{\det A_n}{\det A}$$

where, for each k, A_k is the matrix obtained from A by replacing column k by $\mathbf{b}$.

[5]Gabriel Cramer (1704–1752) was a Swiss mathematician who wrote an introductory work on algebraic curves. He popularized the rule that bears his name, but the idea was known earlier.

Example 3.2.9

Find x_1, given the following system of equations.

$$\begin{aligned} 5x_1 + x_2 - \ \ x_3 &= 4 \\ 9x_1 + x_2 - \ \ x_3 &= 1 \\ x_1 - x_2 + 5x_3 &= 2 \end{aligned}$$

Solution. Compute the determinants of the coefficient matrix A and the matrix A_1 obtained from it by replacing the first column by the column of constants.

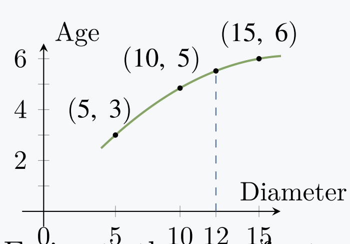
$$\det A = \det \begin{bmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{bmatrix} = -16$$
$$\det A_1 = \det \begin{bmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix} = 12$$

Hence, $x_1 = \frac{\det A_1}{\det A} = -\frac{3}{4}$ by Cramer's rule.

Cramer's rule is *not* an efficient way to solve linear systems or invert matrices. True, it enabled us to calculate x_1 here without computing x_2 or x_3. Although this might seem an advantage, the truth of the matter is that, for large systems of equations, the number of computations needed to find *all* the variables by the gaussian algorithm is comparable to the number required to find *one* of the determinants involved in Cramer's rule. Furthermore, the algorithm works when the matrix of the system is not invertible and even when the coefficient matrix is not square. Like the adjugate formula, then, Cramer's rule is *not* a practical numerical technique; its virtue is theoretical.

Polynomial Interpolation

Example 3.2.10



A forester wants to estimate the age (in years) of a tree by measuring the diameter of the trunk (in cm). She obtains the following data:

	Tree 1	Tree 2	Tree 3
Trunk Diameter	5	10	15
Age	3	5	6

Estimate the age of a tree with a trunk diameter of 12 cm.

Solution.
The forester decides to "fit" a quadratic polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2$$

to the data, that is choose the coefficients r_0, r_1, and r_2 so that $p(5)=3$, $p(10)=5$, and $p(15)=6$, and then use $p(12)$ as the estimate. These conditions give three linear equations:

$$\begin{aligned} r_0 + 5r_1 + 25r_2 &= 3 \\ r_0 + 10r_1 + 100r_2 &= 5 \\ r_0 + 15r_1 + 225r_2 &= 6 \end{aligned}$$

The (unique) solution is $r_0=0$, $r_1=\frac{7}{10}$, and $r_2=-\frac{1}{50}$, so

$$p(x)=\frac{7}{10}x-\frac{1}{50}x^2=\frac{1}{50}x(35-x)$$

Hence the estimate is $p(12)=5.52$.

As in Example 3.2.10, it often happens that two variables x and y are related but the actual functional form $y=f(x)$ of the relationship is unknown. Suppose that for certain values $x_1, x_2, \dots, x_n$ of x the corresponding values $y_1, y_2, \dots, y_n$ are known (say from experimental measurements). One way to estimate the value of y corresponding to some other value a of x is to find a polynomial[6]

$$p(x)=r_0+r_1x+r_2x^2+\cdots+r_{n-1}x^{n-1}$$

that "fits" the data, that is $p(x_i)=y_i$ holds for each $i=1, 2, \dots, n$. Then the estimate for y is $p(a)$. As we will see, such a polynomial always exists if the x_i are distinct.

The conditions that $p(x_i)=y_i$ are

$$\begin{array}{ccccccccc} r_0 &+& r_1x_1 &+& r_2x_1^2 &+\cdots+& r_{n-1}x_1^{n-1} &=& y_1 \\ r_0 &+& r_1x_2 &+& r_2x_2^2 &+\cdots+& r_{n-1}x_2^{n-1} &=& y_2 \\ \vdots && \vdots && \vdots && \vdots && \vdots \\ r_0 &+& r_1x_n &+& r_2x_n^2 &+\cdots+& r_{n-1}x_n^{n-1} &=& y_n \end{array}$$

In matrix form, this is

$$\left[\begin{array}{ccccc} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{array}\right] \left[\begin{array}{c} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{array}\right] = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right] \tag{3.3}$$

It can be shown (see Theorem 3.2.7) that the determinant of the coefficient matrix equals the product of all terms (x_i-x_j) with $i>j$ and so is nonzero (because the x_i are distinct). Hence the equations have a unique solution $r_0, r_1, \dots, r_{n-1}$. This proves

[6] A **polynomial** is an expression of the form $a_0+a_1x+a_2x^2+\cdots+a_nx^n$ where the a_i are numbers and x is a variable. If $a_n\neq 0$, the integer n is called the degree of the polynomial, and a_n is called the leading coefficient. See Appendix **??**.

Theorem 3.2.6

Let n data pairs (x_1, y_1), (x_2, y_2), $\dots$, (x_n, y_n) be given, and assume that the x_i are distinct. Then there exists a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for each $i = 1, 2, \dots, n$.

The polynomial in Theorem 3.2.6 is called the **interpolating polynomial** for the data.

We conclude by evaluating the determinant of the coefficient matrix in Equation 3.3. If $a_1, a_2, \dots, a_n$ are numbers, the determinant

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$

is called a **Vandermonde determinant**.[7] There is a simple formula for this determinant. If $n = 2$, it equals $(a_2 - a_1)$; if $n = 3$, it is $(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$ by Example 3.1.8. The general result is the product

$$\prod_{1 \le j < i \le n} (a_i - a_j)$$

of all factors $(a_i - a_j)$ where $1 \le j < i \le n$. For example, if $n = 4$, it is

$$(a_4 - a_3)(a_4 - a_2)(a_4 - a_1)(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$$

Theorem 3.2.7

Let $a_1, a_2, \dots, a_n$ be numbers where $n \ge 2$. Then the corresponding Vandermonde determinant is given by

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1 \le j < i \le n} (a_i - a_j)$$

Proof. We may assume that the a_i are distinct; otherwise both sides are zero. We proceed by induction on $n \ge 2$; we have it for $n = 2, 3$. So assume it holds for $n - 1$. The trick is to replace a_n

[7]Alexandre Théophile Vandermonde (1735–1796) was a French mathematician who made contributions to the theory of equations.

by a variable x, and consider the determinant

$$p(x) = \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix}$$

Then $p(x)$ is a polynomial of degree at most $n-1$ (expand along the last row), and $p(a_i) = 0$ for each $i = 1, 2, \dots, n-1$ because in each case there are two identical rows in the determinant. In particular, $p(a_1) = 0$, so we have $p(x) = (x-a_1)p_1(x)$ by the factor theorem (see Appendix ??). Since $a_2 \neq a_1$, we obtain $p_1(a_2) = 0$, and so $p_1(x) = (x-a_2)p_2(x)$. Thus $p(x) = (x-a_1)(x-a_2)p_2(x)$. As the a_i are distinct, this process continues to obtain

$$p(x) = (x-a_1)(x-a_2)\cdots(x-a_{n-1})d \tag{3.4}$$

where d is the coefficient of x^{n-1} in $p(x)$. By the cofactor expansion of $p(x)$ along the last row we get

$$d = (-1)^{n+n} \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-2} \end{bmatrix}$$

Because $(-1)^{n+n} = 1$ the induction hypothesis shows that d is the product of all factors $(a_i - a_j)$ where $1 \leq j < i \leq n-1$. The result now follows from Equation 3.4 by substituting a_n for x in $p(x)$. □

Proof of Theorem 3.2.1. If A and B are $n \times n$ matrices we must show that

$$\det(AB) = \det A \det B \tag{3.5}$$

Recall that if E is an elementary matrix obtained by doing one row operation to I_n, then doing that operation to a matrix C (Lemma 2.5.1) results in EC. By looking at the three types of elementary matrices separately, Theorem 3.1.2 shows that

$$\det(EC) = \det E \det C \quad \text{for any matrix } C \tag{3.6}$$

Thus if $E_1, E_2, \dots, E_k$ are all elementary matrices, it follows by induction that

$$\det(E_k \cdots E_2 E_1 C) = \det E_k \cdots \det E_2 \det E_1 \det C \text{ for any matrix } C \tag{3.7}$$

Lemma. If A has no inverse, then $\det A = 0$.

Proof. Let $A \to R$ where R is reduced row-echelon, say $E_n \cdots E_2 E_1 A = R$. Then R has a row of zeros by Part (4) of Theorem 2.4.5, and hence $\det R = 0$. But then Equation 3.7 gives $\det A = 0$ because $\det E \neq 0$ for any elementary matrix E. This proves the Lemma.

Now we can prove Equation 3.5 by considering two cases.

Case 1. A has no inverse. Then AB also has no inverse (otherwise $A[B(AB)^{-1}] = I$) so A is invertible by Corollary 2.4.2 to Theorem 2.4.5. Hence the above Lemma (twice) gives

$$\det(AB) = 0 = 0 \det B = \det A \det B$$

proving Equation 3.5 in this case.

Case 2. A has an inverse. Then A is a product of elementary matrices by Theorem 2.5.2, say $A = E_1E_2\cdots E_k$. Then Equation 3.7 with $C = I$ gives

$$\det A = \det (E_1E_2\cdots E_k) = \det E_1 \det E_2 \cdots \det E_k$$

But then Equation 3.7 with $C = B$ gives

$$\det (AB) = \det [(E_1E_2\cdots E_k)B] = \det E_1 \det E_2 \cdots \det E_k \det B = \det A \det B$$

and Equation 3.5 holds in this case too. □

Exercises for 3.2

Exercise 3.2.1 Find the adjugate of each of the following matrices.

a) $\begin{bmatrix} 5 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ d) $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

---

b. $\begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 6 \\ -3 & 1 & 4 \end{bmatrix}$

d. $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = A$

Exercise 3.2.2 Use determinants to find which real values of c make each of the following matrices invertible.

a) $\begin{bmatrix} 1 & 0 & 3 \\ 3 & -4 & c \\ 2 & 5 & 8 \end{bmatrix}$ b) $\begin{bmatrix} 0 & c & -c \\ -1 & 2 & 1 \\ c & -c & c \end{bmatrix}$

c) $\begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ d) $\begin{bmatrix} 4 & c & 3 \\ c & 2 & c \\ 5 & c & 4 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{bmatrix}$ f) $\begin{bmatrix} 1 & c & -1 \\ c & 1 & 1 \\ 0 & 1 & c \end{bmatrix}$

---

b. $c \neq 0$

d. any c

f. $c \neq -1$

Exercise 3.2.3 Let A, B, and C denote $n \times n$ matrices and assume that $\det A = -1$, $\det B = 2$, and $\det C = 3$. Evaluate:

a) $\det (A^3BC^TB^{-1})$ b) $\det (B^2C^{-1}AB^{-1}C^T)$

---

b. -2

Exercise 3.2.4 Let A and B be invertible $n \times n$ matrices. Evaluate:

a) $\det (B^{-1}AB)$ b) $\det (A^{-1}B^{-1}AB)$

---

b. 1

Exercise 3.2.5 If A is 3×3 and $\det(2A^{-1}) = -4$ and $\det(A^3(B^{-1})^T) = -4$, find $\det A$ and $\det B$.

Exercise 3.2.6 Let $A = \left[\begin{array}{ccc} a & b & c \\ p & q & r \\ u & v & w \end{array}\right]$ and assume that $\det A = 3$. Compute:

a. $\det(2B^{-1})$ where $B = \left[\begin{array}{ccc} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{array}\right]$

b. $\det(2C^{-1})$ where $C = \left[\begin{array}{ccc} 2p & -a+u & 3u \\ 2q & -b+v & 3v \\ 2r & -c+w & 3w \end{array}\right]$

---

b. $\frac{4}{9}$

Exercise 3.2.7 If $\det \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = -2$ calculate:

a. $\det \left[\begin{array}{ccc} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{array}\right]$

b. $\det \left[\begin{array}{ccc} 2b & 0 & 4d \\ 1 & 2 & -2 \\ a+1 & 2 & 2(c-1) \end{array}\right]$

c. $\det(3A^{-1})$ where $A = \left[\begin{array}{cc} 3c & a+c \\ 3d & b+d \end{array}\right]$

---

b. 16

Exercise 3.2.8 Solve each of the following by Cramer's rule:

a) $\begin{array}{rl} 2x + y = & 1 \\ 3x + 7y = & -2 \end{array}$

b) $\begin{array}{rl} 3x + 4y = & 9 \\ 2x - y = & -1 \end{array}$

c) $\begin{array}{rl} 5x + y - z = & -7 \\ 2x - y - 2z = & 6 \\ 3x + 2z = & -7 \end{array}$

d) $\begin{array}{rl} 4x - y + 3z = & 1 \\ 6x + 2y - z = & 0 \\ 3x + 3y + 2z = & -1 \end{array}$

---

b. $\frac{1}{11}\left[\begin{array}{c} 5 \\ 21 \end{array}\right]$

d. $\frac{1}{79}\left[\begin{array}{c} 12 \\ -37 \\ -2 \end{array}\right]$

Exercise 3.2.9 Use Theorem 3.2.4 to find the (2, 3)-entry of A^{-1} if:

a) $A = \left[\begin{array}{ccc} 3 & 2 & 1 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{array}\right]$

b) $A = \left[\begin{array}{ccc} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 0 & 4 & 7 \end{array}\right]$

---

b. $\frac{4}{51}$

Exercise 3.2.10 Explain what can be said about $\det A$ if:

a) $A^2 = A$

b) $A^2 = I$

c) $A^3 = A$

d) $PA = P$ and P is invertible

e) $A^2 = uA$ and A is $n \times n$

f) $A = -A^T$ and A is $n \times n$

g) $A^2 + I = 0$ and A is $n \times n$

---

b. $\det A = 1, -1$

d. $\det A = 1$

f. $\det A = 0$ if n is odd; nothing can be said if n is even

Exercise 3.2.11 Let A be $n \times n$. Show that $uA = (uI)A$, and use this with Theorem 3.2.1 to deduce the result in Theorem 3.1.3: $\det(uA) = u^n \det A$.

Exercise 3.2.12 If A and B are $n \times n$ matrices, if $AB = -BA$, and if n is odd, show that either A or B has no inverse.

Exercise 3.2.13 Show that $\det AB = \det BA$ holds for any two $n \times n$ matrices A and B.

Exercise 3.2.14 If $A^k = 0$ for some $k \geq 1$, show that A is not invertible.

Exercise 3.2.15 If $A^{-1} = A^T$, describe the cofactor matrix of A in terms of A.

---

dA where $d = \det A$

Exercise 3.2.16 Show that no 3×3 matrix A exists such that $A^2 + I = 0$. Find a 2×2 matrix A with this property.

Exercise 3.2.17 Show that $\det(A + B^T) = \det(A^T + B)$ for any $n \times n$ matrices A and B.

Exercise 3.2.18 Let A and B be invertible $n \times n$ matrices. Show that $\det A = \det B$ if and only if $A = UB$ where U is a matrix with $\det U = 1$.

Exercise 3.2.19 For each of the matrices in Exercise 2, find the inverse for those values of c for which it exists.

---

b. $\frac{1}{c}\begin{bmatrix} 1 & 0 & 1 \\ 0 & c & 1 \\ -1 & c & 1 \end{bmatrix}, c \neq 0$

d. $\frac{1}{2}\begin{bmatrix} 8-c^2 & -c & c^2-6 \\ c & 1 & -c \\ c^2-10 & c & 8-c^2 \end{bmatrix}$

f. $\frac{1}{c^3+1}\begin{bmatrix} 1-c & c^2+1 & -c-1 \\ c^2 & -c & c+1 \\ -c & 1 & c^2-1 \end{bmatrix}, c \neq -1$

Exercise 3.2.20 In each case either prove the statement or give an example showing that it is false:

a. If $\operatorname{adj} A$ exists, then A is invertible.

b. If A is invertible and $\operatorname{adj} A = A^{-1}$, then $\det A = 1$.

c. $\det(AB) = \det(B^T A)$.

d. If $\det A \neq 0$ and $AB = AC$, then $B = C$.

e. If $A^T = -A$, then $\det A = -1$.

f. If $\operatorname{adj} A = 0$, then $A = 0$.

g. If A is invertible, then $\operatorname{adj} A$ is invertible.

h. If A has a row of zeros, so also does $\operatorname{adj} A$.

i. $\det(A^T A) > 0$ for all square matrices A.

j. $\det(I + A) = 1 + \det A$.

k. If AB is invertible, then A and B are invertible.

l. If $\det A = 1$, then $\operatorname{adj} A = A$.

m. If A is invertible and $\det A = d$, then $\operatorname{adj} A = dA^{-1}$.

---

b. T. $\det AB = \det A \det B = \det B \det A = \det BA$.

d. T. $\det A \neq 0$ means A^{-1} exists, so $AB = AC$ implies that $B = C$.

f. F. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then $\operatorname{adj} A = 0$.

h. F. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then $\operatorname{adj} A = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$

j. F. If $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ then $\det(I + A) = -1$ but $1 + \det A = 1$.

l. F. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then $\det A = 1$ but $\operatorname{adj} A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \neq A$

Exercise 3.2.21 If A is 2×2 and $\det A = 0$, show that one column of A is a scalar multiple of the other. [*Hint*: Definition 2.5 and Part (2) of Theorem 2.4.5.]

Exercise 3.2.22 Find a polynomial $p(x)$ of degree 2 such that:

a. $p(0) = 2,\ p(1) = 3,\ p(3) = 8$

b. $p(0) = 5,\ p(1) = 3,\ p(2) = 5$

---

b. $5 - 4x + 2x^2$.

Exercise 3.2.23 Find a polynomial $p(x)$ of degree 3 such that:

a. $p(0) = p(1) = 1,\ p(-1) = 4,\ p(2) = -5$

b. $p(0) = p(1) = 1,\ p(-1) = 2,\ p(-2) = -3$

---

b. $1 - \frac{5}{3}x + \frac{1}{2}x^2 + \frac{7}{6}x^3$

Exercise 3.2.24 Given the following data pairs, find the interpolating polynomial of degree 3 and estimate the value of y corresponding to $x = 1.5$.

a. (0, 1), (1, 2), (2, 5), (3, 10)

b. (0, 1), (1, 1.49), (2, −0.42), (3, −11.33)

c. (0, 2), (1, 2.03), (2, −0.40), (−1, 0.89)

---

b. $1 - 0.51x + 2.1x^2 - 1.1x^3; 1.25$, so $y = 1.25$

Exercise 3.2.25 If $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ show that $\det A = 1 + a^2 + b^2 + c^2$. Hence, find A^{-1} for any a, b, and c.

Exercise 3.2.26

a. Show that $A = \begin{bmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{bmatrix}$ has an inverse if and only if $abc \neq 0$, and find A^{-1} in that case.

b. Show that if an upper triangular matrix is invertible, the inverse is also upper triangular.

---

b. Use induction on n where A is $n \times n$. It is clear if $n = 1$. If $n > 1$, write $A = \begin{bmatrix} a & X \\ 0 & B \end{bmatrix}$ in block form where B is $(n-1) \times (n-1)$. Then $A^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}XB^{-1} \\ 0 & B^{-1} \end{bmatrix}$, and this is upper triangular because B is upper triangular by induction.

Exercise 3.2.27 Let A be a matrix each of whose entries are integers. Show that each of the following conditions implies the other.

1. A is invertible and A^{-1} has integer entries.

2. $\det A = 1$ or -1.

Exercise 3.2.28 If $A^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$ find adj A.

$-\frac{1}{21}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$

Exercise 3.2.29 If A is 3×3 and $\det A = 2$, find $\det(A^{-1} + 4 \operatorname{adj} A)$.

Exercise 3.2.30 Show that $\det \begin{bmatrix} 0 & A \\ B & X \end{bmatrix} = \det A \det B$ when A and B are 2×2. What if A and B are 3×3? [*Hint*: Block multiply by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.]

Exercise 3.2.31 Let A be $n \times n$, $n \geq 2$, and assume one column of A consists of zeros. Find the possible values of $\operatorname{rank}(\operatorname{adj} A)$.

Exercise 3.2.32 If A is 3×3 and invertible, compute $\det(-A^2(\operatorname{adj} A)^{-1})$.

Exercise 3.2.33 Show that $\operatorname{adj}(uA) = u^{n-1} \operatorname{adj} A$ for all $n \times n$ matrices A.

Exercise 3.2.34 Let A and B denote invertible $n \times n$ matrices. Show that:

a. $\operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-2}A$ (here $n \geq 2$) [*Hint*: See Example 3.2.8.]

b. $\operatorname{adj}(A^{-1}) = (\operatorname{adj} A)^{-1}$

c. $\operatorname{adj}(A^T) = (\operatorname{adj} A)^T$

d. $\operatorname{adj}(AB) = (\operatorname{adj} B)(\operatorname{adj} A)$ [*Hint*: Show that $AB \operatorname{adj}(AB) = AB \operatorname{adj} B \operatorname{adj} A$.]

---

b. Have $(\operatorname{adj} A)A = (\det A)I$; so taking inverses, $A^{-1} \cdot (\operatorname{adj} A)^{-1} = \frac{1}{\det A}I$. On the other hand, $A^{-1} \operatorname{adj}(A^{-1}) = \det(A^{-1})I = \frac{1}{\det A}I$. Comparison yields $A^{-1}(\operatorname{adj} A)^{-1} = A^{-1} \operatorname{adj}(A^{-1})$, and part **(b)** follows.

d. Write $\det A = d$, $\det B = e$. By the adjugate formula $AB \operatorname{adj}(AB) = deI$, and $AB \operatorname{adj} B \operatorname{adj} A = A[eI] \operatorname{adj} A = (eI)(dI) = deI$. Done as AB is invertible.

3.3 Diagonalization and Eigenvalues

The world is filled with examples of systems that evolve in time—the weather in a region, the economy of a nation, the diversity of an ecosystem, etc. Describing such systems is difficult in general and various methods have been developed in special cases. In this section we describe one such method, called *diagonalization,* which is one of the most important techniques in linear algebra. A very fertile example of this procedure is in modelling the growth of the population of an animal species. This has attracted more attention in recent years with the ever increasing awareness that many species are endangered. To motivate the technique, we begin by setting up a simple model of a bird population in which we make assumptions about survival and reproduction rates.

Example 3.3.1

Consider the evolution of the population of a species of birds. Because the number of males and females are nearly equal, we count only females. We assume that each female remains a juvenile for one year and then becomes an adult, and that only adults have offspring. We make three assumptions about reproduction and survival rates:

1. The number of juvenile females hatched in any year is twice the number of adult females alive the year before (we say the **reproduction rate** is 2).
2. Half of the adult females in any year survive to the next year (the **adult survival rate** is $\frac{1}{2}$).
3. One quarter of the juvenile females in any year survive into adulthood (the **juvenile survival rate** is $\frac{1}{4}$).

If there were 100 adult females and 40 juvenile females alive initially, compute the population of females k years later.

Solution. Let a_k and j_k denote, respectively, the number of adult and juvenile females after k years, so that the total female population is the sum $a_k + j_k$. Assumption 1 shows that $j_{k+1} = 2a_k$, while assumptions 2 and 3 show that $a_{k+1} = \frac{1}{2}a_k + \frac{1}{4}j_k$. Hence the numbers a_k and j_k in successive years are related by the following equations:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{2}a_k + \tfrac{1}{4}j_k \\ j_{k+1} &= 2a_k \end{aligned}$$

If we write $\mathbf{v}_k = \left[\begin{array}{c} a_k \\ j_k \end{array} \right]$ and $A = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{array} \right]$ these equations take the matrix form

$$\mathbf{v}_{k+1} = A\mathbf{v}_k, \quad \text{for each } k = 0,\ 1,\ 2,\ \dots$$

Taking $k = 0$ gives $\mathbf{v}_1 = A\mathbf{v}_0$, then taking $k = 1$ gives $\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$, and taking $k = 2$ gives $\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$. Continuing in this way, we get

$$\mathbf{v}_k = A^k\mathbf{v}_0, \quad \text{for each } k = 0,\ 1,\ 2,\ \dots$$

Since $\mathbf{v}_0 = \begin{bmatrix} a_0 \\ j_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$ is known, finding the population profile $\mathbf{v}_k$ amounts to computing A^k for all $k \geq 0$. We will complete this calculation in Example 3.3.12 after some new techniques have been developed.

Let A be a fixed $n \times n$ matrix. A sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ of column vectors in $\mathbb{R}^n$ is called a **linear dynamical system**[8] if $\mathbf{v}_0$ is known and the other $\mathbf{v}_k$ are determined (as in Example 3.3.1) by the conditions

$$\mathbf{v}_{k+1} = A\mathbf{v}_k \text{ for each } k = 0, 1, 2, \dots$$

These conditions are called a **matrix recurrence** for the vectors $\mathbf{v}_k$. As in Example 3.3.1, they imply that

$$\mathbf{v}_k = A^k\mathbf{v}_0 \text{ for all } k \geq 0$$

so finding the columns $\mathbf{v}_k$ amounts to calculating A^k for $k \geq 0$.

Direct computation of the powers A^k of a square matrix A can be time-consuming, so we adopt an indirect method that is commonly used. The idea is to first **diagonalize** the matrix A, that is, to find an invertible matrix P such that

$$P^{-1}AP = D \text{ is a diagonal matrix} \tag{3.8}$$

This works because the powers D^k of the diagonal matrix D are easy to compute, and Equation 3.8 enables us to compute powers A^k of the matrix A in terms of powers D^k of D. Indeed, we can solve Equation 3.8 for A to get $A = PDP^{-1}$. Squaring this gives

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

Using this we can compute A^3 as follows:

$$A^3 = AA^2 = (PDP^{-1})(PD^2P^{-1}) = PD^3P^{-1}$$

Continuing in this way we obtain Theorem 3.3.1 (even if D is not diagonal).

Theorem 3.3.1

If $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$ for each $k = 1, 2, \dots$.

Hence computing A^k comes down to finding an invertible matrix P as in equation Equation 3.8. To do this it is necessary to first compute certain numbers (called eigenvalues) associated with the matrix A.

[8]More precisely, this is *a linear discrete* dynamical system. Many models regard $\mathbf{v}_t$ as a continuous function of the time t, and replace our condition between $\mathbf{b}_{k+1}$ and $A\mathbf{v}_k$ with a differential relationship viewed as functions of time.

Eigenvalues and Eigenvectors

Definition 3.4 Eigenvalues and Eigenvectors of a Matrix

If A is an $n \times n$ matrix, a number λ is called an ***eigenvalue*** *of A if*

$$A\mathbf{x} = \lambda \mathbf{x} \text{ for some column } \mathbf{x} \neq \mathbf{0} \text{ in } \mathbb{R}^n$$

In this case, $\mathbf{x}$ is called an ***eigenvector*** *of A corresponding to the eigenvalue λ, or a* ***λ-eigenvector*** *for short.*

Example 3.3.2

If $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ then $A\mathbf{x} = 4\mathbf{x}$ so $\lambda = 4$ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$.

The matrix A in Example 3.3.2 has another eigenvalue in addition to $\lambda = 4$. To find it, we develop a general procedure for *any* $n \times n$ matrix A.

By definition a number λ is an eigenvalue of the $n \times n$ matrix A if and only if $A\mathbf{x} = \lambda\mathbf{x}$ for some column $\mathbf{x} \neq \mathbf{0}$. This is equivalent to asking that the homogeneous system

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$. By Theorem 2.4.5 this happens if and only if the matrix $\lambda I - A$ is not invertible and this, in turn, holds if and only if the determinant of the coefficient matrix is zero:

$$\det(\lambda I - A) = 0$$

This last condition prompts the following definition:

Definition 3.5 Characteristic Polynomial of a Matrix

If A is an $n \times n$ matrix, the ***characteristic polynomial*** *$c_A(x)$ of A is defined by*

$$c_A(x) = \det(xI - A)$$

Note that $c_A(x)$ is indeed a polynomial in the variable x, and it has degree n when A is an $n \times n$ matrix (this is illustrated in the examples below). The above discussion shows that a number λ is an eigenvalue of A if and only if $c_A(\lambda) = 0$, that is if and only if λ is a **root** of the characteristic polynomial $c_A(x)$. We record these observations in

Theorem 3.3.2

Let A be an $n \times n$ matrix.

1. *The eigenvalues λ of A are the roots of the characteristic polynomial $c_A(x)$ of A.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nonzero solutions to the homogeneous system*

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations with $\lambda I - A$ as coefficient matrix.

In practice, solving the equations in part 2 of Theorem 3.3.2 is a routine application of gaussian elimination, but finding the eigenvalues can be difficult, often requiring computers (see Section 8.5). For now, the examples and exercises will be constructed so that the roots of the characteristic polynomials are relatively easy to find (usually integers). However, the reader should not be misled by this into thinking that eigenvalues are so easily obtained for the matrices that occur in practical applications!

Example 3.3.3

Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ discussed in Example 3.3.2, and then find all the eigenvalues and their eigenvectors.

Solution. Since $xI - A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x-3 & -5 \\ -1 & x+1 \end{bmatrix}$ we get

$$c_A(x) = \det \begin{bmatrix} x-3 & -5 \\ -1 & x+1 \end{bmatrix} = x^2 - 2x - 8 = (x-4)(x+2)$$

Hence, the roots of $c_A(x)$ are $\lambda_1 = 4$ and $\lambda_2 = -2$, so these are the eigenvalues of A. Note that $\lambda_1 = 4$ was the eigenvalue mentioned in Example 3.3.2, but we have found a new one: $\lambda_2 = -2$.

To find the eigenvectors corresponding to $\lambda_2 = -2$, observe that in this case

$$(\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} \lambda_2 - 3 & -5 \\ -1 & \lambda_2 + 1 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ -1 & -1 \end{bmatrix}$$

so the general solution to $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where t is an arbitrary real number.

Hence, the eigenvectors $\mathbf{x}$ corresponding to λ_2 are $\mathbf{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $t \neq 0$ is arbitrary.

Similarly, $\lambda_1 = 4$ gives rise to the eigenvectors $\mathbf{x} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $t \neq 0$ which includes the observation in Example 3.3.2.

Note that a square matrix A has *many* eigenvectors associated with any given eigenvalue λ.

In fact *every* nonzero solution $\mathbf{x}$ of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is an eigenvector. Recall that these solutions are all linear combinations of certain basic solutions determined by the gaussian algorithm (see Theorem 1.3.2). Observe that any nonzero multiple of an eigenvector is again an eigenvector,[9] and such multiples are often more convenient.[10] Any set of nonzero multiples of the basic solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ will be called a set of **basic eigenvectors** corresponding to λ.

Example 3.3.4

Find the characteristic polynomial, eigenvalues, and basic eigenvectors for

$$A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{array} \right]$$

Solution. Here the characteristic polynomial is given by

$$c_A(x) = \det \left[\begin{array}{ccc} x-2 & 0 & 0 \\ -1 & x-2 & 1 \\ -1 & -3 & x+2 \end{array} \right] = (x-2)(x-1)(x+1)$$

so the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$. To find all eigenvectors for $\lambda_1 = 2$, compute

$$\lambda_1 I - A = \left[\begin{array}{ccc} \lambda_1 - 2 & 0 & 0 \\ -1 & \lambda_1 - 2 & 1 \\ -1 & -3 & \lambda_1 + 2 \end{array} \right] = \left[\begin{array}{rrr} 0 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -3 & 4 \end{array} \right]$$

We want the (nonzero) solutions to $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$. The augmented matrix becomes

$$\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -3 & 4 & 0 \end{array} \right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

using row operations. Hence, the general solution $\mathbf{x}$ to $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right]$ where t is arbitrary, so we can use $\mathbf{x}_1 = \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right]$ as the basic eigenvector corresponding to $\lambda_1 = 2$. As the reader can verify, the gaussian algorithm gives basic eigenvectors $\mathbf{x}_2 = \left[\begin{array}{r} 0 \\ 1 \\ 1 \end{array} \right]$ and $\mathbf{x}_3 = \left[\begin{array}{c} 0 \\ \frac{1}{3} \\ 1 \end{array} \right]$ corresponding to $\lambda_2 = 1$ and $\lambda_3 = -1$, respectively. Note that to eliminate fractions, we could instead use $3\mathbf{x}_3 = \left[\begin{array}{r} 0 \\ 1 \\ 3 \end{array} \right]$ as the basic λ_3-eigenvector.

[9] In fact, any nonzero linear combination of λ-eigenvectors is again a λ-eigenvector.

[10] Allowing nonzero multiples helps eliminate round-off error when the eigenvectors involve fractions.

Example 3.3.5

If A is a square matrix, show that A and A^T have the same characteristic polynomial, and hence the same eigenvalues.

Solution. We use the fact that $xI - A^T = (xI - A)^T$. Then

$$c_{A^T}(x) = \det\left(xI - A^T\right) = \det\left[(xI - A)^T\right] = \det(xI - A) = c_A(x)$$

by Theorem 3.2.3. Hence $c_{A^T}(x)$ and $c_A(x)$ have the same roots, and so A^T and A have the same eigenvalues (by Theorem 3.3.2).

The eigenvalues of a matrix need not be distinct. For example, if $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$ the characteristic polynomial is $(x-1)^2$ so the eigenvalue 1 occurs twice. Furthermore, eigenvalues are usually not computed as the roots of the characteristic polynomial. There are iterative, numerical methods (for example the QR-algorithm in Section 8.5) that are much more efficient for large matrices.

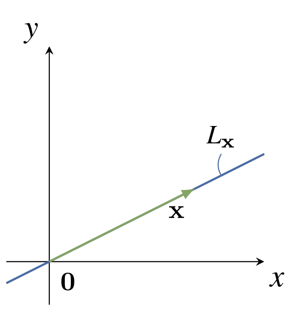
A-Invariance

If A is a 2×2 matrix, we can describe the eigenvectors of A geometrically using the following concept. A line L through the origin in $\mathbb{R}^2$ is called **A-invariant** if $A\mathbf{x}$ is in L whenever $\mathbf{x}$ is in L. If we think of A as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, this asks that A carries L into itself, that is the image $A\mathbf{x}$ of each vector $\mathbf{x}$ in L is again in L.

Example 3.3.6

The x axis $L = \left\{ \left[\begin{array}{c} x \\ 0 \end{array}\right] \mid x \text{ in } \mathbb{R} \right\}$ is A-invariant for any matrix of the form

$$A = \left[\begin{array}{cc} a & b \\ 0 & c \end{array}\right] \text{ because } \left[\begin{array}{cc} a & b \\ 0 & c \end{array}\right]\left[\begin{array}{c} x \\ 0 \end{array}\right] = \left[\begin{array}{c} ax \\ 0 \end{array}\right] \text{ is } L \text{ for all } \mathbf{x} = \left[\begin{array}{c} x \\ 0 \end{array}\right] \text{ in } L$$



To see the connection with eigenvectors, let $\mathbf{x} \neq \mathbf{0}$ be any nonzero vector in $\mathbb{R}^2$ and let $L_\mathbf{x}$ denote the unique line through the origin containing $\mathbf{x}$ (see the diagram). By the definition of scalar multiplication in Section 2.6, we see that $L_\mathbf{x}$ consists of all scalar multiples of $\mathbf{x}$, that is

$$L_\mathbf{x} = \mathbb{R}\mathbf{x} = \{t\mathbf{x} \mid t \text{ in } \mathbb{R}\}$$

Now suppose that $\mathbf{x}$ is an eigenvector of A, say $A\mathbf{x} = \lambda\mathbf{x}$ for some λ in $\mathbb{R}$. Then if $t\mathbf{x}$ is in $L_\mathbf{x}$ then

$$A(t\mathbf{x}) = t(A\mathbf{x}) = t(\lambda\mathbf{x}) = (t\lambda)\mathbf{x} \text{ is again in } L_\mathbf{x}$$

That is, $L_\mathbf{x}$ is A-invariant. On the other hand, if $L_\mathbf{x}$ is A-invariant then $A\mathbf{x}$ is in $L_\mathbf{x}$ (since $\mathbf{x}$ is in $L_\mathbf{x}$). Hence $A\mathbf{x} = t\mathbf{x}$ for some t in $\mathbb{R}$, so $\mathbf{x}$ is an eigenvector for A (with eigenvalue t). This proves:

Theorem 3.3.3

Let A be a 2×2 matrix, let $\mathbf{x} \neq \mathbf{0}$ be a vector in $\mathbb{R}^2$, and let $L_{\mathbf{x}}$ be the line through the origin in $\mathbb{R}^2$ containing $\mathbf{x}$. Then

$$\mathbf{x} \text{ is an eigenvector of } A \quad \text{if and only if} \quad L_{\mathbf{x}} \text{ is } A\text{-invariant}$$

Example 3.3.7

1. If θ is not a multiple of π, show that $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has no real eigenvalue.

2. If m is real show that $B = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$ has a 1 as an eigenvalue.

Solution.

1. A induces rotation about the origin through the angle θ (Theorem 2.6.4). Since θ is not a multiple of π, this shows that no line through the origin is A-invariant. Hence A has no eigenvector by Theorem 3.3.3, and so has no eigenvalue.

2. B induces reflection Q_m in the line through the origin with slope m by Theorem 2.6.5. If $\mathbf{x}$ is any nonzero point on this line then it is clear that $Q_m\mathbf{x} = \mathbf{x}$, that is $Q_m\mathbf{x} = 1\mathbf{x}$. Hence 1 is an eigenvalue (with eigenvector $\mathbf{x}$).

If $\theta = \frac{\pi}{2}$ in Example 3.3.7, then $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so $c_A(x) = x^2 + 1$. This polynomial has no root in $\mathbb{R}$, so A has no (real) eigenvalue, and hence no eigenvector. In fact its eigenvalues are the complex numbers i and $-i$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ In other words, A *has* eigenvalues and eigenvectors, just not real ones.

Note that *every* polynomial has complex roots,[11] so every matrix has complex eigenvalues. While these eigenvalues may very well be real, this suggests that we really should be doing linear algebra over the complex numbers. Indeed, everything we have done (gaussian elimination, matrix algebra, determinants, etc.) works if all the scalars are complex.

[11] This is called the *Fundamental Theorem of Algebra* and was first proved by Gauss in his doctoral dissertation.

Diagonalization

An $n \times n$ matrix D is called a **diagonal matrix** if all its entries off the main diagonal are zero, that is if D has the form

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \text{diag}\,(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n)$$

where $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ are numbers. Calculations with diagonal matrices are very easy. Indeed, if $D = \text{diag}\,(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n)$ and $E = \text{diag}\,(\mu_1,\ \mu_2,\ \ldots,\ \mu_n)$ are two diagonal matrices, their product DE and sum $D+E$ are again diagonal, and are obtained by doing the same operations to corresponding diagonal elements:

$$\begin{aligned} DE &= \text{diag}\,(\lambda_1\mu_1,\ \lambda_2\mu_2,\ \ldots,\ \lambda_n\mu_n) \\ D+E &= \text{diag}\,(\lambda_1+\mu_1,\ \lambda_2+\mu_2,\ \ldots,\ \lambda_n+\mu_n) \end{aligned}$$

Because of the simplicity of these formulas, and with an eye on Theorem 3.3.1 and the discussion preceding it, we make another definition:

Definition 3.6 Diagonalizable Matrices

An $n \times n$ matrix A is called ***diagonalizable*** *if*

$$P^{-1}AP \text{ is diagonal for some invertible } n \times n \text{ matrix } P$$

Here the invertible matrix P is called a ***diagonalizing matrix*** *for A.*

To discover when such a matrix P exists, we let $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n$ denote the columns of P and look for ways to determine when such $\mathbf{x}_i$ exist and how to compute them. To this end, write P in terms of its columns as follows:

$$P = [\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots,\ \mathbf{x}_n]$$

Observe that $P^{-1}AP = D$ for some diagonal matrix D holds if and only if

$$AP = PD$$

If we write $D = \text{diag}\,(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n)$, where the λ_i are numbers to be determined, the equation $AP = PD$ becomes

$$A\,[\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots,\ \mathbf{x}_n] = [\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots,\ \mathbf{x}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

By the definition of matrix multiplication, each side simplifies as follows

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \end{bmatrix}$$

Comparing columns shows that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for each i, so

$$P^{-1}AP = D \quad \text{ if and only if } A\mathbf{x}_i = \lambda_i\mathbf{x}_i \text{ for each } i$$

In other words, $P^{-1}AP = D$ holds if and only if the diagonal entries of D are eigenvalues of A and the columns of P are corresponding eigenvectors. This proves the following fundamental result.

Theorem 3.3.4

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ such that the matrix $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{bmatrix}$ is invertible.*
2. *When this is the case, $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where, for each i, λ_i is the eigenvalue of A corresponding to $\mathbf{x}_i$.*

Example 3.3.8

Diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$ in Example 3.3.4.

Solution. By Example 3.3.4, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, with corresponding basic eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ respectively. Since the matrix $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ is invertible, Theorem 3.3.4 guarantees that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

The reader can verify this directly—easier to check $AP = PD$.

In Example 3.3.8, suppose we let $Q = \begin{bmatrix} \mathbf{x}_2 & \mathbf{x}_1 & \mathbf{x}_3 \end{bmatrix}$ be the matrix formed from the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ of A, but in a *different order* than that used to form P. Then $Q^{-1}AQ = \text{diag}(\lambda_2, \lambda_1, \lambda_3)$ is diagonal by Theorem 3.3.4, but the eigenvalues are in the *new* order. Hence we can choose the diagonalizing matrix P so that the eigenvalues λ_i appear in any order we want along the main diagonal of D.

In every example above each eigenvalue has had only one basic eigenvector. Here is a diagonalizable matrix where this is not the case.

Example 3.3.9

Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution. To compute the characteristic polynomial of A first add rows 2 and 3 of $xI - A$ to row 1:

$$\begin{aligned} c_A(x) &= \det \begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} = \det \begin{bmatrix} x-2 & x-2 & x-2 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} \\ &= \det \begin{bmatrix} x-2 & 0 & 0 \\ -1 & x+1 & 0 \\ -1 & 0 & x+1 \end{bmatrix} = (x-2)(x+1)^2 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, with λ_2 repeated twice (we say that λ_2 has *multiplicity* two). However, A *is* diagonalizable. For $\lambda_1 = 2$, the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ has general solution $\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the reader can verify, so a basic λ_1-eigenvector is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Turning to the repeated eigenvalue $\lambda_2 = -1$, we must solve $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$. By gaussian elimination, the general solution is $\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where s and t are arbitrary. Hence the gaussian algorithm produces *two* basic λ_2-eigenvectors $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ If we take $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ we find that P is invertible. Hence $P^{-1}AP = \text{diag}(2, -1, -1)$ by Theorem 3.3.4.

Example 3.3.9 typifies every diagonalizable matrix. To describe the general case, we need some terminology.

Definition 3.7 Multiplicity of an Eigenvalue

An eigenvalue λ of a square matrix A is said to have **multiplicity** *m if it occurs m times as a root of the characteristic polynomial $c_A(x)$.*

For example, the eigenvalue $\lambda_2 = -1$ in Example 3.3.9 has multiplicity 2. In that example the gaussian algorithm yields two basic λ_2-eigenvectors, the same number as the multiplicity. This works in general.

Theorem 3.3.5

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors; that is, if and only if the general solution of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has exactly m parameters.

One case of Theorem 3.3.5 deserves mention.

Theorem 3.3.6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The proofs of Theorem 3.3.5 and Theorem 3.3.6 require more advanced techniques and are given in Chapter 5. The following procedure summarizes the method.

Diagonalization Algorithm

To diagonalize an $n \times n$ matrix A:

Step 1. Find the distinct eigenvalues λ of A.

Step 2. Compute a set of basic eigenvectors corresponding to each of these eigenvalues λ as basic solutions of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Step 3. The matrix A is diagonalizable if and only if there are n basic eigenvectors in all.

Step 4. If A is diagonalizable, the $n \times n$ matrix P with these basic eigenvectors as its columns is a diagonalizing matrix for A, that is, P is invertible and $P^{-1}AP$ is diagonal.

The diagonalization algorithm is valid even if the eigenvalues are nonreal complex numbers. In this case the eigenvectors will also have complex entries, but we will not pursue this here.

Example 3.3.10

Show that $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$ is not diagonalizable.

Solution 1. The characteristic polynomial is $c_A(x) = (x-1)^2$, so A has only one eigenvalue $\lambda_1 = 1$ of multiplicity 2. But the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ has general solution $t \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$, so there is only one parameter, and so only one basic eigenvector $\left[\begin{array}{c} 1 \\ 2 \end{array} \right]$. Hence A is not diagonalizable.

Solution 2. We have $c_A(x) = (x-1)^2$ so the only eigenvalue of A is $\lambda = 1$. Hence, if A were diagonalizable, Theorem 3.3.4 would give $P^{-1}AP = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I$ for some invertible matrix

P. But then $A = PIP^{-1} = I$, which is not the case. So A cannot be diagonalizable.

Diagonalizable matrices share many properties of their eigenvalues. The following example illustrates why.

Example 3.3.11

If $\lambda^3 = 5\lambda$ for every eigenvalue of the diagonalizable matrix A, show that $A^3 = 5A$.

Solution. Let $P^{-1}AP = D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Because $\lambda_i^3 = 5\lambda_i$ for each i, we obtain

$$D^3 = \text{diag}(\lambda_1^3, \dots, \lambda_n^3) = \text{diag}(5\lambda_1, \dots, 5\lambda_n) = 5D$$

Hence $A^3 = (PDP^{-1})^3 = PD^3P^{-1} = P(5D)P^{-1} = 5(PDP^{-1}) = 5A$ using Theorem 3.3.1. This is what we wanted.

If $p(x)$ is any polynomial and $p(\lambda) = 0$ for every eigenvalue of the diagonalizable matrix A, an argument similar to that in Example 3.3.11 shows that $p(A) = 0$. Thus Example 3.3.11 deals with the case $p(x) = x^3 - 5x$. In general, $p(A)$ is called the *evaluation* of the polynomial $p(x)$ at the matrix A. For example, if $p(x) = 2x^3 - 3x + 5$, then $p(A) = 2A^3 - 3A + 5I$—note the use of the identity matrix.

In particular, if $c_A(x)$ denotes the characteristic polynomial of A, we certainly have $c_A(\lambda) = 0$ for each eigenvalue λ of A (Theorem 3.3.2). Hence $c_A(A) = 0$ for every diagonalizable matrix A. This is, in fact, true for *any* square matrix, diagonalizable or not, and the general result is called the Cayley-Hamilton theorem. It is proved in Section **??** and again in Section **??**.

Linear Dynamical Systems

We began Section 3.3 with an example from ecology which models the evolution of the population of a species of birds as time goes on. As promised, we now complete the example—Example 3.3.12 below.

The bird population was described by computing the female population profile $\mathbf{v}_k = \begin{bmatrix} a_k \\ j_k \end{bmatrix}$ of the species, where a_k and j_k represent the number of adult and juvenile females present k years after the initial values a_0 and j_0 were observed. The model assumes that these numbers are related by the following equations:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{2}a_k + \tfrac{1}{4}j_k \\ j_{k+1} &= 2a_k \end{aligned}$$

If we write $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$ the columns $\mathbf{v}_k$ satisfy $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for each $k = 0, 1, 2, \dots$.

Hence $\mathbf{v}_k = A^k\mathbf{v}_0$ for each $k = 1, 2, \dots$. We can now use our diagonalization techniques to determine the population profile $\mathbf{v}_k$ for all values of k in terms of the initial values.

Example 3.3.12

Assuming that the initial values were $a_0 = 100$ adult females and $j_0 = 40$ juvenile females, compute a_k and j_k for $k = 1, 2, \dots$.

Solution. The characteristic polynomial of the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$ is $c_A(x) = x^2 - \frac{1}{2}x - \frac{1}{2} = (x-1)(x+\frac{1}{2})$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$ and gaussian elimination gives corresponding basic eigenvectors $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$. For convenience, we can use multiples $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ respectively. Hence a diagonalizing matrix is $P = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and we obtain

$$P^{-1}AP = D \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

This gives $A = PDP^{-1}$ so, for each $k \geq 0$, we can compute A^k explicitly:

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^k \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{bmatrix} \end{aligned}$$

Hence we obtain

$$\begin{aligned} \begin{bmatrix} a_k \\ j_k \end{bmatrix} = \mathbf{v}_k = A^k\mathbf{v}_0 &= \frac{1}{6} \begin{bmatrix} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} 100 \\ 40 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 440+160(-\frac{1}{2})^k \\ 880-640(-\frac{1}{2})^k \end{bmatrix} \end{aligned}$$

Equating top and bottom entries, we obtain exact formulas for a_k and j_k:

$$a_k = \frac{220}{3} + \frac{80}{3}\left(-\frac{1}{2}\right)^k \text{ and } j_k = \frac{440}{3} + \frac{320}{3}\left(-\frac{1}{2}\right)^k \text{ for } k = 1, 2, \cdots$$

In practice, the exact values of a_k and j_k are not usually required. What is needed is a measure of how these numbers behave for large values of k. This is easy to obtain here. Since $(-\frac{1}{2})^k$ is nearly zero for large k, we have the following approximate values

$$a_k \approx \frac{220}{3} \text{ and } j_k \approx \frac{440}{3} \text{ if } k \text{ is large}$$

Hence, in the long term, the female population stabilizes with approximately twice as many juveniles as adults.

Definition 3.8 Linear Dynamical System

If A is an $n \times n$ matrix, a sequence $\mathbf{v}_0,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \ldots$ of columns in $\mathbb{R}^n$ is called a ***linear dynamical system*** *if $\mathbf{v}_0$ is specified and $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots$ are given by the matrix recurrence $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for each $k \geq 0$. We call A the* ***migration*** *matrix of the system.*

We have $\mathbf{v}_1 = A\mathbf{v}_0$, then $\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$, and continuing we find

$$\mathbf{v}_k = A^k\mathbf{v}_0 \text{ for each } k = 1,\ 2,\ \cdots \tag{3.9}$$

Hence the columns $\mathbf{v}_k$ are determined by the powers A^k of the matrix A and, as we have seen, these powers can be efficiently computed if A is diagonalizable. In fact Equation 3.9 can be used to give a nice "formula" for the columns $\mathbf{v}_k$ in this case.

Assume that A is diagonalizable with eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ and corresponding basic eigenvectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n$. If $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{array}\right]$ is a diagonalizing matrix with the $\mathbf{x}_i$ as columns, then P is invertible and

$$P^{-1}AP = D = \text{diag}\,(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n)$$

by Theorem 3.3.4. Hence $A = PDP^{-1}$ so Equation 3.9 and Theorem 3.3.1 give

$$\mathbf{v}_k = A^k\mathbf{v}_0 = (PDP^{-1})^k\mathbf{v}_0 = (PD^kP^{-1})\mathbf{v}_0 = PD^k(P^{-1}\mathbf{v}_0)$$

for each $k = 1,\ 2,\ \ldots$. For convenience, we denote the column $P^{-1}\mathbf{v}_0$ arising here as follows:

$$\mathbf{b} = P^{-1}\mathbf{v}_0 = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right]$$

Then matrix multiplication gives

$$\begin{aligned}
\mathbf{v}_k &= PD^k(P^{-1}\mathbf{v}_0) \\
&= \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array}\right] \left[\begin{array}{cccc} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{array}\right] \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right] \\
&= \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array}\right] \left[\begin{array}{c} b_1\lambda_1^k \\ b_2\lambda_2^k \\ \vdots \\ b_3\lambda_n^k \end{array}\right] \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n
\end{aligned} \tag{3.10}$$

for each $k \geq 0$. This is a useful **exact formula** for the columns $\mathbf{v}_k$. Note that, in particular,

$$\mathbf{v}_0 = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n$$

However, such an exact formula for $\mathbf{v}_k$ is often not required in practice; all that is needed is to *estimate* $\mathbf{v}_k$ for large values of k (as was done in Example 3.3.12). This can be easily done if A has a largest eigenvalue. An eigenvalue λ of a matrix A is called a **dominant eigenvalue** of A if it has multiplicity 1 and

$$|\lambda| > |\mu| \text{ for all eigenvalues } \mu \neq \lambda$$

where $|\lambda|$ denotes the absolute value of the number λ. For example, $\lambda_1 = 1$ is dominant in Example 3.3.12.

Returning to the above discussion, suppose that A has a dominant eigenvalue. By choosing the order in which the columns $\mathbf{x}_i$ are placed in P, we may assume that λ_1 is dominant among the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A (see the discussion following Example 3.3.8). Now recall the exact expression for $\mathbf{v}_k$ in Equation 3.10 above:

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n$$

Take λ_1^k out as a common factor in this equation to get

$$\mathbf{v}_k = \lambda_1^k\left[b_1\mathbf{x}_1 + b_2\left(\tfrac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + \cdots + b_n\left(\tfrac{\lambda_n}{\lambda_1}\right)^k\mathbf{x}_n\right]$$

for each $k \geq 0$. Since λ_1 is dominant, we have $|\lambda_i| < |\lambda_1|$ for each $i \geq 2$, so each of the numbers $(\lambda_i/\lambda_1)^k$ become small in absolute value as k increases. Hence $\mathbf{v}_k$ is approximately equal to the first term $\lambda_1^k b_1\mathbf{x}_1$, and we write this as $\mathbf{v}_k \approx \lambda_1^k b_1\mathbf{x}_1$. These observations are summarized in the following theorem (together with the above exact formula for $\mathbf{v}_k$).

Theorem 3.3.7

Consider the dynamical system $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ *with matrix recurrence*

$$\mathbf{v}_{k+1} = A\mathbf{v}_k \text{ for } k \geq 0$$

where A *and* $\mathbf{v}_0$ *are given. Assume that* A *is a diagonalizable* $n \times n$ *matrix with eigenvalues* $\lambda_1, \lambda_2, \dots, \lambda_n$ *and corresponding basic eigenvectors* $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$*, and let* $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{array}\right]$ *be the diagonalizing matrix. Then an exact formula for* $\mathbf{v}_k$ *is*

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \text{ for each } k \geq 0$$

where the coefficients b_i *come from*

$$\mathbf{b} = P^{-1}\mathbf{v}_0 = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right]$$

Moreover, if A *has dominant*[12] *eigenvalue* λ_1*, then* $\mathbf{v}_k$ *is approximated by*

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 \text{ for sufficiently large } k.$$

Example 3.3.13

Returning to Example 3.3.12, we see that $\lambda_1 = 1$ is the dominant eigenvalue, with eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Here $P = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{v}_0 = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$ so $P^{-1}\mathbf{v}_0 = \frac{1}{3}\begin{bmatrix} 220 \\ -80 \end{bmatrix}$. Hence $b_1 = \frac{220}{3}$ in the notation of Theorem 3.3.7, so

$$\begin{bmatrix} a_k \\ j_k \end{bmatrix} = \mathbf{v}_k \approx b_1\lambda_1^k\mathbf{x}_1 = \frac{220}{3}1^k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where k is large. Hence $a_k \approx \frac{220}{3}$ and $j_k \approx \frac{440}{3}$ as in Example 3.3.12.

This next example uses Theorem 3.3.7 to solve a "linear recurrence." See also Section **??**.

Example 3.3.14

Suppose a sequence $x_0,\ x_1,\ x_2,\ \dots$ is determined by insisting that

$$x_0 = 1,\ x_1 = -1, \quad \text{and } x_{k+2} = 2x_k - x_{k+1} \text{ for every } k \geq 0$$

Find a formula for x_k in terms of k.

Solution. Using the linear recurrence $x_{k+2} = 2x_k - x_{k+1}$ repeatedly gives

$$x_2 = 2x_0 - x_1 = 3, \quad x_3 = 2x_1 - x_2 = -5, \quad x_4 = 11, \quad x_5 = -21,\ \dots$$

so the x_i are determined but no pattern is apparent. The idea is to find $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ for each k instead, and then retrieve x_k as the top component of $\mathbf{v}_k$. The reason this works is that the linear recurrence guarantees that these $\mathbf{v}_k$ are a dynamical system:

$$\mathbf{v}_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 2x_k - x_{k+1} \end{bmatrix} = A\mathbf{v}_k \text{ where } A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 1$ with eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the diagonalizing matrix is $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$.

Moreover, $\mathbf{b} = P_0^{-1}\mathbf{v}_0 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ so the exact formula for $\mathbf{v}_k$ is

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \frac{2}{3}(-2)^k\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{1}{3}1^k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Equating top entries gives the desired formula for x_k:

$$x_k = \frac{1}{3}\left[2(-2)^k + 1\right] \text{ for all } k = 0,\ 1,\ 2,\ \dots$$

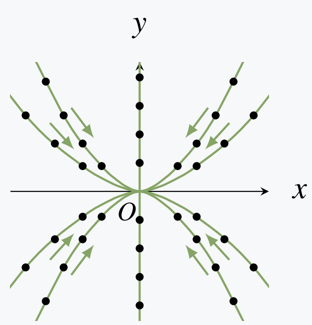
The reader should check this for the first few values of k.

[12]Similar results can be found in other situations. If for example, eigenvalues λ_1 and λ_2 (possibly equal) satisfy $|\lambda_1| = |\lambda_2| > |\lambda_i|$ for all $i > 2$, then we obtain $\mathbf{v}_k \approx b_1\lambda_1^k x_1 + b_2\lambda_2^k x_2$ for large k.

Graphical Description of Dynamical Systems

If a dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ is given, the sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ is called the **trajectory** of the system starting at $\mathbf{v}_0$. It is instructive to obtain a graphical plot of the system by writing $\mathbf{v}_k = \left[\begin{array}{c} x_k \\ y_k \end{array} \right]$ and plotting the successive values as points in the plane, identifying $\mathbf{v}_k$ with the point (x_k, y_k) in the plane. We give several examples which illustrate properties of dynamical systems. For ease of calculation we assume that the matrix A is simple, usually diagonal.

Example 3.3.15

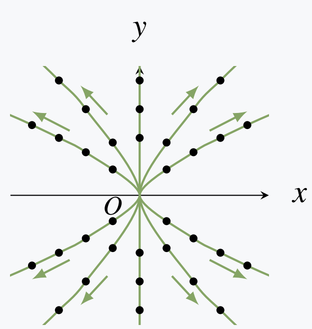


Let $A = \left[\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{array} \right]$ Then the eigenvalues are $\frac{1}{2}$ and $\frac{1}{3}$, with corresponding eigenvectors $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$. The exact formula is

$$\mathbf{v}_k = b_1 \left(\tfrac{1}{2}\right)^k \left[\begin{array}{c} 1 \\ 0 \end{array} \right] + b_2 \left(\tfrac{1}{3}\right)^k \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$$

for $k = 0, 1, 2, \dots$ by Theorem 3.3.7, where the coefficients b_1 and b_2 depend on the initial point $\mathbf{v}_0$. Several trajectories are plotted in the diagram and, for each choice of $\mathbf{v}_0$, the trajectories converge toward the origin because both eigenvalues are less than 1 in absolute value. For this reason, the origin is called an **attractor** for the system.

Example 3.3.16

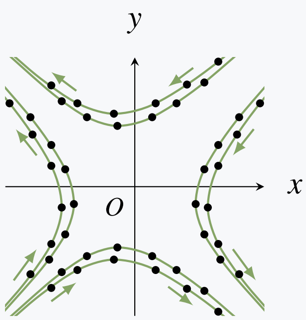


Let $A = \left[\begin{array}{cc} \frac{3}{2} & 0 \\ 0 & \frac{4}{3} \end{array} \right]$. Here the eigenvalues are $\frac{3}{2}$ and $\frac{4}{3}$, with corresponding eigenvectors $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$ as before. The exact formula is

$$\mathbf{v}_k = b_1 \left(\tfrac{3}{2}\right)^k \left[\begin{array}{c} 1 \\ 0 \end{array} \right] + b_2 \left(\tfrac{4}{3}\right)^k \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$$

for $k = 0, 1, 2, \dots$. Since both eigenvalues are greater than 1 in absolute value, the trajectories diverge away from the origin for every choice of initial point V_0. For this reason, the origin is called a **repellor** for the system.

Example 3.3.17



Let $A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$. Now the eigenvalues are $\frac{3}{2}$ and $\frac{1}{2}$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The exact formula is

$$\mathbf{v}_k = b_1 \left(\frac{3}{2}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b_2 \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

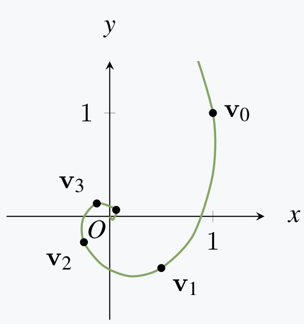
for $k = 0, 1, 2, \dots$. In this case $\frac{3}{2}$ is the dominant eigenvalue so, if $b_1 \neq 0$, we have $\mathbf{v}_k \approx b_1 \left(\frac{3}{2}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for large k and $\mathbf{v}_k$ is approaching the line $y = -x$.

However, if $b_1 = 0$, then $\mathbf{v}_k = b_2 \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and so approaches the origin along the line $y = x$. In general the trajectories appear as in the diagram, and the origin is called a **saddle point** for the dynamical system in this case.

Example 3.3.18

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$. Now the characteristic polynomial is $c_A(x) = x^2 + \frac{1}{4}$, so the eigenvalues are the complex numbers $\frac{i}{2}$ and $-\frac{i}{2}$ where $i^2 = -1$. Hence A is not diagonalizable as a real matrix. However, the trajectories are not difficult to describe. If we start with $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then the trajectory begins as

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{8} \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{16} \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} \frac{1}{32} \\ -\frac{1}{32} \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} -\frac{1}{64} \\ -\frac{1}{64} \end{bmatrix}, \dots$$



The first five of these points are plotted in the diagram. Here each trajectory spirals in toward the origin, so the origin is an attractor. Note that the two (complex) eigenvalues have absolute value less than 1 here. If they had absolute value greater than 1, the trajectories would spiral out from the origin.

Google PageRank

Dominant eigenvalues are useful to the Google search engine for finding information on the Web. If an information query comes in from a client, Google has a sophisticated method of establishing the "relevance" of each site to that query. When the relevant sites have been determined, they are placed in order of importance using a ranking of *all* sites called the PageRank. The relevant sites with the highest PageRank are the ones presented to the client. It is the construction of the PageRank that is our interest here.

The Web contains many links from one site to another. Google interprets a link from site j to site i as a "vote" for the importance of site i. Hence if site i has more links to it than does site j, then i is regarded as more "important" and assigned a higher PageRank. One way to look at this is to view the sites as vertices in a huge directed graph (see Section 2.2). Then if site j links to site i there is an edge from j to i, and hence the (i, j)-entry is a 1 in the associated adjacency matrix (called the *connectivity* matrix in this context). Thus a large number of 1s in row i of this matrix is a measure of the PageRank of site i.[13]

However this does not take into account the PageRank of the sites that link to i. Intuitively, the higher the rank of these sites, the higher the rank of site i. One approach is to compute a dominant eigenvector $\mathbf{x}$ for the connectivity matrix. In most cases the entries of $\mathbf{x}$ can be chosen to be positive with sum 1. Each site corresponds to an entry of $\mathbf{x}$, so the sum of the entries of sites linking to a given site i is a measure of the rank of site i. In fact, Google chooses the PageRank of a site so that it is proportional to this sum.[14]

Exercises for 3.3

Exercise 3.3.1 In each case find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix P such that $P^{-1}AP$ is diagonal.

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

c) $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$

e) $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$

f) $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

g) $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$

h) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

i) $A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$, $\lambda \neq \mu$

---

b. $(x-3)(x+2); 3; -2; \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix};$
$P = \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}; P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$

[13]For more on PageRank, visit https://en.wikipedia.org/wiki/PageRank.

[14]See the articles "Searching the web with eigenvectors" by Herbert S. Wilf, UMAP Journal 23(2), 2002, pages 101–103, and "The worlds largest matrix computation: Google's PageRank is an eigenvector of a matrix of order 2.7 billion" by Cleve Moler, Matlab News and Notes, October 2002, pages 12–13.

d. $(x-2)^3; 2; \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$; No such P; Not diagonalizable.

f. $(x+1)^2(x-2); -1, \ -2; \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$; No such P; Not diagonalizable. Note that this matrix and the matrix in Example 3.3.9 have the same characteristic polynomial, but that matrix is diagonalizable.

h. $(x-1)^2(x-3); 1, \ 3; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ No such P; Not diagonalizable.

Exercise 3.3.2 Consider a linear dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for $k \geq 0$. In each case approximate $\mathbf{v}_k$ using Theorem 3.3.7.

a. $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

---

b. $V_k = \frac{7}{3} 2^k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

d. $V_k = \frac{3}{2} 3^k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Exercise 3.3.3 Show that A has $\lambda = 0$ as an eigenvalue if and only if A is not invertible.

Exercise 3.3.4 Let A denote an $n \times n$ matrix and put $A_1 = A - \alpha I$, α in $\mathbb{R}$. Show that λ is an eigenvalue of A if and only if $\lambda - \alpha$ is an eigenvalue of A_1. (Hence, the eigenvalues of A_1 are just those of A "shifted" by α.) How do the eigenvectors compare?

$A\mathbf{x} = \lambda \mathbf{x}$ if and only if $(A - \alpha I)\mathbf{x} = (\lambda - \alpha)\mathbf{x}$. Same eigenvectors.

Exercise 3.3.5 Show that the eigenvalues of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $e^{i\theta}$ and $e^{-i\theta}$. (See Appendix ??)

Exercise 3.3.6 Find the characteristic polynomial of the $n \times n$ identity matrix I. Show that I has exactly one eigenvalue and find the eigenvectors.

Exercise 3.3.7 Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that:

a. $c_A(x) = x^2 - \operatorname{tr} Ax + \det A$, where $\operatorname{tr} A = a + d$ is called the **trace** of A.

b. The eigenvalues are $\frac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\right]$.

Exercise 3.3.8 In each case, find $P^{-1}AP$ and then compute A^n.

a. $A = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}, P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} -7 & -12 \\ 6 & -10 \end{bmatrix}, P = \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}$ [*Hint*: $(PDP^{-1})^n = PD^nP^{-1}$ for each $n = 1, \ 2, \ \dots$.]

---

b. $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so $A^n = P \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} P^{-1} = \begin{bmatrix} 9 - 8 \cdot 2^n & 12(1-2^n) \\ 6(2^n - 1) & 9 \cdot 2^n - 8 \end{bmatrix}$

Exercise 3.3.9

a. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ verify that A and B are diagonalizable, but AB is not.

b. If $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ find a diagonalizable matrix A such that $D + A$ is not diagonalizable.

---

b. $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

Exercise 3.3.10 If A is an $n \times n$ matrix, show that A is diagonalizable if and only if A^T is diagonalizable.

Exercise 3.3.11 If A is diagonalizable, show that each of the following is also diagonalizable.

a. A^n, $n \geq 1$

b. kA, k any scalar.

c. $p(A)$, $p(x)$ any polynomial (Theorem 3.3.1)

d. $U^{-1}AU$ for any invertible matrix U.

e. $kI + A$ for any scalar k.

---

b. and d. $PAP^{-1} = D$ is diagonal, then b. $P^{-1}(kA)P = kD$ is diagonal, and d. $Q(U^{-1}AU)Q = D$ where $Q = PU$.

Exercise 3.3.12 Give an example of two diagonalizable matrices A and B whose sum $A + B$ is not diagonalizable.

$\left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$ is not diagonalizable by Example 3.3.8. But $\left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right] = \left[\begin{array}{rr} 2 & 1 \\ 0 & -1 \end{array} \right] + \left[\begin{array}{rr} -1 & 0 \\ 0 & 2 \end{array} \right]$ where $\left[\begin{array}{rr} 2 & 1 \\ 0 & -1 \end{array} \right]$ has diagonalizing matrix $P = \left[\begin{array}{rr} 1 & -1 \\ 0 & 3 \end{array} \right]$ and $\left[\begin{array}{rr} -1 & 0 \\ 0 & 2 \end{array} \right]$ is already diagonal.

Exercise 3.3.13 If A is diagonalizable and 1 and -1 are the only eigenvalues, show that $A^{-1} = A$.

Exercise 3.3.14 If A is diagonalizable and 0 and 1 are the only eigenvalues, show that $A^2 = A$.

We have $\lambda^2 = \lambda$ for every eigenvalue λ (as $\lambda = 0, 1$) so $D^2 = D$, and so $A^2 = A$ as in Example 3.3.9.

Exercise 3.3.15 If A is diagonalizable and $\lambda \geq 0$ for each eigenvalue of A, show that $A = B^2$ for some matrix B.

Exercise 3.3.16 If $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, show that $AB = BA$. [*Hint*: Diagonal matrices commute.]

Exercise 3.3.17 A square matrix A is called **nilpotent** if $A^n = 0$ for some $n \geq 1$. Find all nilpotent diagonalizable matrices. [*Hint*: Theorem 3.3.1.]

Exercise 3.3.18 Let A be any $n \times n$ matrix and $r \neq 0$ a real number.

a. Show that the eigenvalues of rA are precisely the numbers $r\lambda$, where λ is an eigenvalue of A.

b. Show that $c_{rA}(x) = r^n c_A\left(\frac{x}{r}\right)$.

---

b. $c_{rA}(x) = \det\left[xI - rA\right]$
$= r^n \det\left[\frac{x}{r}I - A\right] = r^n c_A\left[\frac{x}{r}\right]$

Exercise 3.3.19

a. If all rows of A have the same sum s, show that s is an eigenvalue.

b. If all columns of A have the same sum s, show that s is an eigenvalue.

Exercise 3.3.20 Let A be an invertible $n \times n$ matrix.

a. Show that the eigenvalues of A are nonzero.

b. Show that the eigenvalues of A^{-1} are precisely the numbers $1/\lambda$, where λ is an eigenvalue of A.

c. Show that $c_{A^{-1}}(x) = \frac{(-x)^n}{\det A} c_A\left(\frac{1}{x}\right)$.

---

b. If $\lambda \neq 0$, $A\mathbf{x} = \lambda\mathbf{x}$ if and only if $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. The result follows.

Exercise 3.3.21 Suppose λ is an eigenvalue of a square matrix A with eigenvector $\mathbf{x} \neq \mathbf{0}$.

a. Show that λ^2 is an eigenvalue of A^2 (with the same $\mathbf{x}$).

b. Show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$.

c. Show that $p(\lambda)$ is an eigenvalue of $p(A)$ for any nonzero polynomial $p(x)$.

---

b. $(A^3 - 2A - 3I)\mathbf{x} = A^3\mathbf{x} - 2A\mathbf{x} + 3\mathbf{x} = \lambda^3\mathbf{x} - 2\lambda\mathbf{x} + 3\mathbf{x} = (\lambda^3 - 2\lambda - 3)\mathbf{x}$.

Exercise 3.3.22 If A is an $n \times n$ matrix, show that $c_{A^2}(x^2) = (-1)^n c_A(x) c_A(-x)$.

Exercise 3.3.23 An $n \times n$ matrix A is called nilpotent if $A^m = 0$ for some $m \geq 1$.

a. Show that every triangular matrix with zeros on the main diagonal is nilpotent.

b. If A is nilpotent, show that $\lambda = 0$ is the only eigenvalue (even complex) of A.

c. Deduce that $c_A(x) = x^n$, if A is $n \times n$ and nilpotent.

---

b. If $A^m = 0$ and $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. In general, $A^k\mathbf{x} = \lambda^k\mathbf{x}$ for all $k \geq 1$. Hence, $\lambda^m\mathbf{x} = A^m\mathbf{x} = 0\mathbf{x} = \mathbf{0}$, so $\lambda = 0$ (because $\mathbf{x} \neq \mathbf{0}$).

Exercise 3.3.24 Let A be diagonalizable with real eigenvalues and assume that $A^m = I$ for some $m \geq 1$.

a. Show that $A^2 = I$.

b. If m is odd, show that $A = I$. [*Hint*: Theorem **??**]

---

a. If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^k\mathbf{x} = \lambda^k\mathbf{x}$ for each k. Hence $\lambda^m\mathbf{x} = A^m\mathbf{x} = \mathbf{x}$, so $\lambda^m = 1$. As λ is real, $\lambda = \pm 1$ by the Hint. So if $P^{-1}AP = D$ is diagonal, then $D^2 = I$ by Theorem 3.3.4. Hence $A^2 = PD^2P = I$.

Exercise 3.3.25 Let $A^2 = I$, and assume that $A \neq I$ and $A \neq -I$.

a. Show that the only eigenvalues of A are $\lambda = 1$ and $\lambda = -1$.

b. Show that A is diagonalizable. [*Hint*: Verify that $A(A+I) = A+I$ and $A(A-I) = -(A-I)$, and then look at nonzero columns of $A+I$ and of $A-I$.]

c. If $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = mx$ where $m \neq 0$, use (b) to show that the matrix of Q_m is diagonalizable for each m.

d. Now prove (c) geometrically using Theorem 3.3.3.

Exercise 3.3.26 Let $A = \begin{bmatrix} 2 & 3 & -3 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$. Show that $c_A(x) = c_B(x) = (x+1)^2(x-2)$, but A is diagonalizable and B is not.

Exercise 3.3.27

a. Show that the only diagonalizable matrix A that has only one eigenvalue λ is the scalar matrix $A = \lambda I$.

b. Is $\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$ diagonalizable?

---

a. We have $P^{-1}AP = \lambda I$ by the diagonalization algorithm, so $A = P(\lambda I)P^{-1} = \lambda PP^{-1} = \lambda I$.

b. No. $\lambda = 1$ is the only eigenvalue.

Exercise 3.3.28 Characterize the diagonalizable $n \times n$ matrices A such that $A^2 - 3A + 2I = 0$ in terms of their eigenvalues. [*Hint*: Theorem 3.3.1.]

Exercise 3.3.29 Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ where B and C are square matrices.

a. If B and C are diagonalizable via Q and R (that is, $Q^{-1}BQ$ and $R^{-1}CR$ are diagonal), show that A is diagonalizable via $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$

b. Use (a) to diagonalize A if $B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}$.

Exercise 3.3.30 Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ where B and C are square matrices.

a. Show that $c_A(x) = c_B(x)c_C(x)$.

b. If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of B and C, respectively, show that $\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix}$ are eigenvectors of A, and show how every eigenvector of A arises from such eigenvectors.

Exercise 3.3.31 Referring to the model in Example 3.3.1, determine if the population stabilizes, becomes extinct, or becomes large in each case. Denote the adult and juvenile survival rates as A and J, and the reproduction rate as R.

	R	A	J
a.	2	$\frac{1}{2}$	$\frac{1}{2}$
b.	3	$\frac{1}{4}$	$\frac{1}{4}$
c.	2	$\frac{1}{4}$	$\frac{1}{3}$
d.	3	$\frac{3}{5}$	$\frac{1}{5}$

---

b. $\lambda_1 = 1$, stabilizes.

d. $\lambda_1 = \frac{1}{24}(3+\sqrt{69}) = 1.13$, diverges.

Exercise 3.3.32 In the model of Example 3.3.1, does the final outcome depend on the initial population of adult and juvenile females? Support your answer.

Exercise 3.3.33 In Example 3.3.1, keep the same reproduction rate of 2 and the same adult survival rate of $\frac{1}{2}$, but suppose that the juvenile survival rate is ρ. Determine which values of ρ cause the population to become extinct or to become large.

Exercise 3.3.34 In Example 3.3.1, let the juvenile survival rate be $\frac{2}{5}$ and let the reproduction rate be 2. What values of the adult survival rate α will ensure that the population stabilizes?

Extinct if $\alpha < \frac{1}{5}$, stable if $\alpha = \frac{1}{5}$, diverges if $\alpha > \frac{1}{5}$.

Supplementary Exercises for Chapter 3

Exercise 3.1 Show that

$$\det \begin{bmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{bmatrix} = (1+x^3) \det \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$$

Exercise 3.2

a. Show that $(A_{ij})^T = (A^T)_{ji}$ for all i, j, and all square matrices A.

b. Use (a) to prove that $\det A^T = \det A$. [*Hint*: Induction on n where A is $n \times n$.]

---

b. If A is 1×1, then $A^T = A$. In general, $\det [A_{ij}] = \det \left[(A_{ij})^T\right] = \det \left[(A^T)_{ji}\right]$ by (a) and induction. Write $A^T = \left[a'_{ij}\right]$ where $a'_{ij} = a_{ji}$, and expand $\det A^T$ along column 1.

$$\det A^T = \sum_{j=1}^{n} a'_{j1}(-1)^{j+1} \det [(A^T)_{j1}]$$
$$= \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det [A_{1j}] = \det A$$

where the last equality is the expansion of $\det A$ along row 1.

Exercise 3.3 Show that $\det \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix} = (-1)^{nm}$ for all $n \geq 1$ and $m \geq 1$.

Exercise 3.4 Show that

$$\det \begin{bmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{bmatrix} = (b-a)(c-a)(c-b)(a+b+c)$$

Exercise 3.5 Let $A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ be a 2×2 matrix with rows R_1 and R_2. If $\det A = 5$, find $\det B$ where

$$B = \begin{bmatrix} 3R_1 + 2R_3 \\ 2R_1 + 5R_2 \end{bmatrix}$$

Exercise 3.6 Let $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ and let $\mathbf{v}_k = A^k \mathbf{v}_0$ for each $k \geq 0$.

a. Show that A has no dominant eigenvalue.

b. Find $\mathbf{v}_k$ if $\mathbf{v}_0$ equals:

i. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ii. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

iii. $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. Vector Geometry

Contents

4.1 Vectors and Lines

In this chapter we study the geometry of 3-dimensional space. We view a point in 3-space as an arrow from the origin to that point. Doing so provides a "picture" of the point that is truly worth a thousand words. We used this idea earlier, in Section 2.6, to describe rotations, reflections, and projections of the plane $\mathbb{R}^2$. We now apply the same techniques to 3-space to examine similar transformations of $\mathbb{R}^3$. Moreover, the method enables us to completely describe all lines and planes in space.

Vectors in $\mathbb{R}^3$

Introduce a coordinate system in 3-dimensional space in the usual way. First choose a point O called the *origin*, then choose three mutually perpendicular lines through O, called the x, y, and z *axes*, and establish a number scale on each axis with zero at the origin. Given a point P in 3-space we associate three numbers x, y, and z with P, as described in Figure 4.1.1. These numbers are called the *coordinates* of P, and we denote the point as (x, y, z), or $P(x, y, z)$ to emphasize the label P. The result is called a *cartesian*[1] coordinate system for 3-space, and the resulting description of 3-space is called *cartesian geometry*.

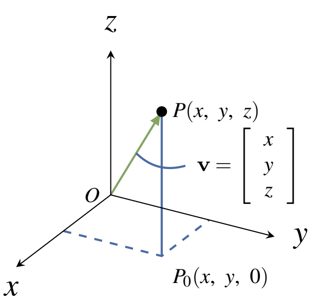


Figure 4.1.1

As in the plane, we introduce vectors by identifying each point $P(x, y, z)$ with the vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, represented by the **arrow** from the origin to P as in Figure 4.1.1. Informally, we say that the point P *has vector* $\mathbf{v}$, and that vector $\mathbf{v}$ *has point* P. In this way 3-space is identified with $\mathbb{R}^3$, and this identification will be made throughout this chapter, often without comment. In particular, the terms "vector" and "point" are interchangeable.[2] The resulting description of 3-space is called **vector geometry**. Note that the origin is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

[1]Named after René Descartes who introduced the idea in 1637.

[2]Recall that we defined $\mathbb{R}^n$ as the set of all ordered n-tuples of real numbers, and reserved the right to denote them as rows or as columns.

Length and Direction

We are going to discuss two fundamental geometric properties of vectors in $\mathbb{R}^3$: length and direction. First, if $\mathbf{v}$ is a vector with point P, the **length** $\|\mathbf{v}\|$ of vector $\mathbf{v}$ is defined to be the distance from the origin to P, that is the length of the arrow representing $\mathbf{v}$. The following properties of length will be used frequently.

Theorem 4.1.1

Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *be a vector.*

1. $\|\mathbf{v}\| = \sqrt{x^2+y^2+z^2}$. [3]
2. $\mathbf{v} = \mathbf{0}$ *if and only if* $\|\mathbf{v}\| = 0$
3. $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$ *for all scalars* a. [4]

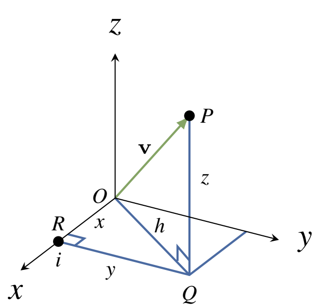


Figure 4.1.2

Proof. Let $\mathbf{v}$ have point $P(x,\ y,\ z)$.

1. In Figure 4.1.2, $\|\mathbf{v}\|$ is the hypotenuse of the right triangle OQP, and so $\|\mathbf{v}\|^2 = h^2+z^2$ by Pythagoras' theorem.[5] But h is the hypotenuse of the right triangle ORQ, so $h^2 = x^2+y^2$. Now (1) follows by eliminating h^2 and taking positive square roots.
2. If $\|\mathbf{v}\| = 0$, then $x^2+y^2+z^2 = 0$ by (1). Because squares of real numbers are nonnegative, it follows that $x = y = z = 0$, and hence that $\mathbf{v} = \mathbf{0}$. The converse is because $\|\mathbf{0}\| = 0$.
3. We have $a\mathbf{v} = \begin{bmatrix} ax & ay & az \end{bmatrix}^T$ so (1) gives

$$\|a\mathbf{v}\|^2 = (ax)^2+(ay)^2+(az)^2 = a^2\|\mathbf{v}\|^2$$

Hence $\|a\mathbf{v}\| = \sqrt{a^2}\|\mathbf{v}\|$, and we are done because $\sqrt{a^2} = |a|$ for any real number a. □

Of course the $\mathbb{R}^2$-version of Theorem 4.1.1 also holds.

[3] When we write $\sqrt{p}$ we mean the positive square root of p.

[4] Recall that the absolute value $|a|$ of a real number is defined by $|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$.

[5] Pythagoras' theorem states that if a and b are sides of right triangle with hypotenuse c, then $a^2+b^2 = c^2$. A proof is given at the end of this section.

Example 4.1.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ then $\|\mathbf{v}\| = \sqrt{4+1+9} = \sqrt{14}$. Similarly if $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ in 2-space then $\|\mathbf{v}\| = \sqrt{9+16} = 5$.

When we view two nonzero vectors as arrows emanating from the origin, it is clear geometrically what we mean by saying that they have the same or opposite **direction**. This leads to a fundamental new description of vectors.

Theorem 4.1.2

Let $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$ be vectors in $\mathbb{R}^3$. Then $\mathbf{v} = \mathbf{w}$ as matrices if and only if $\mathbf{v}$ and $\mathbf{w}$ have the same direction and the same length.[6]

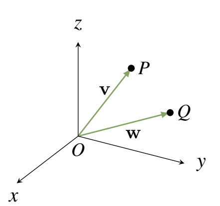


Figure 4.1.3

Proof. If $\mathbf{v} = \mathbf{w}$, they clearly have the same direction and length. Conversely, let $\mathbf{v}$ and $\mathbf{w}$ be vectors with points $P(x, y, z)$ and $Q(x_1, y_1, z_1)$ respectively. If $\mathbf{v}$ and $\mathbf{w}$ have the same length and direction then, geometrically, P and Q must be the same point (see Figure 4.1.3). Hence $x = x_1$, $y = y_1$, and $z = z_1$, that is $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \mathbf{w}$. □

A characterization of a vector in terms of its length and direction only is called an **intrinsic** description of the vector. The point to note is that such a description does *not* depend on the choice of coordinate system in $\mathbb{R}^3$. Such descriptions are important in applications because physical laws are often stated in terms of vectors, and these laws cannot depend on the particular coordinate system used to describe the situation.

Geometric Vectors

If A and B are distinct points in space, the arrow from A to B has length and direction.

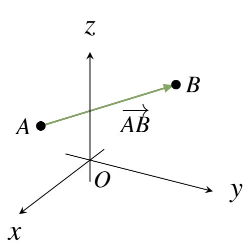


Figure 4.1.4

[6]It is Theorem 4.1.2 that gives vectors their power in science and engineering because many physical quantities are determined by their length and magnitude (and are called **vector quantities**). For example, saying that an airplane is flying at 200 km/h does not describe where it is going; the direction must also be specified. The speed and direction comprise the **velocity** of the airplane, a vector quantity.

Hence:

Definition 4.1 Geometric Vectors

Suppose that A and B are any two points in $\mathbb{R}^3$. In Figure 4.1.4 the line segment from A to B is denoted $\overrightarrow{AB}$ and is called the ***geometric vector*** *from A to B. Point A is called the* ***tail*** *of $\overrightarrow{AB}$, B is called the* ***tip*** *of $\overrightarrow{AB}$, and the* ***length*** *of $\overrightarrow{AB}$ is denoted $\|\overrightarrow{AB}\|$.*

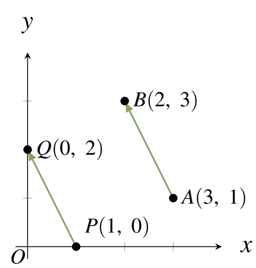


Figure 4.1.5

Note that if $\mathbf{v}$ is any vector in $\mathbb{R}^3$ with point P then $\mathbf{v} = \overrightarrow{OP}$ is itself a geometric vector where O is the origin. Referring to $\overrightarrow{AB}$ as a "vector" seems justified by Theorem 4.1.2 because it has a direction (from A to B) and a length $\|\overrightarrow{AB}\|$. However there appears to be a problem because two geometric vectors can have the same length and direction even if the tips and tails are different. For example $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in Figure 4.1.5 have the same length $\sqrt{5}$ and the same direction (1 unit left and 2 units up) so, by Theorem 4.1.2, they are the same vector! The best way to understand this apparent paradox is to see $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ as different *representations* of the same[7] underlying vector $\left[\begin{array}{r} -1 \\ 2 \end{array}\right]$. Once it is clarified, this phenomenon is a great benefit because, thanks to Theorem 4.1.2, it means that the same geometric vector can be positioned anywhere in space; what is important is the length and direction, not the location of the tip and tail. This ability to move geometric vectors about is very useful as we shall soon see.

The Parallelogram Law

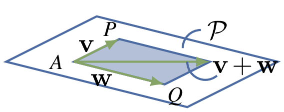


Figure 4.1.6

We now give an intrinsic description of the sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, that is a description that depends only on the lengths and directions of $\mathbf{v}$ and $\mathbf{w}$ and not on the choice of coordinate system. Using Theorem 4.1.2 we can think of these vectors as having a common tail A. If their tips are P and Q respectively, then they both lie in a plane $\mathcal{P}$ containing A, P, and Q, as shown in Figure 4.1.6. The vectors $\mathbf{v}$ and $\mathbf{w}$ create a parallelogram[8] in $\mathcal{P}$, shaded in Figure 4.1.6, called the parallelogram **determined** by $\mathbf{v}$ and $\mathbf{w}$.

If we now choose a coordinate system in the plane $\mathcal{P}$ with A as origin, then the parallelogram law in the plane (Section 2.6) shows that their sum $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram they determine with tail A. This is an intrinsic description of the sum $\mathbf{v} + \mathbf{w}$ because it makes no reference to coordinates. This discussion proves:

[7]Fractions provide another example of quantities that can be the same but *look* different. For example $\frac{6}{9}$ and $\frac{14}{21}$ certainly appear different, but they are equal fractions—both equal $\frac{2}{3}$ in "lowest terms".

[8]Recall that a parallelogram is a four-sided figure whose opposite sides are parallel and of equal length.

The Parallelogram Law

In the parallelogram determined by two vectors $\mathbf{v}$ *and* $\mathbf{w}$*, the vector* $\mathbf{v}+\mathbf{w}$ *is the diagonal with the same tail as* $\mathbf{v}$ *and* $\mathbf{w}$*.*

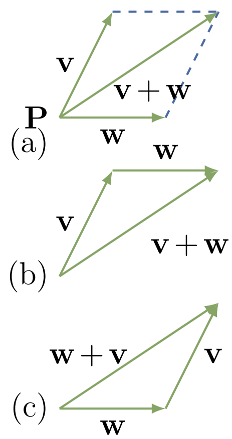


Figure 4.1.7

Because a vector can be positioned with its tail at any point, the parallelogram law leads to another way to view vector addition. In Figure 4.1.7(a) the sum $\mathbf{v}+\mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is shown as given by the parallelogram law. If $\mathbf{w}$ is moved so its tail coincides with the tip of $\mathbf{v}$ (Figure 4.1.7(b)) then the sum $\mathbf{v}+\mathbf{w}$ is seen as "first $\mathbf{v}$ and then $\mathbf{w}$. Similarly, moving the tail of $\mathbf{v}$ to the tip of $\mathbf{w}$ shows in Figure 4.1.7(c) that $\mathbf{v}+\mathbf{w}$ is "first $\mathbf{w}$ and then $\mathbf{v}$." This will be referred to as the **tip-to-tail rule**, and it gives a graphic illustration of why $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

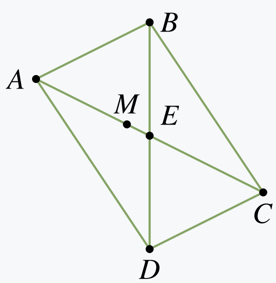
Since $\overrightarrow{AB}$ denotes the vector from a point A to a point B, the tip-to-tail rule takes the easily remembered form

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

for any points A, B, and C. The next example uses this to derive a theorem in geometry without using coordinates.

Example 4.1.2

Show that the diagonals of a parallelogram bisect each other.



Solution. Let the parallelogram have vertices A, B, C, and D, as shown; let E denote the intersection of the two diagonals; and let M denote the midpoint of diagonal AC. We must show that $M=E$ and that this is the midpoint of diagonal BD. This is accomplished by showing that $\overrightarrow{BM}=\overrightarrow{MD}$. (Then the fact that these vectors have the same direction means that $M=E$, and the fact that they have the same length means that $M=E$ is the midpoint of BD.) Now $\overrightarrow{AM}=\overrightarrow{MC}$ because M is the midpoint of AC, and $\overrightarrow{BA}=\overrightarrow{CD}$ because the figure is a parallelogram. Hence

$$\overrightarrow{BM}=\overrightarrow{BA}+\overrightarrow{AM}=\overrightarrow{CD}+\overrightarrow{MC}=\overrightarrow{MC}+\overrightarrow{CD}=\overrightarrow{MD}$$

where the first and last equalities use the tip-to-tail rule of vector addition.

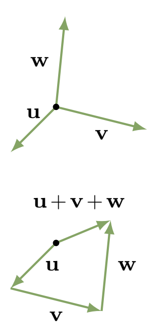


Figure 4.1.8

One reason for the importance of the tip-to-tail rule is that it means two or more vectors can be added by placing them tip-to-tail in sequence. This gives a useful "picture" of the sum of several vectors, and is illustrated for three vectors in Figure 4.1.8 where $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is viewed as first $\mathbf{u}$, then $\mathbf{v}$, then $\mathbf{w}$.

There is a simple geometrical way to visualize the (matrix) **difference** $\mathbf{v}-\mathbf{w}$ of two vectors. If $\mathbf{v}$ and $\mathbf{w}$ are positioned so that they have a common tail A (see Figure 4.1.9), and if B and C are their respective tips, then the tip-to-tail rule gives $\mathbf{w} + \overrightarrow{CB} = \mathbf{v}$. Hence $\mathbf{v} - \mathbf{w} = \overrightarrow{CB}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$. Thus both $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ appear as diagonals in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ (see Figure 4.1.9). We record this for reference.

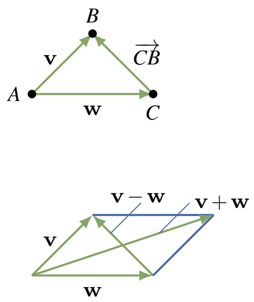


Figure 4.1.9

Theorem 4.1.3

If $\mathbf{v}$ and $\mathbf{w}$ have a common tail, then $\mathbf{v} - \mathbf{w}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$.

One of the most useful applications of vector subtraction is that it gives a simple formula for the vector from one point to another, and for the distance between the points.

Theorem 4.1.4

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then:

1. $\overrightarrow{P_1P_2} = \left[\begin{array}{c} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{array}\right]$.

2. *The distance between P_1 and P_2 is* $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

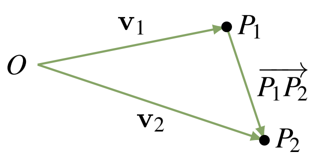


Figure 4.1.10

Proof. If O is the origin, write

$$\mathbf{v}_1 = \overrightarrow{OP}_1 = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right] \text{ and } \mathbf{v}_2 = \overrightarrow{OP}_2 = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$$

as in Figure 4.1.10.

Then Theorem 4.1.3 gives $\overrightarrow{P_1P_2} = \mathbf{v}_2 - \mathbf{v}_1$, and (1) follows. But the distance between P_1 and P_2 is $\|\overrightarrow{P_1P_2}\|$, so (2) follows from (1) and Theorem 4.1.1. □

Of course the $\mathbb{R}^2$-version of Theorem 4.1.4 is also valid: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in $\mathbb{R}^2$, then $\overrightarrow{P_1P_2} = \left[\begin{array}{c} x_2 - x_1 \\ y_2 - y_1 \end{array}\right]$, and the distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 4.1.3

The distance between $P_1(2, -1, 3)$ and $P_2(1, 1, 4)$ is $\sqrt{(-1)^2 + (2)^2 + (1)^2} = \sqrt{6}$, and the vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

As for the parallelogram law, the intrinsic rule for finding the length and direction of a scalar multiple of a vector in $\mathbb{R}^3$ follows easily from the same situation in $\mathbb{R}^2$.

Scalar Multiple Law

If a is a real number and $\mathbf{v} \neq \mathbf{0}$ is a vector then:

1. *The length of $a\mathbf{v}$ is $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.*
2. *If[9] $a\mathbf{v} \neq \mathbf{0}$, the direction of $a\mathbf{v}$ is* $\begin{cases} \text{the same as } \mathbf{v} \text{ if } a > 0, \\ \text{opposite to } \mathbf{v} \text{ if } a < 0. \end{cases}$

Proof.

1. This is part of Theorem 4.1.1.
2. Let O denote the origin in $\mathbb{R}^3$, let $\mathbf{v}$ have point P, and choose any plane containing O and P. If we set up a coordinate system in this plane with O as origin, then $\mathbf{v} = \overrightarrow{OP}$ so the result in (2) follows from the scalar multiple law in the plane (Section 2.6). □

Figure 4.1.11 gives several examples of scalar multiples of a vector $\mathbf{v}$.

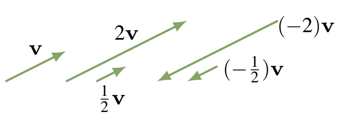


Figure 4.1.11

Consider a line L through the origin, let P be any point on L other than the origin O, and let $\mathbf{p} = \overrightarrow{OP}$. If $t \neq 0$, then $t\mathbf{p}$ is a point on L because it has direction the same or opposite as that of $\mathbf{p}$. Moreover $t > 0$ or $t < 0$ according as the point $t\mathbf{p}$ lies on the same or opposite side of the origin as P. This is illustrated in Figure 4.1.12.

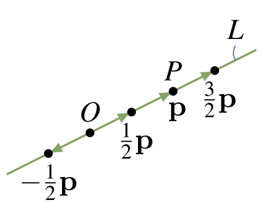


Figure 4.1.12

A vector $\mathbf{u}$ is called a **unit vector** if $\|\mathbf{u}\| = 1$. Then $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are unit vectors, called the **coordinate** vectors. We discuss them in more detail in Section 4.2.

Example 4.1.4

If $\mathbf{v} \neq \mathbf{0}$ show that $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is the unique unit vector in the same direction as $\mathbf{v}$.

[9] Since the zero vector has no direction, we deal only with the case $a\mathbf{v} \neq \mathbf{0}$.

Solution. The vectors in the same direction as $\mathbf{v}$ are the scalar multiples $a\mathbf{v}$ where $a > 0$. But $\|a\mathbf{v}\| = |a|\|\mathbf{v}\| = a\|\mathbf{v}\|$ when $a > 0$, so $a\mathbf{v}$ is a unit vector if and only if $a = \frac{1}{\|\mathbf{v}\|}$.

The next example shows how to find the coordinates of a point on the line segment between two given points. The technique is important and will be used again below.

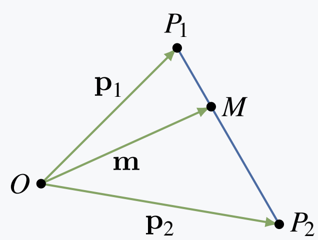
Example 4.1.5

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors of two points P_1 and P_2. If M is the point one third the way from P_1 to P_2, show that the vector $\mathbf{m}$ of M is given by

$$\mathbf{m} = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$$

Conclude that if $P_1 = P_1(x_1,\ y_1,\ z_1)$ and $P_2 = P_2(x_2,\ y_2,\ z_2)$, then M has coordinates

$$M = M\left(\frac{2}{3}x_1 + \frac{1}{3}x_2,\ \frac{2}{3}y_1 + \frac{1}{3}y_2,\ \frac{2}{3}z_1 + \frac{1}{3}z_2\right)$$



Solution. The vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{m}$ are shown in the diagram. We have $\overrightarrow{P_1M} = \frac{1}{3}\overrightarrow{P_1P_2}$ because $\overrightarrow{P_1M}$ is in the same direction as $\overrightarrow{P_1P_2}$ and $\frac{1}{3}$ as long. By Theorem 4.1.3 we have $\overrightarrow{P_1P_2} = \mathbf{p}_2 - \mathbf{p}_1$, so tip-to-tail addition gives

$$\mathbf{m} = \mathbf{p}_1 + \overrightarrow{P_1M} = \mathbf{p}_1 + \frac{1}{3}(\mathbf{p}_2 - \mathbf{p}_1) = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$$

as required. For the coordinates, we have $\mathbf{p}_1 = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]$ and $\mathbf{p}_2 = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$, so

$$\mathbf{m} = \frac{2}{3}\left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right] + \frac{1}{3}\left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right] = \left[\begin{array}{c} \frac{2}{3}x_1 + \frac{1}{3}x_2 \\ \frac{2}{3}y_1 + \frac{1}{3}y_2 \\ \frac{2}{3}z_1 + \frac{1}{3}z_2 \end{array}\right]$$

by matrix addition. The last statement follows.

Note that in Example 4.1.5 $\mathbf{m} = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$ is a "weighted average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ with more weight on $\mathbf{p}_1$ because $\mathbf{m}$ is closer to $\mathbf{p}_1$.

The point M halfway between points P_1 and P_2 is called the **midpoint** between these points. In the same way, the vector $\mathbf{m}$ of M is

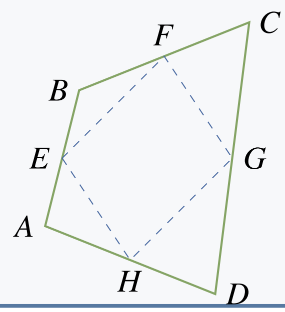
$$\mathbf{m} = \frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2 = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$$

as the reader can verify, so $\mathbf{m}$ is the "average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ in this case.

Example 4.1.6

Show that the midpoints of the four sides of any quadrilateral are the vertices of a parallelogram. Here a quadrilateral is any figure with four vertices and straight sides.

Solution. Suppose that the vertices of the quadrilateral are A, B, C, and D (in that order) and that E, F, G, and H are the midpoints of the sides as shown in the diagram. It suffices to show $\overrightarrow{EF} = \overrightarrow{HG}$ (because then sides EF and HG are parallel and of equal length).



Now the fact that E is the midpoint of AB means that $\overrightarrow{EB} = \frac{1}{2}\overrightarrow{AB}$. Similarly, $\overrightarrow{BF} = \frac{1}{2}\overrightarrow{BC}$, so

$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{2}\overrightarrow{AC}$$

A similar argument shows that $\overrightarrow{HG} = \frac{1}{2}\overrightarrow{AC}$ too, so $\overrightarrow{EF} = \overrightarrow{HG}$ as required.

Definition 4.2 Parallel Vectors in $\mathbb{R}^3$

Two nonzero vectors are called ***parallel*** *if they have the same or opposite direction.*

Many geometrical propositions involve this notion, so the following theorem will be referred to repeatedly.

Theorem 4.1.5

Two nonzero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel if and only if one is a scalar multiple of the other.*

Proof. If one of them is a scalar multiple of the other, they are parallel by the scalar multiple law.

Conversely, assume that $\mathbf{v}$ and $\mathbf{w}$ are parallel and write $d = \frac{\|\mathbf{v}\|}{\|\mathbf{w}\|}$ for convenience. Then $\mathbf{v}$ and $\mathbf{w}$ have the same or opposite direction. If they have the same direction we show that $\mathbf{v} = d\mathbf{w}$ by showing that $\mathbf{v}$ and $d\mathbf{w}$ have the same length and direction. In fact, $\|d\mathbf{w}\| = |d|\|\mathbf{w}\| = \|\mathbf{v}\|$ by Theorem 4.1.1; as to the direction, $d\mathbf{w}$ and $\mathbf{w}$ have the same direction because $d > 0$, and this is the direction of $\mathbf{v}$ by assumption. Hence $\mathbf{v} = d\mathbf{w}$ in this case by Theorem 4.1.2. In the other case, $\mathbf{v}$ and $\mathbf{w}$ have opposite direction and a similar argument shows that $\mathbf{v} = -d\mathbf{w}$. We leave the details to the reader. □

Example 4.1.7

Given points $P(2, -1, 4)$, $Q(3, -1, 3)$, $A(0, 2, 1)$, and $B(1, 3, 0)$, determine if $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are parallel.

Solution. By Theorem 4.1.3, $\overrightarrow{PQ} = (1,\ 0,\ -1)$ and $\overrightarrow{AB} = (1,\ 1,\ -1)$. If $\overrightarrow{PQ} = t\overrightarrow{AB}$ then $(1,\ 0,\ -1) = (t,\ t,\ -t)$, so $1 = t$ and $0 = t$, which is impossible. Hence $\overrightarrow{PQ}$ is *not* a scalar multiple of $\overrightarrow{AB}$, so these vectors are not parallel by Theorem 4.1.5.

Lines in Space

These vector techniques can be used to give a very simple way of describing straight lines in space. In order to do this, we first need a way to specify the orientation of such a line, much as the slope does in the plane.

Definition 4.3 Direction Vector of a Line

With this in mind, we call a nonzero vector $\mathbf{d} \neq \mathbf{0}$ a ***direction vector*** *for the line if it is parallel to $\overrightarrow{AB}$ for some pair of distinct points A and B on the line.*

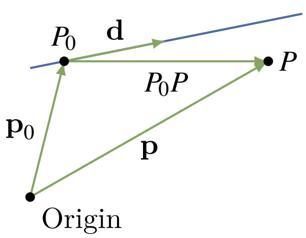


Figure 4.1.13

Of course it is then parallel to $\overrightarrow{CD}$ for *any* distinct points C and D on the line. In particular, any nonzero scalar multiple of $\mathbf{d}$ will also serve as a direction vector of the line.

We use the fact that there is exactly one line that passes through a particular point $P_0(x_0,\ y_0,\ z_0)$ and has a given direction vector $\mathbf{d} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right]$. We want to describe this line by giving a condition on x, y, and z that the point $P(x,\ y,\ z)$ lies on this line. Let $\mathbf{p}_0 = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array}\right]$ and $\mathbf{p} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ denote the vectors of P_0 and P, respectively (see Figure 4.1.13). Then

$$\mathbf{p} = \mathbf{p}_0 + \overrightarrow{P_0P}$$

Hence P lies on the line if and only if $\overrightarrow{P_0P}$ is parallel to $\mathbf{d}$—that is, if and only if $\overrightarrow{P_0P} = t\mathbf{d}$ for some scalar t by Theorem 4.1.5. Thus $\mathbf{p}$ is the vector of a point on the line if and only if $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$ for some scalar t. This discussion is summed up as follows.

Vector Equation of a Line

The line parallel to $\mathbf{d} \neq \mathbf{0}$ through the point with vector $\mathbf{p}_0$ is given by

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \quad t \text{ any scalar}$$

In other words, the point P with vector $\mathbf{p}$ is on this line if and only if a real number t exists such that $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$.

In component form the vector equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Equating components gives a different description of the line.

Parametric Equations of a Line

The line through $P_0(x_0,\ y_0,\ z_0)$ *with direction vector* $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *is given by*

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \quad t \text{ any scalar} \\ z &= z_0 + tc \end{aligned}$$

In other words, the point $P(x,\ y,\ z)$ *is on this line if and only if a real number* t *exists such that* $x = x_0 + ta$, $y = y_0 + tb$, *and* $z = z_0 + tc$.

Example 4.1.8

Find the equations of the line through the points $P_0(2,\ 0,\ 1)$ and $P_1(4,\ -1,\ 1)$.

Solution. Let $\mathbf{d} = \overrightarrow{P_0P_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ denote the vector from P_0 to P_1. Then $\mathbf{d}$ is parallel to the line (P_0 and P_1 are *on* the line), so $\mathbf{d}$ serves as a direction vector for the line. Using P_0 as the point on the line leads to the parametric equations

$$\begin{aligned} x &= 2 + 2t \\ y &= -t \qquad t \text{ a parameter} \\ z &= 1 \end{aligned}$$

Note that if P_1 is used (rather than P_0), the equations are

$$\begin{aligned} x &= 4 + 2s \\ y &= -1 - s \quad s \text{ a parameter} \\ z &= 1 \end{aligned}$$

These are different from the preceding equations, but this is merely the result of a change of parameter. In fact, $s = t - 1$.

Example 4.1.9

Find the equations of the line through $P_0(3,\ -1,\ 2)$ parallel to the line with equations

$$\begin{aligned} x &= -1+2t \\ y &= 1+t \\ z &= -3+4t \end{aligned}$$

Solution. The coefficients of t give a direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ of the given line. Because the line we seek is parallel to this line, $\mathbf{d}$ also serves as a direction vector for the new line. It passes through P_0, so the parametric equations are

$$\begin{aligned} x &= 3+2t \\ y &= -1+t \\ z &= 2+4t \end{aligned}$$

Example 4.1.10

Determine whether the following lines intersect and, if so, find the point of intersection.

$$\begin{aligned} x &= 1-3t & \quad x &= -1+s \\ y &= 2+5t & \quad y &= 3-4s \\ z &= 1+t & \quad z &= 1-s \end{aligned}$$

Solution. Suppose $P(x,\ y,\ z)$ with vector $\mathbf{p}$ lies on both lines. Then

$$\begin{bmatrix} 1-3t \\ 2+5t \\ 1+t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1+s \\ 3-4s \\ 1-s \end{bmatrix} \text{ for some } t \text{ and } s,$$

where the first (second) equation is because P lies on the first (second) line. Hence the lines intersect if and only if the three equations

$$\begin{aligned} 1-3t &= -1+s \\ 2+5t &= 3-4s \\ 1+t &= 1-s \end{aligned}$$

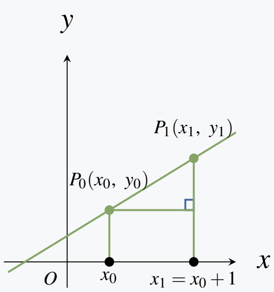
have a solution. In this case, $t = 1$ and $s = -1$ satisfy all three equations, so the lines *do* intersect and the point of intersection is

$$\mathbf{p} = \begin{bmatrix} 1-3t \\ 2+5t \\ 1+t \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$$

using $t = 1$. Of course, this point can also be found from $\mathbf{p} = \begin{bmatrix} -1+s \\ 3-4s \\ 1-s \end{bmatrix}$ using $s = -1$.

Example 4.1.11

Show that the line through $P_0(x_0, y_0)$ with slope m has direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and equation $y - y_0 = m(x - x_0)$. This equation is called the *point-slope* formula.



Solution. Let $P_1(x_1, y_1)$ be the point on the line one unit to the right of P_0 (see the diagram). Hence $x_1 = x_0 + 1$. Then $\mathbf{d} = \overrightarrow{P_0P_1}$ serves as direction vector of the line, and $\mathbf{d} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ y_1 - y_0 \end{bmatrix}$. But the slope m can be computed as follows:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{1} = y_1 - y_0$$

Hence $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and the parametric equations are $x = x_0 + t$, $y = y_0 + mt$. Eliminating t gives $y - y_0 = mt = m(x - x_0)$, as asserted.

Note that the vertical line through $P_0(x_0, y_0)$ has a direction vector $\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ that is *not* of the form $\begin{bmatrix} 1 \\ m \end{bmatrix}$ for any m. This result confirms that the notion of slope makes no sense in this case. However, the vector method gives parametric equations for the line:

$$\begin{aligned} x &= x_0 \\ y &= y_0 + t \end{aligned}$$

Because y is arbitrary here (t is arbitrary), this is usually written simply as $x = x_0$.

Pythagoras' Theorem

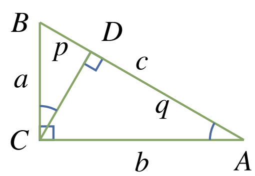


Figure 4.1.14

The Pythagorean theorem was known earlier, but Pythagoras (c. 550 B.C.) is credited with giving the first rigorous, logical, deductive proof of the result. The proof we give depends on a basic property of similar triangles: ratios of corresponding sides are equal.

Theorem 4.1.6: Pythagoras' Theorem

Given a right-angled triangle with hypotenuse c and sides a and b, then $a^2 + b^2 = c^2$.

Proof. Let A, B, and C be the vertices of the triangle as in Figure 4.1.14. Draw a perpendicular line from C to the point D on the hypotenuse, and let p and q be the lengths of BD and DA respectively. Then DBC and CBA are similar triangles so $\frac{p}{a} = \frac{a}{c}$. This means $a^2 = pc$. In the same way, the similarity of DCA and CBA gives $\frac{q}{b} = \frac{b}{c}$, whence $b^2 = qc$. But then

$$a^2 + b^2 = pc + qc = (p+q)c = c^2$$

because $p + q = c$. This proves Pythagoras' theorem[10]. □

Exercises for 4.1

Exercise 4.1.1 Compute $\|\mathbf{v}\|$ if $\mathbf{v}$ equals:

a) $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ d) $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

e) $2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ f) $-3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

---

b. $\sqrt{6}$

d. $\sqrt{5}$

f. $3\sqrt{6}$

Exercise 4.1.2 Find a unit vector in the direction of:

a) $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$ b) $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

---

b. $\frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Exercise 4.1.3

a. Find a unit vector in the direction from $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

b. If $\mathbf{u} \neq \mathbf{0}$, for which values of a is $a\mathbf{u}$ a unit vector?

Exercise 4.1.4 Find the distance between the following pairs of points.

a) $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ d) $\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

---

b. $\sqrt{2}$

d. 3

[10]There is an intuitive geometrical proof of Pythagoras' theorem in Example ??.

Exercise 4.1.5 Use vectors to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half as long.

Exercise 4.1.6 Let A, B, and C denote the three vertices of a triangle.

a. If E is the midpoint of side BC, show that

$$\overrightarrow{AE} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

b. If F is the midpoint of side AC, show that

$$\overrightarrow{FE} = \frac{1}{2}\overrightarrow{AB}$$

---

b. $\overrightarrow{FE} = \overrightarrow{FC} + \overrightarrow{CE} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{CB}) = \frac{1}{2}\overrightarrow{AB}$

Exercise 4.1.7 Determine whether $\mathbf{u}$ and $\mathbf{v}$ are parallel in each of the following cases.

a. $\mathbf{u} = \begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -8 \\ 0 \\ 4 \end{bmatrix}$

---

b. Yes

d. Yes

Exercise 4.1.8 Let $\mathbf{p}$ and $\mathbf{q}$ be the vectors of points P and Q, respectively, and let R be the point whose vector is $\mathbf{p}+\mathbf{q}$. Express the following in terms of $\mathbf{p}$ and $\mathbf{q}$.

a) $\overrightarrow{QP}$

b) $\overrightarrow{QR}$

c) $\overrightarrow{RP}$

d) $\overrightarrow{RO}$ where O is the origin

---

b. $\mathbf{p}$

d. $-(\mathbf{p}+\mathbf{q})$.

Exercise 4.1.9 In each case, find $\overrightarrow{PQ}$ and $\|\overrightarrow{PQ}\|$.

a. $P(1,\ -1,\ 3)$, $Q(3,\ 1,\ 0)$

b. $P(2,\ 0,\ 1)$, $Q(1,\ -1,\ 6)$

c. $P(1,\ 0,\ 1)$, $Q(1,\ 0,\ -3)$

d. $P(1,\ -1,\ 2)$, $Q(1,\ -1,\ 2)$

e. $P(1,\ 0,\ -3)$, $Q(-1,\ 0,\ 3)$

f. $P(3,\ -1,\ 6)$, $Q(1,\ 1,\ 4)$

---

b. $\begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$, $\sqrt{27}$

d. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, 0

f. $\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$, $\sqrt{12}$

Exercise 4.1.10 In each case, find a point Q such that $\overrightarrow{PQ}$ has (i) the same direction as $\mathbf{v}$; (ii) the opposite direction to $\mathbf{v}$.

a. $P(-1,\ 2,\ 2)$, $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

b. $P(3,\ 0,\ -1)$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

---

b. **(i)** $Q(5,\ -1,\ 2)$ **(ii)** $Q(1,\ 1,\ -4)$.

Exercise 4.1.11 Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$. In each case, find $\mathbf{x}$ such that:

a. $3(2\mathbf{u}+\mathbf{x})+\mathbf{w} = 2\mathbf{x}-\mathbf{v}$

b. $2(3\mathbf{v}-\mathbf{x}) = 5\mathbf{w}+\mathbf{u}-3\mathbf{x}$

---

b. $\mathbf{x} = \mathbf{u}-6\mathbf{v}+5\mathbf{w} = \begin{bmatrix} -26 \\ 4 \\ 19 \end{bmatrix}$

Exercise 4.1.12 Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. In each case, find numbers a, b, and c such that $\mathbf{x} = a\mathbf{u}+b\mathbf{v}+c\mathbf{w}$.

a) $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

b) $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

---

b. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 6 \end{bmatrix}$

Exercise 4.1.13 Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. In each case, show that there are no numbers a, b, and c such that:

a. $a\mathbf{u}+b\mathbf{v}+c\mathbf{z} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

b. $a\mathbf{u}+b\mathbf{v}+c\mathbf{z} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$

---

b. If it holds then $\begin{bmatrix} 3a+4b+c \\ -a+c \\ b+c \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$\left[\begin{array}{rrr|r} 3 & 4 & 1 & x_1 \\ -1 & 0 & 1 & x_2 \\ 0 & 1 & 1 & x_3 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 4 & 4 & x_1+3x_2 \\ -1 & 0 & 1 & x_2 \\ 0 & 1 & 1 & x_3 \end{array}\right]$ If there is to be a solution then $x_1+3x_2 = 4x_3$ must hold. This is not satisfied.

Exercise 4.1.14 Given $P_1(2, 1, -2)$ and $P_2(1, -2, 0)$. Find the coordinates of the point P:

a. $\frac{1}{5}$ the way from P_1 to P_2

b. $\frac{1}{4}$ the way from P_2 to P_1

---

b. $\frac{1}{4}\begin{bmatrix} 5 \\ -5 \\ -2 \end{bmatrix}$

Exercise 4.1.15 Find the two points trisecting the segment between $P(2, 3, 5)$ and $Q(8, -6, 2)$.

Exercise 4.1.16 Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points with vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If r and s are positive integers, show that the point P lying $\frac{r}{r+s}$ the way from P_1 to P_2 has vector

$$\mathbf{p} = \left(\frac{s}{r+s}\right)\mathbf{p}_1 + \left(\frac{r}{r+s}\right)\mathbf{p}_2$$

Exercise 4.1.17 In each case, find the point Q:

a. $\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $P = P(2, -3, 1)$

b. $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}$ and $P = P(1, 3, -4)$

---

b. $Q(0, 7, 3)$.

Exercise 4.1.18 Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

In each case find $\mathbf{x}$:

a. $2\mathbf{u} - \|\mathbf{v}\|\mathbf{v} = \frac{3}{2}(\mathbf{u} - 2\mathbf{x})$

b. $3\mathbf{u} + 7\mathbf{v} = \|\mathbf{u}\|^2(2\mathbf{x} + \mathbf{v})$

---

b. $\mathbf{x} = \frac{1}{40}\begin{bmatrix} -20 \\ -13 \\ 14 \end{bmatrix}$

Exercise 4.1.19 Find all vectors $\mathbf{u}$ that are parallel to $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and satisfy $\|\mathbf{u}\| = 3\|\mathbf{v}\|$.

Exercise 4.1.20 Let P, Q, and R be the vertices of a parallelogram with adjacent sides PQ and PR. In each case, find the other vertex S.

a. $P(3,\ -1,\ -1)$, $Q(1,\ -2,\ 0)$, $R(1,\ -1,\ 2)$

b. $P(2,\ 0,\ -1)$, $Q(-2,\ 4,\ 1)$, $R(3,\ -1,\ 0)$

---

b. $S(-1,\ 3,\ 2)$.

Exercise 4.1.21 In each case either prove the statement or give an example showing that it is false.

a. The zero vector $\mathbf{0}$ is the only vector of length 0.

b. If $\|\mathbf{v} - \mathbf{w}\| = 0$, then $\mathbf{v} = \mathbf{w}$.

c. If $\mathbf{v} = -\mathbf{v}$, then $\mathbf{v} = \mathbf{0}$.

d. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \mathbf{w}$.

e. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \pm\mathbf{w}$.

f. If $\mathbf{v} = t\mathbf{w}$ for some scalar t, then $\mathbf{v}$ and $\mathbf{w}$ have the same direction.

g. If $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$ are nonzero, and $\mathbf{v}$ and $\mathbf{v} + \mathbf{w}$ parallel, then $\mathbf{v}$ and $\mathbf{w}$ are parallel.

h. $\|-5\mathbf{v}\| = -5\|\mathbf{v}\|$, for all $\mathbf{v}$.

i. If $\|\mathbf{v}\| = \|2\mathbf{v}\|$, then $\mathbf{v} = \mathbf{0}$.

j. $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$, for all $\mathbf{v}$ and $\mathbf{w}$.

---

b. T. $\|\mathbf{v} - \mathbf{w}\| = 0$ implies that $\mathbf{v} - \mathbf{w} = \mathbf{0}$.

d. F. $\|\mathbf{v}\| = \|-\mathbf{v}\|$ for all $\mathbf{v}$ but $\mathbf{v} = -\mathbf{v}$ only holds if $\mathbf{v} = \mathbf{0}$.

f. F. If $t < 0$ they have the *opposite* direction.

h. F. $\|-5\mathbf{v}\| = 5\|\mathbf{v}\|$ for all $\mathbf{v}$, so it fails if $\mathbf{v} \neq \mathbf{0}$.

j. F. Take $\mathbf{w} = -\mathbf{v}$ where $\mathbf{v} \neq \mathbf{0}$.

Exercise 4.1.22 Find the vector and parametric equations of the following lines.

a. The line parallel to $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and passing through $P(1,\ -1,\ 3)$.

b. The line passing through $P(3,\ -1,\ 4)$ and $Q(1,\ 0,\ -1)$.

c. The line passing through $P(3,\ -1,\ 4)$ and $Q(3,\ -1,\ 5)$.

d. The line parallel to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and passing through $P(1,\ 1,\ 1)$.

e. The line passing through $P(1,\ 0,\ -3)$ and parallel to the line with parametric equations $x = -1 + 2t$, $y = 2 - t$, and $z = 3 + 3t$.

f. The line passing through $P(2,\ -1,\ 1)$ and parallel to the line with parametric equations $x = 2 - t$, $y = 1$, and $z = t$.

g. The lines through $P(1,\ 0,\ 1)$ that meet the line with vector equation $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ at points at distance 3 from $P_0(1,\ 2,\ 0)$.

---

b. $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$; $x = 3 + 2t$, $y = -1 - t$, $z = 4 + 5t$

d. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $x = y = z = 1 + t$

f. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $x = 2 - t$, $y = -1$, $z = 1 + t$

Exercise 4.1.23 In each case, verify that the points P and Q lie on the line.

a. $x = 3 - 4t$, $y = 2 + t$, $z = 1 - t$ $P(-1, 3, 0)$, $Q(11, 0, 3)$

b. $x = 4 - t$, $y = 3$, $z = 1 - 2t$ $P(2, 3, -3)$, $Q(-1, 3, -9)$

---

b. P corresponds to $t = 2$; Q corresponds to $t = 5$.

Exercise 4.1.24 Find the point of intersection (if any) of the following pairs of lines.

a. $x = 3 + t$, $y = 1 - 2t$, $z = 3 + 3t$ $x = 4 + 2s$, $y = 6 + 3s$, $z = 1 + s$

b. $x = 1 - t$, $y = 2 + 2t$, $z = -1 + 3t$ $x = 2s$, $y = 1 + s$, $z = 3$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$

---

b. No intersection

d. $P(2, -1, 3)$; $t = -2$, $s = -3$

Exercise 4.1.25 Show that if a line passes through the origin, the vectors of points on the line are all scalar multiples of some fixed nonzero vector.

Exercise 4.1.26 Show that every line parallel to the z axis has parametric equations $x = x_0$, $y = y_0$, $z = t$ for some fixed numbers x_0 and y_0.

Exercise 4.1.27 Let $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector where a, b, and c are *all* nonzero. Show that the equations of the line through $P_0(x_0, y_0, z_0)$ with direction vector $\mathbf{d}$ can be written in the form

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This is called the **symmetric form** of the equations.

Exercise 4.1.28 A parallelogram has sides AB, BC, CD, and DA. Given $A(1, -1, 2)$, $C(2, 1, 0)$, and the midpoint $M(1, 0, -3)$ of AB, find $\overrightarrow{BD}$.

Exercise 4.1.29 Find all points C on the line through $A(1, -1, 2)$ and $B = (2, 0, 1)$ such that $\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\|$.

$P(3, 1, 0)$ or $P(\frac{5}{3}, \frac{-1}{3}, \frac{4}{3})$

Exercise 4.1.30 Let A, B, C, D, E, and F be the vertices of a regular hexagon, taken in order. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

Exercise 4.1.31

a. Let P_1, P_2, P_3, P_4, P_5, and P_6 be six points equally spaced on a circle with centre C. Show that

$$\overrightarrow{CP}_1 + \overrightarrow{CP}_2 + \overrightarrow{CP}_3 + \overrightarrow{CP}_4 + \overrightarrow{CP}_5 + \overrightarrow{CP}_6 = \mathbf{0}$$

b. Show that the conclusion in part (a) holds for any *even* set of points evenly spaced on the circle.

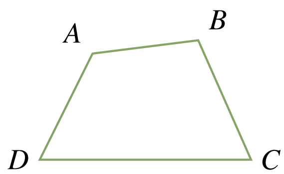
c. Show that the conclusion in part (a) holds for *three* points.

d. Do you think it works for *any* finite set of points evenly spaced around the circle?

---

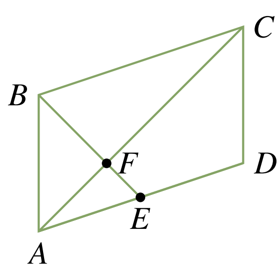
b. $\overrightarrow{CP}_k = -\overrightarrow{CP}_{n+k}$ if $1 \leq k \leq n$, where there are $2n$ points.

Exercise 4.1.32 Consider a quadrilateral with vertices A, B, C, and D in order (as shown in the diagram).



If the diagonals AC and BD bisect each other, show that the quadrilateral is a parallelogram. (This is the converse of Example 4.1.2.) [*Hint*: Let E be the intersection of the diagonals. Show that $\overrightarrow{AB} = \overrightarrow{DC}$ by writing $\overrightarrow{AB} = \overrightarrow{AE} + \overrightarrow{EB}$.]

Exercise 4.1.33 Consider the parallelogram $ABCD$ (see diagram), and let E be the midpoint of side AD.



Show that BE and AC trisect each other; that is, show that the intersection point is one-third of the way from E to B and from A to C. [*Hint*: If F is one-third of the way from A to C, show that $2\overrightarrow{EF} = \overrightarrow{FB}$ and argue as in Example 4.1.2.]

$\overrightarrow{DA} = 2\overrightarrow{EA}$ and $2\overrightarrow{AF} = \overrightarrow{FC}$, so $2\overrightarrow{EF} = 2(\overrightarrow{EF} + \overrightarrow{AF}) = \overrightarrow{DA} + \overrightarrow{FC} = \overrightarrow{CB} + \overrightarrow{FC} = \overrightarrow{FC} + \overrightarrow{CB} = \overrightarrow{FB}$. Hence $\overrightarrow{EF} = \frac{1}{2}\overrightarrow{FB}$. So F is the trisection point of both AC and EB.

Exercise 4.1.34 The line from a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. If the vertices of a triangle have vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, show that the point on each median that is $\frac{1}{3}$ the way from the midpoint to the vertex has vector $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$. Conclude that the point C with vector $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$ lies on all three medians. This point C is called the **centroid** of the triangle.

Exercise 4.1.35 Given four noncoplanar points in space, the figure with these points as vertices is called a **tetrahedron**. The line from a vertex through the centroid (see previous exercise) of the triangle formed by the remaining vertices is called a **median** of the tetrahedron. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$ are the vectors of the four vertices, show that the point on a median one-fourth the way from the centroid to the vertex has vector $\frac{1}{4}(\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{x})$. Conclude that the four medians are concurrent.

4.2 Projections and Planes

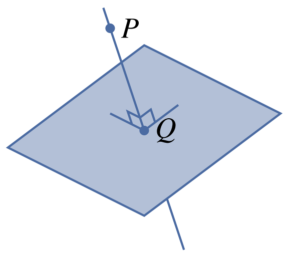


Figure 4.2.1

Any student of geometry soon realizes that the notion of perpendicular lines is fundamental. As an illustration, suppose a point P and a plane are given and it is desired to find the point Q that lies in the plane and is closest to P, as shown in Figure 4.2.1. Clearly, what is required is to find the line through P that is perpendicular to the plane and then to obtain Q as the point of intersection of this line with the plane. Finding the line *perpendicular* to the plane requires a way to determine when two vectors are perpendicular. This can be done using the idea of the dot product of two vectors.

The Dot Product and Angles

Definition 4.4 Dot Product in $\mathbb{R}^3$

Given vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *their* ***dot product*** $\mathbf{v}\cdot\mathbf{w}$ *is a number defined*

$$\mathbf{v}\cdot\mathbf{w} = x_1x_2 + y_1y_2 + z_1z_2 = \mathbf{v}^T\mathbf{w}$$

Because $\mathbf{v}\cdot\mathbf{w}$ is a number, it is sometimes called the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$.[11]

Example 4.2.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, then $\mathbf{v}\cdot\mathbf{w} = 2\cdot 1 + (-1)\cdot 4 + 3\cdot(-1) = -5$.

The next theorem lists several basic properties of the dot product.

Theorem 4.2.1

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

1. $\mathbf{v}\cdot\mathbf{w}$ *is a real number.*
2. $\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$.
3. $\mathbf{v}\cdot\mathbf{0} = 0 = \mathbf{0}\cdot\mathbf{v}$.
4. $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$.

[11] Similarly, if $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, then $\mathbf{v}\cdot\mathbf{w} = x_1x_2 + y_1y_2$.

5. $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{v}) = \mathbf{v} \cdot (k\mathbf{w})$ *for all scalars* k.

6. $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$

Proof. (1), (2), and (3) are easily verified, and (4) comes from Theorem 4.1.1. The rest are properties of matrix arithmetic (because $\mathbf{w} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{w}$), and are left to the reader. □

The properties in Theorem 4.2.1 enable us to do calculations like

$$3\mathbf{u} \cdot (2\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}) = 6(\mathbf{u} \cdot \mathbf{v}) - 9(\mathbf{u} \cdot \mathbf{w}) + 12(\mathbf{u} \cdot \mathbf{z})$$

and such computations will be used without comment below. Here is an example.

Example 4.2.2

Verify that $\|\mathbf{v} - 3\mathbf{w}\|^2 = 1$ when $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 1$, and $\mathbf{v} \cdot \mathbf{w} = 2$.

Solution. We apply Theorem 4.2.1 several times:

$$\begin{aligned}
\|\mathbf{v} - 3\mathbf{w}\|^2 &= (\mathbf{v} - 3\mathbf{w}) \cdot (\mathbf{v} - 3\mathbf{w}) \\
&= \mathbf{v} \cdot (\mathbf{v} - 3\mathbf{w}) - 3\mathbf{w} \cdot (\mathbf{v} - 3\mathbf{w}) \\
&= \mathbf{v} \cdot \mathbf{v} - 3(\mathbf{v} \cdot \mathbf{w}) - 3(\mathbf{w} \cdot \mathbf{v}) + 9(\mathbf{w} \cdot \mathbf{w}) \\
&= \|\mathbf{v}\|^2 - 6(\mathbf{v} \cdot \mathbf{w}) + 9\|\mathbf{w}\|^2 \\
&= 4 - 12 + 9 = 1
\end{aligned}$$

There is an intrinsic description of the dot product of two nonzero vectors in $\mathbb{R}^3$. To understand it we require the following result from trigonometry.

Law of Cosines

If a triangle has sides a, b, *and* c, *and if* θ *is the interior angle opposite* c *then*

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

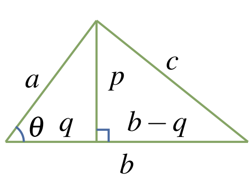


Figure 4.2.2

Proof. We prove it when is θ acute, that is $0 \leq \theta < \frac{\pi}{2}$; the obtuse case is similar. In Figure 4.2.2 we have $p = a\sin\theta$ and $q = a\cos\theta$. Hence Pythagoras' theorem gives

$$\begin{aligned}
c^2 = p^2 + (b-q)^2 &= a^2\sin^2\theta + (b - a\cos\theta)^2 \\
&= a^2(\sin^2\theta + \cos^2\theta) + b^2 - 2ab\cos\theta
\end{aligned}$$

The law of cosines follows because $\sin^2\theta + \cos^2\theta = 1$ for any angle θ. □

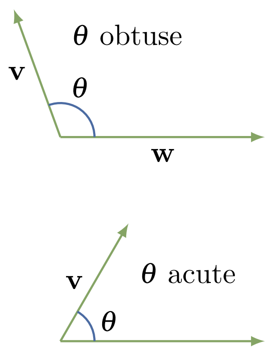


Figure 4.2.3

Note that the law of cosines reduces to Pythagoras' theorem if θ is a right angle (because $\cos\frac{\pi}{2} = 0$).

Now let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors positioned with a common tail as in Figure 4.2.3. Then they determine a unique angle θ in the range

$$0 \leq \theta \leq \pi$$

This angle θ will be called the **angle between** $\mathbf{v}$ and $\mathbf{w}$. Figure 4.2.3 illustrates when θ is acute (less than $\frac{\pi}{2}$) and obtuse (greater than $\frac{\pi}{2}$). Clearly $\mathbf{v}$ and $\mathbf{w}$ are parallel if θ is either 0 or π. Note that we do not define the angle between $\mathbf{v}$ and $\mathbf{w}$ if one of these vectors is $\mathbf{0}$.

The next result gives an easy way to compute the angle between two nonzero vectors using the dot product.

Theorem 4.2.2

Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$$

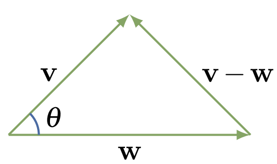


Figure 4.2.4

Proof. We calculate $\|\mathbf{v}-\mathbf{w}\|^2$ in two ways. First apply the law of cosines to the triangle in Figure 4.2.4 to obtain:

$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

On the other hand, we use Theorem 4.2.1:

$$\begin{aligned} \|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}-\mathbf{w}) \cdot (\mathbf{v}-\mathbf{w}) \\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} \\ &= \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2 \end{aligned}$$

Comparing these we see that $-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = -2(\mathbf{v}\cdot\mathbf{w})$, and the result follows. □

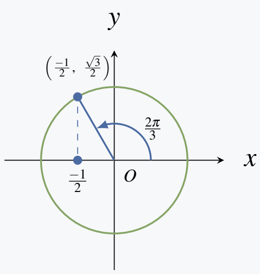
If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, Theorem 4.2.2 gives an intrinsic description of $\mathbf{v}\cdot\mathbf{w}$ because $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and the angle θ between $\mathbf{v}$ and $\mathbf{w}$ do not depend on the choice of coordinate system. Moreover, since $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ are nonzero ($\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors), it gives a formula for the cosine of the angle θ:

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \tag{4.1}$$

Since $0 \leq \theta \leq \pi$, this can be used to find θ.

Example 4.2.3

Compute the angle between $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.



Solution. Compute $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-2+1-2}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}$. Now recall that $\cos\theta$ and $\sin\theta$ are defined so that $(\cos\theta, \sin\theta)$ is the point on the unit circle determined by the angle θ (drawn counterclockwise, starting from the positive x axis). In the present case, we know that $\cos\theta = -\frac{1}{2}$ and that $0 \leq \theta \leq \pi$. Because $\cos\frac{\pi}{3} = \frac{1}{2}$, it follows that $\theta = \frac{2\pi}{3}$ (see the diagram).

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, equation (4.1) shows that $\cos\theta$ has the same sign as $\mathbf{v}\cdot\mathbf{w}$, so

$$\begin{array}{lll} \mathbf{v}\cdot\mathbf{w} > 0 & \text{if and only if} & \theta \text{ is acute } (0 \leq \theta < \frac{\pi}{2}) \\ \mathbf{v}\cdot\mathbf{w} < 0 & \text{if and only if} & \theta \text{ is obtuse } (\frac{\pi}{2} < \theta \leq 0) \\ \mathbf{v}\cdot\mathbf{w} = 0 & \text{if and only if} & \theta = \frac{\pi}{2} \end{array}$$

In this last case, the (nonzero) vectors are perpendicular. The following terminology is used in linear algebra:

Definition 4.5 Orthogonal Vectors in $\mathbb{R}^3$

*Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **orthogonal** if $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or the angle between them is $\frac{\pi}{2}$.*

Since $\mathbf{v}\cdot\mathbf{w} = 0$ if either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, we have the following theorem:

Theorem 4.2.3

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

Example 4.2.4

Show that the points $P(3, -1, 1)$, $Q(4, 1, 4)$, and $R(6, 0, 4)$ are the vertices of a right triangle.

Solution. The vectors along the sides of the triangle are

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \overrightarrow{PR} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and } \overrightarrow{QR} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

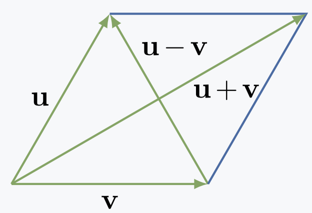
Evidently $\overrightarrow{PQ}\cdot\overrightarrow{QR} = 2 - 2 + 0 = 0$, so $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are orthogonal vectors. This means sides

PQ and *QR* are perpendicular—that is, the angle at *Q* is a right angle.

Example 4.2.5 demonstrates how the dot product can be used to verify geometrical theorems involving perpendicular lines.

Example 4.2.5

A parallelogram with sides of equal length is called a **rhombus**. Show that the diagonals of a rhombus are perpendicular.



Solution. Let $\mathbf{u}$ and $\mathbf{v}$ denote vectors along two adjacent sides of a rhombus, as shown in the diagram. Then the diagonals are $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$, and we compute

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v})-\mathbf{v}\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2-\|\mathbf{v}\|^2\\ &= 0\end{aligned}$$

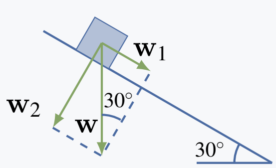
because $\|\mathbf{u}\| = \|\mathbf{v}\|$ (it is a rhombus). Hence $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are orthogonal.

Projections

In applications of vectors, it is frequently useful to write a vector as the sum of two orthogonal vectors. Here is an example.

Example 4.2.6

Suppose a ten-kilogram block is placed on a flat surface inclined 30° to the horizontal as in the diagram. Neglecting friction, how much force is required to keep the block from sliding down the surface?



Solution. Let $\mathbf{w}$ denote the weight (force due to gravity) exerted on the block. Then $\|\mathbf{w}\| = 10$ kilograms and the direction of $\mathbf{w}$ is vertically down as in the diagram. The idea is to write $\mathbf{w}$ as a sum $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is parallel to the inclined surface and $\mathbf{w}_2$ is perpendicular to the surface. Since there is no friction, the force required is $-\mathbf{w}_1$ because the force $\mathbf{w}_2$ has no effect parallel to the surface. As the angle between $\mathbf{w}$ and $\mathbf{w}_2$ is 30° in the diagram, we have $\frac{\|\mathbf{w}_1\|}{\|\mathbf{w}\|} = \sin 30° = \frac{1}{2}$. Hence $\|\mathbf{w}_1\| = \frac{1}{2}\|\mathbf{w}\| = \frac{1}{2}10 = 5$. Thus the required force has a magnitude of 5 kilograms weight directed up the surface.

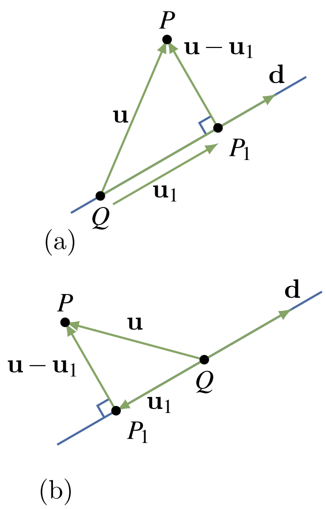


Figure 4.2.5

If a nonzero vector $\mathbf{d}$ is specified, the key idea in Example 4.2.6 is to be able to write an arbitrary vector $\mathbf{u}$ as a sum of two vectors,

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ is orthogonal to $\mathbf{d}$. Suppose that $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ emanate from a common tail Q (see Figure 4.2.5). Let P be the tip of $\mathbf{u}$, and let P_1 denote the foot of the perpendicular from P to the line through Q parallel to $\mathbf{d}$.

Then $\mathbf{u}_1 = \overrightarrow{QP_1}$ has the required properties:

1. $\mathbf{u}_1$ *is parallel to* $\mathbf{d}$.
2. $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ *is orthogonal to* $\mathbf{d}$.
3. $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

Definition 4.6 Projection in $\mathbb{R}^3$

The vector $\mathbf{u}_1 = \overrightarrow{QP_1}$ *in Figure 4.2.5 is called* ***the projection*** *of* $\mathbf{u}$ *on* $\mathbf{d}$. *It is denoted*

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$$

In Figure 4.2.5(a) the vector $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ has the same direction as $\mathbf{d}$; however, $\mathbf{u}_1$ and $\mathbf{d}$ have opposite directions if the angle between $\mathbf{u}$ and $\mathbf{d}$ is greater than $\frac{\pi}{2}$ (Figure 4.2.5(b)). Note that the projection $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ is zero if and only if $\mathbf{u}$ and $\mathbf{d}$ are orthogonal.

Calculating the projection of $\mathbf{u}$ on $\mathbf{d} \neq \mathbf{0}$ is remarkably easy.

Theorem 4.2.4

Let $\mathbf{u}$ *and* $\mathbf{d} \neq \mathbf{0}$ *be vectors.*

1. *The projection of* $\mathbf{u}$ *on* $\mathbf{d}$ *is given by* $\text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$.
2. *The vector* $\mathbf{u} - \text{proj}_{\mathbf{d}}\, \mathbf{u}$ *is orthogonal to* $\mathbf{d}$.

Proof. The vector $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ is parallel to $\mathbf{d}$ and so has the form $\mathbf{u}_1 = t\mathbf{d}$ for some scalar t. The requirement that $\mathbf{u} - \mathbf{u}_1$ and $\mathbf{d}$ are orthogonal determines t. In fact, it means that $(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{d} = 0$ by Theorem 4.2.3. If $\mathbf{u}_1 = t\mathbf{d}$ is substituted here, the condition is

$$0 = (\mathbf{u} - t\mathbf{d}) \cdot \mathbf{d} = \mathbf{u} \cdot \mathbf{d} - t(\mathbf{d} \cdot \mathbf{d}) = \mathbf{u} \cdot \mathbf{d} - t\|\mathbf{d}\|^2$$

It follows that $t = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$, where the assumption that $\mathbf{d} \neq \mathbf{0}$ guarantees that $\|\mathbf{d}\|^2 \neq 0$. □

Example 4.2.7

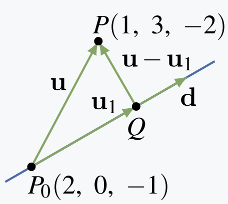
Find the projection of $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ on $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and express $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{d}$.

Solution. The projection $\mathbf{u}_1$ of $\mathbf{u}$ on $\mathbf{d}$ is

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{2+3+3}{1^2+(-1)^2+3^2}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{8}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Hence $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \frac{1}{11}\begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix}$, and this is orthogonal to $\mathbf{d}$ by Theorem 4.2.4 (alternatively, observe that $\mathbf{d}\cdot\mathbf{u}_2 = 0$). Since $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, we are done.

Example 4.2.8



Find the shortest distance (see diagram) from the point $P(1,\ 3,\ -2)$ to the line through $P_0(2,\ 0,\ -1)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Also find the point Q that lies on the line and is closest to P.

Solution. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ denote the vector from P_0 to P, and let $\mathbf{u}_1$ denote the projection of $\mathbf{u}$ on $\mathbf{d}$. Thus

$$\mathbf{u}_1 = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{-1-3+0}{1^2+(-1)^2+0^2}\mathbf{d} = -2\mathbf{d} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

by Theorem 4.2.4. We see geometrically that the point Q on the line is closest to P, so the distance is

$$\|\overrightarrow{QP}\| = \|\mathbf{u} - \mathbf{u}_1\| = \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{3}$$

To find the coordinates of Q, let $\mathbf{p}_0$ and $\mathbf{q}$ denote the vectors of P_0 and Q, respectively. Then $\mathbf{p}_0 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \mathbf{p}_0 + \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Hence $Q(0,\ 2,\ -1)$ is the required point. It can be checked that the distance from Q to P is $\sqrt{3}$, as expected.

Planes

It is evident geometrically that among all planes that are perpendicular to a given straight line there is exactly one containing any given point. This fact can be used to give a very simple description of a plane. To do this, it is necessary to introduce the following notion:

Definition 4.7 Normal Vector in a Plane

A nonzero vector $\mathbf{n}$ *is called a* ***normal*** *for a plane if it is orthogonal to every vector in the plane.*

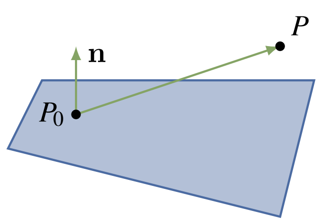


Figure 4.2.6

For example, the coordinate vector $\mathbf{k}$ is a normal for the x-y plane.

Given a point $P_0 = P_0(x_0,\ y_0,\ z_0)$ and a nonzero vector $\mathbf{n}$, there is a unique plane through P_0 with normal $\mathbf{n}$, shaded in Figure 4.2.6. A point $P = P(x,\ y,\ z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$—that is, if and only if $\mathbf{n}\cdot\overrightarrow{P_0P} = 0$. Because $\overrightarrow{P_0P} = \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix}$ this gives the following result:

Scalar Equation of a Plane

The plane through $P_0(x_0,\ y_0,\ z_0)$ *with normal* $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *as a normal vector is given by*

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

In other words, a point $P(x,\ y,\ z)$ *is on this plane if and only if* x, y, *and* z *satisfy this equation.*

Example 4.2.9

Find an equation of the plane through $P_0(1,\ -1,\ 3)$ with $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ as normal.

Solution. Here the general scalar equation becomes

$$3(x-1)-(y+1)+2(z-3) = 0$$

This simplifies to $3x-y+2z = 10$.

If we write $d = ax_0 + by_0 + cz_0$, the scalar equation shows that every plane with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a linear equation of the form

$$ax + by + cz = d \tag{4.2}$$

for some constant d. Conversely, the graph of this equation is a plane with $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a normal vector (assuming that a, b, and c are not all zero).

Example 4.2.10

Find an equation of the plane through $P_0(3, -1, 2)$ that is parallel to the plane with equation $2x - 3y = 6$.

Solution. The plane with equation $2x - 3y = 6$ has normal $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$. Because the two planes are parallel, $\mathbf{n}$ serves as a normal for the plane we seek, so the equation is $2x - 3y = d$ for some d by Equation 4.2. Insisting that $P_0(3, -1, 2)$ lies on the plane determines d; that is, $d = 2 \cdot 3 - 3(-1) = 9$. Hence, the equation is $2x - 3y = 9$.

Consider points $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$ with vectors $\mathbf{p}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Given a nonzero vector $\mathbf{n}$, the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ takes the vector form:

Vector Equation of a Plane

The plane with normal $\mathbf{n} \neq \mathbf{0}$ *through the point with vector* $\mathbf{p}_0$ *is given by*

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$$

In other words, the point with vector $\mathbf{p}$ *is on the plane if and only if* $\mathbf{p}$ *satisfies this condition.*

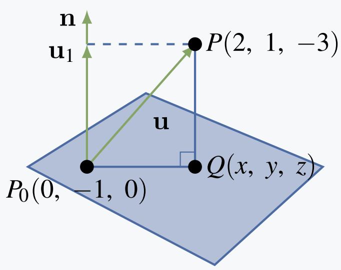
Moreover, Equation 4.2 translates as follows:

Every plane with normal $\mathbf{n}$ has vector equation $\mathbf{n} \cdot \mathbf{p} = d$ for some number d.

This is useful in the second solution of Example 4.2.11.

Example 4.2.11

Find the shortest distance from the point $P(2,\ 1,\ -3)$ to the plane with equation $3x - y + 4z = 1$. Also find the point Q on this plane closest to P.



Solution 1. The plane in question has normal $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$. Choose any point P_0 on the plane—say $P_0(0,\ -1,\ 0)$—and let $Q(x,\ y,\ z)$ be the point on the plane closest to P (see the diagram). The vector from P_0 to P is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. Now erect $\mathbf{n}$ with its tail at P_0. Then $\overrightarrow{QP} = \mathbf{u}_1$ and $\mathbf{u}_1$ is the projection of $\mathbf{u}$ on $\mathbf{n}$:

$$\mathbf{u}_1 = \frac{\mathbf{n}\cdot\mathbf{u}}{\|\mathbf{n}\|^2}\mathbf{n} = \frac{-8}{26}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \frac{-4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

Hence the distance is $\|\overrightarrow{QP}\| = \|\mathbf{u}_1\| = \frac{4\sqrt{26}}{13}$. To calculate the point Q, let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p}_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ be the vectors of Q and P_0. Then

$$\mathbf{q} = \mathbf{p}_0 + \mathbf{u} - \mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{38}{13} \\ \frac{9}{13} \\ \frac{-23}{13} \end{bmatrix}$$

This gives the coordinates of $Q(\frac{38}{13},\ \frac{9}{13},\ \frac{-23}{13})$.

Solution 2. Let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ be the vectors of Q and P. Then Q is on the line through P with direction vector $\mathbf{n}$, so $\mathbf{q} = \mathbf{p} + t\mathbf{n}$ for some scalar t. In addition, Q lies on the plane, so $\mathbf{n}\cdot\mathbf{q} = 1$. This determines t:

$$1 = \mathbf{n}\cdot\mathbf{q} = \mathbf{n}\cdot(\mathbf{p} + t\mathbf{n}) = \mathbf{n}\cdot\mathbf{p} + t\|\mathbf{n}\|^2 = -7 + t(26)$$

This gives $t = \frac{8}{26} = \frac{4}{13}$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{q} = \mathbf{p} + t\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \frac{1}{13}\begin{bmatrix} 38 \\ 9 \\ -23 \end{bmatrix}$$

as before. This determines Q (in the diagram), and the reader can verify that the required distance is $\|\overrightarrow{QP}\| = \frac{4}{13}\sqrt{26}$, as before.

The Cross Product

If P, Q, and R are three distinct points in $\mathbb{R}^3$ that are not all on some line, it is clear geometrically that there is a unique plane containing all three. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ both lie in this plane, so finding a normal amounts to finding a nonzero vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The cross product provides a systematic way to do this.

Definition 4.8 Cross Product

Given vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *define the* ***cross product*** $\mathbf{v}_1 \times \mathbf{v}_2$ *by*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1z_2 - z_1y_2 \\ -(x_1z_2 - z_1x_2) \\ x_1y_2 - y_1x_2 \end{bmatrix}$$

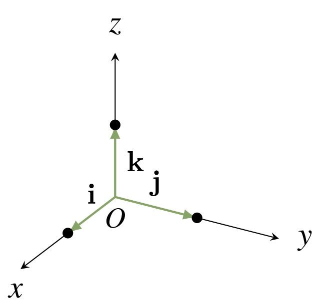


Figure 4.2.7

(Because it is a vector, $\mathbf{v}_1 \times \mathbf{v}_2$ is often called the **vector product**.) There is an easy way to remember this definition using the **coordinate vectors**:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

They are vectors of length 1 pointing along the positive x, y, and z axes, respectively, as in Figure 4.2.7. The reason for the name is that any vector can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

With this, the cross product can be described as follows:

Determinant Form of the Cross Product

If $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *are two vectors, then*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \boldsymbol{i} & x_1 & x_2 \\ \boldsymbol{j} & y_1 & y_2 \\ \boldsymbol{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \boldsymbol{k}$$

where the determinant is expanded along the first column.

Example 4.2.12

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, then

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & -1 & 3 \\ \mathbf{k} & 4 & 7 \end{bmatrix} = \begin{vmatrix} -1 & 3 \\ 4 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{k}$$
$$= -19\mathbf{i} - 10\mathbf{j} + 7\mathbf{k}$$
$$= \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix}$$

Observe that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ in Example 4.2.12. This holds in general as can be verified directly by computing $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})$, and is recorded as the first part of the following theorem. It will follow from a more general result which, together with the second part, will be proved in Section 4.3 where a more detailed study of the cross product will be undertaken.

Theorem 4.2.5

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

1. *$\mathbf{v} \times \mathbf{w}$ is a vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.*
2. *If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.*

It is interesting to contrast Theorem 4.2.5(2) with the assertion (in Theorem 4.2.3) that

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{if and only if } \mathbf{v} \text{ and } \mathbf{w} \text{ are orthogonal.}$$

Example 4.2.13

Find the equation of the plane through $P(1, 3, -2)$, $Q(1, 1, 5)$, and $R(2, -2, 3)$.

Solution. The vectors $\overrightarrow{PQ} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ lie in the plane, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \mathbf{i} & 0 & 1 \\ \mathbf{j} & -2 & -5 \\ \mathbf{k} & 7 & 5 \end{bmatrix} = 25\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$$

is a normal for the plane (being orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$). Hence the plane has equation

$$25x + 7y + 2z = d \qquad \text{for some number } d.$$

Since $P(1, 3, -2)$ lies in the plane we have $25 \cdot 1 + 7 \cdot 3 + 2(-2) = d$. Hence $d = 42$ and the equation is $25x + 7y + 2z = 42$. Incidentally, the same equation is obtained (verify) if $\overrightarrow{QP}$ and $\overrightarrow{QR}$, or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$, are used as the vectors in the plane.

Example 4.2.14

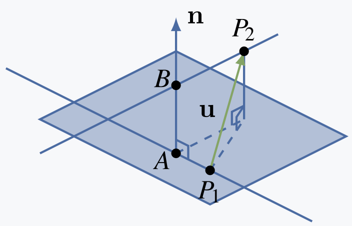
Find the shortest distance between the nonparallel lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Then find the points A and B on the lines that are closest together.

Solution. Direction vectors for the two lines are $\mathbf{d}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, so

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & 0 & 1 \\ \mathbf{k} & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$



is perpendicular to both lines. Consider the plane shaded in the diagram containing the first line with $\mathbf{n}$ as normal. This plane contains $P_1(1, 0, -1)$ and is parallel to the second line. Because $P_2(3, 1, 0)$ is on the second line, the distance in question is just the shortest distance between $P_2(3, 1, 0)$ and this plane. The vector $\mathbf{u}$ from P_1 to P_2 is $\mathbf{u} = \overrightarrow{P_1P_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and so, as in Example 4.2.11, the distance is the length of the projection of $\mathbf{u}$ on $\mathbf{n}$.

$$\text{distance } = \left\| \frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \right\| = \frac{|\mathbf{u} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

Note that it is necessary that $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$ be nonzero for this calculation to be possible. As is shown later (Theorem 4.3.4), this is guaranteed by the fact that $\mathbf{d}_1$ and $\mathbf{d}_2$ are *not* parallel.

The points A and B have coordinates $A(1 + 2t, 0, t - 1)$ and $B(3 + s, 1 + s, -s)$ for some s and t, so $\overrightarrow{AB} = \begin{bmatrix} 2 + s - 2t \\ 1 + s \\ 1 - s - t \end{bmatrix}$. This vector is orthogonal to both $\mathbf{d}_1$ and $\mathbf{d}_2$, and the conditions $\overrightarrow{AB} \cdot \mathbf{d}_1 = 0$ and $\overrightarrow{AB} \cdot \mathbf{d}_2 = 0$ give equations $5t - s = 5$ and $t - 3s = 2$. The solution is $s = \frac{-5}{14}$ and $t = \frac{13}{14}$, so the points are $A(\frac{40}{14}, 0, \frac{-1}{14})$ and $B(\frac{37}{14}, \frac{9}{14}, \frac{5}{14})$. We have $\|\overrightarrow{AB}\| = \frac{3\sqrt{14}}{14}$, as before.

Exercises for 4.2

Exercise 4.2.1 Compute $\mathbf{u}\cdot\mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \mathbf{u}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ -7 \\ -5 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\mathbf{v} = \mathbf{0}$

---

b. 6

d. 0

f. 0

Exercise 4.2.2 Find the angle between the following pairs of vectors.

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5\sqrt{2} \\ -7 \\ -1 \end{bmatrix}$

---

b. π or 180°

d. $\frac{\pi}{3}$ or 60°

f. $\frac{2\pi}{3}$ or 120°

Exercise 4.2.3 Find all real numbers x such that:

a. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

b. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$.

---

b. 1 or -17

Exercise 4.2.4 Find all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both:

a. $\mathbf{u}_1 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

c. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

d. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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b. $t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

d. $s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

Exercise 4.2.5 Find two orthogonal vectors that are both orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Exercise 4.2.6 Consider the triangle with vertices $P(2,\ 0,\ -3)$, $Q(5,\ -2,\ 1)$, and $R(7,\ 5,\ 3)$.

a. Show that it is a right-angled triangle.

b. Find the lengths of the three sides and verify the Pythagorean theorem.

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b. $29 + 57 = 86$

Exercise 4.2.7 Show that the triangle with vertices $A(4,\ -7,\ 9)$, $B(6,\ 4,\ 4)$, and $C(7,\ 10,\ -6)$ is not a right-angled triangle.

Exercise 4.2.8 Find the three internal angles of the triangle with vertices:

a. $A(3,\ 1,\ -2)$, $B(3,\ 0,\ -1)$, and $C(5,\ 2,\ -1)$

b. $A(3,\ 1,\ -2)$, $B(5,\ 2,\ -1)$, and $C(4,\ 3,\ -3)$

---

b. $A = B = C = \frac{\pi}{3}$ or 60°

Exercise 4.2.9 Show that the line through $P_0(3,\ 1,\ 4)$ and $P_1(2,\ 1,\ 3)$ is perpendicular to the line through $P_2(1,\ -1,\ 2)$ and $P_3(0,\ 5,\ 3)$.

Exercise 4.2.10 In each case, compute the projection of $\mathbf{u}$ on $\mathbf{v}$.

a. $\mathbf{u} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

---

b. $\frac{11}{18}\mathbf{v}$

d. $-\frac{1}{2}\mathbf{v}$

Exercise 4.2.11 In each case, write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ -1 \end{bmatrix}$

---

b. $\frac{5}{21} \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} + \frac{1}{21} \begin{bmatrix} 53 \\ 26 \\ 20 \end{bmatrix}$

d. $\frac{27}{53} \begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix} + \frac{1}{53} \begin{bmatrix} -3 \\ 2 \\ 26 \end{bmatrix}$

Exercise 4.2.12 Calculate the distance from the point P to the line in each case and find the point Q on the line closest to P.

a. $P(3,\ 2-1)$

line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

b. $P(1,\ -1,\ 3)$

line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

---

b. $\frac{1}{26}\sqrt{5642}$, $Q(\frac{71}{26},\ \frac{15}{26},\ \frac{34}{26})$

Exercise 4.2.13 Compute $\mathbf{u} \times \mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

---

b. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 4 \\ -15 \\ 8 \end{bmatrix}$

Exercise 4.2.14 Find an equation of each of the following planes.

a. Passing through $A(2,\ 1,\ 3)$, $B(3,\ -1,\ 5)$, and $C(1,\ 2,\ -3)$.

b. Passing through $A(1,\ -1,\ 6)$, $B(0,\ 0,\ 1)$, and $C(4,\ 7,\ -11)$.

c. Passing through $P(2,\ -3,\ 5)$ and parallel to the plane with equation $3x-2y-z=0$.

d. Passing through $P(3,\ 0,\ -1)$ and parallel to the plane with equation $2x-y+z=3$.

e. Containing $P(3,\ 0,\ -1)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

f. Containing $P(2,\ 1,\ 0)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

g. Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

h. Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

i. Each point of which is equidistant from $P(2,\ -1,\ 3)$ and $Q(1,\ 1,\ -1)$.

j. Each point of which is equidistant from $P(0,\ 1,\ -1)$ and $Q(2,\ -1,\ -3)$.

---

b. $-23x+32y+11z=11$

d. $2x-y+z=5$

f. $2x+3y+2z=7$

h. $2x-7y-3z=-1$

j. $x-y-z=3$

Exercise 4.2.15 In each case, find a vector equation of the line.

a. Passing through $P(3, -1, 4)$ and perpendicular to the plane $3x-2y-z=0$.

b. Passing through $P(2, -1, 3)$ and perpendicular to the plane $2x+y=1$.

c. Passing through $P(0, 0, 0)$ and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

d. Passing through $P(1, 1, -1)$, and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

e. Passing through $P(2, 1, -1)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and perpendicular to that line.

f. Passing through $P(1, 1, 2)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and perpendicular to that line.

---

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

f. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}$

Exercise 4.2.16 In each case, find the shortest distance from the point P to the plane and find the point Q on the plane closest to P.

a. $P(2, 3, 0)$; plane with equation $5x+y+z=1$.

b. $P(3, 1, -1)$; plane with equation $2x+y-z=6$.

---

b. $\frac{\sqrt{6}}{3}$, $Q(\frac{7}{3}, \frac{2}{3}, \frac{-2}{3})$

Exercise 4.2.17

a. Does the line through $P(1, 2, -3)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ lie in the plane $2x-y-z=3$? Explain.

b. Does the plane through $P(4, 0, 5)$, $Q(2, 2, 1)$, and $R(1, -1, 2)$ pass through the origin? Explain.

---

b. Yes. The equation is $5x-3y-4z=0$.

Exercise 4.2.18 Show that every plane containing $P(1, 2, -1)$ and $Q(2, 0, 1)$ must also contain $R(-1, 6, -5)$.

Exercise 4.2.19 Find the equations of the line of intersection of the following planes.

a. $2x-3y+2z=5$ and $x+2y-z=4$.

b. $3x+y-2z=1$ and $x+y+z=5$.

---

b. $(-2, 7, 0)+t(3, -5, 2)$

Exercise 4.2.20 In each case, find all points of intersection of the given plane and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

a) $x-3y+2z=4$ b) $2x-y-z=5$

c) $3x-y+z=8$ d) $-x-4y-3z=6$

---

b. None

d. $P(\frac{13}{19}, \frac{-78}{19}, \frac{65}{19})$

Exercise 4.2.21 Find the equation of *all* planes:

a. Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

b. Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

c. Containing the origin.

d. Containing $P(3, 2, -4)$.

e. Containing $P(1, 1, -1)$ and $Q(0, 1, 1)$.

f. Containing $P(2, -1, 1)$ and $Q(1, 0, 0)$.

g. Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

h. Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

---

b. $3x+2z=d$, d arbitrary

d. $a(x-3)+b(y-2)+c(z+4)=0$; a, b, and c not all zero

f. $ax+by+(b-a)z=a$; a and b not both zero

h. $ax+by+(a-2b)z=5a-4b$; a and b not both zero

Exercise 4.2.22 If a plane contains two distinct points P_1 and P_2, show that it contains every point on the line through P_1 and P_2.

Exercise 4.2.23 Find the shortest distance between the following pairs of parallel lines.

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

---

b. $\sqrt{10}$

Exercise 4.2.24 Find the shortest distance between the following pairs of nonparallel lines and find the points on the lines that are closest together.

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

---

b. $\frac{\sqrt{14}}{2}$, $A(3,\ 1,\ 2)$, $B(\frac{7}{2},\ -\frac{1}{2},\ 3)$

d. $\frac{\sqrt{6}}{6}$, $A(\frac{19}{3},\ 2,\ \frac{1}{3})$, $B(\frac{37}{6},\ \frac{13}{6},\ 0)$

Exercise 4.2.25 Show that two lines in the plane with slopes m_1 and m_2 are perpendicular if and only if
$m_1 m_2 = -1$. [*Hint*: Example 4.1.11.]

Exercise 4.2.26

a. Show that, of the four diagonals of a cube, no pair is perpendicular.

b. Show that each diagonal is perpendicular to the face diagonals it does not meet.

---

b. Consider the diagonal $\mathbf{d} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ The six face diagonals in question are $\pm \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix}$, $\pm \begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$, $\pm \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix}$. All of these are orthogonal to $\mathbf{d}$. The result works for the other diagonals by symmetry.

Exercise 4.2.27 Given a rectangular solid with sides of lengths 1, 1, and $\sqrt{2}$, find the angle between a diagonal and one of the longest sides.

Exercise 4.2.28 Consider a rectangular solid with sides of lengths a, b, and c. Show that it has two orthogonal diagonals if and only if the sum of two of a^2, b^2, and c^2 equals the third.

The four diagonals are $(a,\ b,\ c)$, $(-a,\ b,\ c)$, $(a,\ -b,\ c)$ and $(a,\ b,\ -c)$ or their negatives. The dot products are $\pm(-a^2+b^2+c^2)$, $\pm(a^2-b^2+c^2)$, and $\pm(a^2+b^2-c^2)$.

Exercise 4.2.29 Let A, B, and $C(2,\ -1,\ 1)$ be the vertices of a triangle where $\overrightarrow{AB}$ is parallel to $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\overrightarrow{AC}$ is parallel to $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and angle $C = 90^\circ$. Find the equation of the line through B and C.

Exercise 4.2.30 If the diagonals of a parallelogram have equal length, show that the parallelogram is a rectangle.

Exercise 4.2.31 Given $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in component form, show that the projections of $\mathbf{v}$ on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$, respectively.

Exercise 4.2.32

a. Can $\mathbf{u}\cdot\mathbf{v} = -7$ if $\|\mathbf{u}\| = 3$ and $\|\mathbf{v}\| = 2$? Defend your answer.

b. Find $\mathbf{u}\cdot\mathbf{v}$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\|\mathbf{v}\| = 6$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

Exercise 4.2.33 Show $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

Exercise 4.2.34

a. Show $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

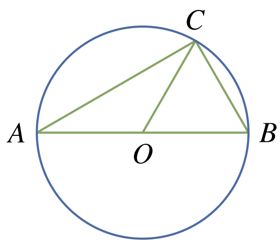
b. What does this say about parallelograms?

---

b. The sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the four sides.

Exercise 4.2.35 Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus. [*Hint*: Example 4.2.5.]

Exercise 4.2.36 Let A and B be the end points of a diameter of a circle (see the diagram). If C is any point on the circle, show that AC and BC are perpendicular. [*Hint*: Express $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$ and $\overrightarrow{BC}$ in terms of $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OC}$, where O is the centre.]



Exercise 4.2.37 Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Exercise 4.2.38 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be pairwise orthogonal vectors.

a. Show that $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

b. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all the same length, show that they all make the same angle with $\mathbf{u}+\mathbf{v}+\mathbf{w}$.

---

b. The angle θ between $\mathbf{u}$ and $(\mathbf{u}+\mathbf{v}+\mathbf{w})$ is given by $\cos\theta = \frac{\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}+\mathbf{w})}{\|\mathbf{u}\|\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|} = \frac{\|\mathbf{u}\|}{\sqrt{\|\mathbf{u}\|^2+\|\mathbf{v}\|^2+\|\mathbf{w}\|^2}} = \frac{1}{\sqrt{3}}$ because $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\|$. Similar remarks apply to the other angles.

Exercise 4.2.39

a. Show that $\mathbf{n} = \left[\begin{array}{c} a \\ b \end{array}\right]$ is orthogonal to every vector along the line $ax+by+c=0$.

b. Show that the shortest distance from $P_0(x_0, y_0)$ to the line is $\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$. [*Hint*: If P_1 is on the line, project $\mathbf{u} = \overrightarrow{P_1P_0}$ on $\mathbf{n}$.]

---

b. Let $\mathbf{p}_0$, $\mathbf{p}_1$ be the vectors of P_0, P_1, so $\mathbf{u} = \mathbf{p}_0 - \mathbf{p}_1$. Then $\mathbf{u}\cdot\mathbf{n} = \mathbf{p}_0\cdot\mathbf{n} - \mathbf{p}_1\cdot\mathbf{n} = (ax_0 + by_0) - (ax_1 + by_1) = ax_0 + by_0 + c$. Hence the distance is

$$\left\|\left(\frac{\mathbf{u}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\mathbf{n}\right\| = \frac{|\mathbf{u}\cdot\mathbf{n}|}{\|\mathbf{n}\|}$$

as required.

Exercise 4.2.40 Assume $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors that are not parallel. Show that $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ is a nonzero vector that bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Exercise 4.2.41 Let α, β, and γ be the angles a vector $\mathbf{v} \neq \mathbf{0}$ makes with the positive x, y, and z axes, respectively. Then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector $\mathbf{v}$.

a. If $\mathbf{v} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right]$, show that $\cos\alpha = \frac{a}{\|\mathbf{v}\|}$, $\cos\beta = \frac{b}{\|\mathbf{v}\|}$, and $\cos\gamma = \frac{c}{\|\mathbf{v}\|}$.

b. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

---

b. This follows from **(a)** because $\|\mathbf{v}\|^2 = a^2 + b^2 + c^2$.

Exercise 4.2.42 Let $\mathbf{v} \neq \mathbf{0}$ be any nonzero vector and suppose that a vector $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p}$ is parallel to $\mathbf{v}$ and $\mathbf{q}$ is orthogonal to $\mathbf{v}$. Show that $\mathbf{p}$ must equal the projection of $\mathbf{u}$ on $\mathbf{v}$. [*Hint*: Argue as in the proof of Theorem 4.2.4.]

Exercise 4.2.43 Let $\mathbf{v} \neq \mathbf{0}$ be a nonzero vector and let $a \neq 0$ be a scalar. If $\mathbf{u}$ is any vector, show that the projection of $\mathbf{u}$ on $\mathbf{v}$ equals the projection of $\mathbf{u}$ on $a\mathbf{v}$.

Exercise 4.2.44

a. Show that the **Cauchy-Schwarz inequality** $|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: $|\cos\theta| \leq 1$ for all angles θ.]

b. Show that $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel. [*Hint*: When is $\cos\theta = \pm 1$?]

c. Show that $|x_1x_2 + y_1y_2 + z_1z_2| \leq \sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}$ holds for all numbers x_1, x_2, y_1, y_2, z_1, and z_2.

d. Show that $|xy + yz + zx| \leq x^2 + y^2 + z^2$ for all x, y, and z.

e. Show that $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ holds for all x, y, and z.

---

d. Take $\left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right] = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ and $\left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right] = \left[\begin{array}{c} y \\ z \\ x \end{array}\right]$ in (**c**).

Exercise 4.2.45 Prove that the **triangle inequality** $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\|+\|\mathbf{v}\|$ holds for all vectors **u** and **v**. [*Hint*: Consider the triangle with **u** and **v** as two sides.]

4.3 More on the Cross Product

The cross product $\mathbf{v} \times \mathbf{w}$ of two $\mathbb{R}^3$-vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ was defined in Section 4.2 where we observed that it can be best remembered using a determinant:

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathbf{k} \tag{4.3}$$

Here $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are the coordinate vectors, and the determinant is expanded along the first column. We observed (but did not prove) in Theorem 4.2.5 that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. This follows easily from the next result.

Theorem 4.3.1

If $\mathbf{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *then* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$.

Proof. Recall that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is computed by multiplying corresponding components of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ and then adding. Using equation (4.3), the result is:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = x_0 \left(\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \right) + y_0 \left(- \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \right) + z_0 \left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$$

where the last determinant is expanded along column 1. □

The result in Theorem 4.3.1 can be succinctly stated as follows: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$$

where $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ denotes the matrix with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns. Now it is clear that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ because the determinant of a matrix is zero if two columns are identical.

Because of (4.3) and Theorem 4.3.1, several of the following properties of the cross product follow from properties of determinants (they can also be verified directly).

Theorem 4.3.2

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote arbitrary vectors in* $\mathbb{R}^3$.

1. $\mathbf{u} \times \mathbf{v}$ *is a vector.*
2. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$.
3. $\mathbf{u} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{u}$.
4. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
5. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.
6. $(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$ *for any scalar* k.
7. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.
8. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$.

Proof. (1) is clear; (2) follows from Theorem 4.3.1; and (3) and (4) follow because the determinant of a matrix is zero if one column is zero or if two columns are identical. If two columns are interchanged, the determinant changes sign, and this proves (5). The proofs of (6), (7), and (8) are left as Exercise 4.3.15. □

We now come to a fundamental relationship between the dot and cross products.

Theorem 4.3.3: Lagrange Identity[12]

If $\mathbf{u}$ *and* $\mathbf{v}$ *are any two vectors in* $\mathbb{R}^3$, *then*

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof. Given $\mathbf{u}$ and $\mathbf{v}$, introduce a coordinate system and write $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ in component form. Then all the terms in the identity can be computed in terms of the components. The detailed proof is left as Exercise 4.3.14. □

An expression for the magnitude of the vector $\mathbf{u} \times \mathbf{v}$ can be easily obtained from the Lagrange identity. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, substituting $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$ into the Lagrange identity gives

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2 \theta = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta$$

using the fact that $1 - \cos^2 \theta = \sin^2 \theta$. But $\sin \theta$ is nonnegative on the range $0 \leq \theta \leq \pi$, so taking the positive square root of both sides gives

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

[12] Joseph Louis Lagrange (1736–1813) was born in Italy and spent his early years in Turin. At the age of 19 he solved a famous problem by inventing an entirely new method, known today as the calculus of variations, and went on to become one of the greatest mathematicians of all time. His work brought a new level of rigour to analysis and his *Mécanique Analytique* is a masterpiece in which he introduced methods still in use. In 1766 he was appointed to the Berlin Academy by Frederik the Great who asserted that the "greatest mathematician in Europe" should be at the court of the "greatest king in Europe." After the death of Frederick, Lagrange went to Paris at the invitation of Louis XVI. He remained there throughout the revolution and was made a count by Napoleon.

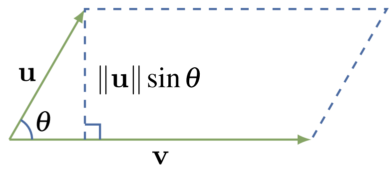


Figure 4.3.1

This expression for $\|\mathbf{u}\times\mathbf{v}\|$ makes no reference to a coordinate system and, moreover, it has a nice geometrical interpretation. The parallelogram determined by the vectors **u** and **v** has base length $\|\mathbf{v}\|$ and altitude $\|\mathbf{u}\|\sin\theta$ (see Figure 4.3.1). Hence the area of the parallelogram formed by **u** and **v** is

$$(\|\mathbf{u}\|\sin\theta)\|\mathbf{v}\| = \|\mathbf{u}\times\mathbf{v}\|$$

This proves the first part of Theorem 4.3.4.

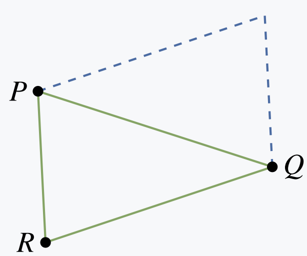
Theorem 4.3.4

If **u** *and* **v** *are two nonzero vectors and* θ *is the angle between* **u** *and* **v**, *then*

1. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta =$ *the area of the parallelogram determined by* **u** *and* **v**.
2. **u** *and* **v** *are parallel if and only if* $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

Proof of (2). By (1), $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if the area of the parallelogram is zero. By Figure 4.3.1 the area vanishes if and only if **u** and **v** have the same or opposite direction—that is, if and only if they are parallel. □

Example 4.3.1



Find the area of the triangle with vertices $P(2,\ 1,\ 0)$, $Q(3,\ -1,\ 1)$, and $R(1,\ 0,\ 1)$.

Solution. We have $\overrightarrow{RP} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\overrightarrow{RQ} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. The area of the triangle is half the area of the parallelogram (see the diagram), and so equals $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\|$. We have

$$\overrightarrow{RP}\times\overrightarrow{RQ} = \det\begin{bmatrix} \mathbf{i} & 1 & 2 \\ \mathbf{j} & 1 & -1 \\ \mathbf{k} & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

so the area of the triangle is $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\| = \frac{1}{2}\sqrt{1+4+9} = \frac{1}{2}\sqrt{14}$.

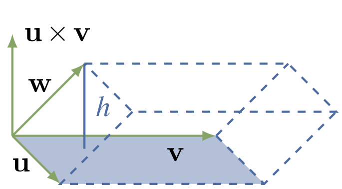


Figure 4.3.2

If three vectors **u**, **v**, and **w** are given, they determine a "squashed" rectangular solid called a **parallelepiped** (Figure 4.3.2), and it is often useful to be able to find the volume of such a solid. The base of the solid is the parallelogram determined by **u** and **v**, so it has area $A = \|\mathbf{u}\times\mathbf{v}\|$ by Theorem 4.3.4. The height of the solid is the length h of the projection of **w** on $\mathbf{u}\times\mathbf{v}$. Hence

$$h = \left|\frac{\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})}{\|\mathbf{u}\times\mathbf{v}\|^2}\right|\|\mathbf{u}\times\mathbf{v}\| = \frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{\|\mathbf{u}\times\mathbf{v}\|} = \frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{A}$$

Thus the volume of the parallelepiped is $hA = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$. This proves

Theorem 4.3.5

The volume of the parallelepiped determined by three vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ (Figure 4.3.2) is given by $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$.

Example 4.3.2

Find the volume of the parallelepiped determined by the vectors

$$\mathbf{w} = \left[\begin{array}{r} 1 \\ 2 \\ -1 \end{array} \right], \ \mathbf{u} = \left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array} \right], \ \mathbf{v} = \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array} \right]$$

Solution. By Theorem 4.3.1, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \det \left[\begin{array}{rrr} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{array} \right] = -3$. Hence the volume is $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| = |-3| = 3$ by Theorem 4.3.5.

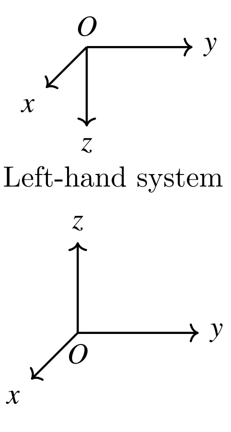


Figure 4.3.3

We can now give an intrinsic description of the cross product $\mathbf{u} \times \mathbf{v}$. Its magnitude $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$ is coordinate-free. If $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, its direction is very nearly determined by the fact that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and so points along the line normal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. It remains only to decide which of the two possible directions is correct.

Before this can be done, the basic issue of how coordinates are assigned must be clarified. When coordinate axes are chosen in space, the procedure is as follows: An origin is selected, two perpendicular lines (the x and y axes) are chosen through the origin, and a positive direction on each of these axes is selected quite arbitrarily. Then the line through the origin normal to this x-y plane is called the z axis, but there is a choice of which direction on this axis is the positive one. The two possibilities are shown in Figure 4.3.3, and it is a standard convention that cartesian coordinates are always **right-hand coordinate systems**. The reason for this terminology is that, in such a system, if the z axis is grasped in the right hand with the thumb pointing in the positive z direction, then the fingers curl around from the positive x axis to the positive y axis (through a right angle).

Suppose now that $\mathbf{u}$ and $\mathbf{v}$ are given and that θ is the angle between them (so $0 \leq \theta \leq \pi$). Then the direction of $\|\mathbf{u} \times \mathbf{v}\|$ is given by the right-hand rule.

Right-hand Rule

If the vector $\mathbf{u}\times\mathbf{v}$ *is grasped in the right hand and the fingers curl around from* $\mathbf{u}$ *to* $\mathbf{v}$ *through the angle* θ, *the thumb points in the direction for* $\mathbf{u}\times\mathbf{v}$.

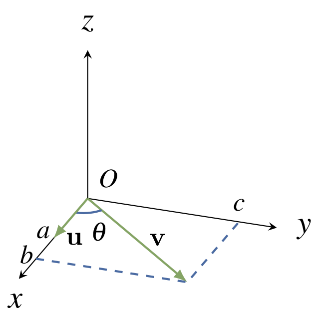


Figure 4.3.4

To indicate why this is true, introduce coordinates in $\mathbb{R}^3$ as follows: Let $\mathbf{u}$ and $\mathbf{v}$ have a common tail O, choose the origin at O, choose the x axis so that $\mathbf{u}$ points in the positive x direction, and then choose the y axis so that $\mathbf{v}$ is in the x-y plane and the positive y axis is on the same side of the x axis as $\mathbf{v}$. Then, in this system, $\mathbf{u}$ and $\mathbf{v}$ have component form $\mathbf{u}=\left[\begin{array}{c} a \\ 0 \\ 0 \end{array}\right]$ and $\mathbf{v}=\left[\begin{array}{c} b \\ c \\ 0 \end{array}\right]$ where $a>0$ and $c>0$. The situation is depicted in Figure 4.3.4. The right-hand rule asserts that $\mathbf{u}\times\mathbf{v}$ should point in the positive z direction. But our definition of $\mathbf{u}\times\mathbf{v}$ gives

$$\mathbf{u}\times\mathbf{v}=\det\left[\begin{array}{ccc} \mathbf{i} & a & b \\ \mathbf{j} & 0 & c \\ \mathbf{k} & 0 & 0 \end{array}\right]=\left[\begin{array}{c} 0 \\ 0 \\ ac \end{array}\right]=(ac)\mathbf{k}$$

and $(ac)\mathbf{k}$ has the positive z direction because $ac>0$.

Exercises for 4.3

Exercise 4.3.1 If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the coordinate vectors, verify that $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$, and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$.

Exercise 4.3.2 Show that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ need not equal $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ by calculating both when

$$\mathbf{u}=\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right],\ \mathbf{v}=\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right],\ \text{and } \mathbf{w}=\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

Exercise 4.3.3 Find two unit vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ if:

a. $\mathbf{u}=\left[\begin{array}{c} 1 \\ 2 \\ 2 \end{array}\right]$, $\mathbf{v}=\left[\begin{array}{r} 2 \\ -1 \\ 2 \end{array}\right]$

b. $\mathbf{u}=\left[\begin{array}{r} 1 \\ 2 \\ -1 \end{array}\right]$, $\mathbf{v}=\left[\begin{array}{c} 3 \\ 1 \\ 2 \end{array}\right]$

---

b. $\pm\frac{\sqrt{3}}{3}\left[\begin{array}{r} 1 \\ -1 \\ -1 \end{array}\right]$.

Exercise 4.3.4 Find the area of the triangle with the following vertices.

a. $A(3,\ -1,\ 2)$, $B(1,\ 1,\ 0)$, and $C(1,\ 2,\ -1)$

b. $A(3,\ 0,\ 1)$, $B(5,\ 1,\ 0)$, and $C(7,\ 2,\ -1)$

c. $A(1,\ 1,\ -1)$, $B(2,\ 0,\ 1)$, and $C(1,\ -1,\ 3)$

d. $A(3,\ -1,\ 1)$, $B(4,\ 1,\ 0)$, and $C(2,\ -3,\ 0)$

---

b. 0

d. $\sqrt{5}$

Exercise 4.3.5 Find the volume of the parallelepiped determined by $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ when:

a. $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

b. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

---

b. 7

Exercise 4.3.6 Let P_0 be a point with vector $\mathbf{p}_0$, and let $ax+by+cz=d$ be the equation of a plane with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

a. Show that the point on the plane closest to P_0 has vector $\mathbf{p}$ given by

$$\mathbf{p} = \mathbf{p}_0 + \frac{d-(\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}.$$

[*Hint*: $\mathbf{p} = \mathbf{p}_0 + t\mathbf{n}$ for some t, and $\mathbf{p} \cdot \mathbf{n} = \mathbf{d}$.]

b. Show that the shortest distance from P_0 to the plane is $\frac{|d-(\mathbf{p}_0 \cdot \mathbf{n})|}{\|\mathbf{n}\|}$.

c. Let P_0' denote the reflection of P_0 in the plane—that is, the point on the opposite side of the plane such that the line through P_0 and P_0' is perpendicular to the plane. Show that $\mathbf{p}_0 + 2\frac{d-(\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}$ is the vector of P_0'.

---

b. The distance is $\|\mathbf{p} - \mathbf{p}_0\|$; use part (a.).

Exercise 4.3.7 Simplify $(a\mathbf{u} + b\mathbf{v}) \times (c\mathbf{u} + d\mathbf{v})$.

Exercise 4.3.8 Show that the shortest distance from a point P to the line through P_0 with direction vector $\mathbf{d}$ is $\frac{\|\overrightarrow{P_0P} \times \mathbf{d}\|}{\|\mathbf{d}\|}$.

Exercise 4.3.9 Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero, nonorthogonal vectors. If θ is the angle between them, show that $\tan\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\mathbf{u} \cdot \mathbf{v}}$.

Exercise 4.3.10 Show that points A, B, and C are all on one line if and only if $\overrightarrow{AB} \times \overrightarrow{AC} = 0$

$\|\overrightarrow{AB} \times \overrightarrow{AC}\|$ is the area of the parallelogram determined by A, B, and C.

Exercise 4.3.11 Show that points A, B, C, and D are all on one plane if and only if $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$

Exercise 4.3.12 Use Theorem 4.3.5 to confirm that, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular, the (rectangular) parallelepiped they determine has volume $\|\mathbf{u}\|\|\mathbf{v}\|\|\mathbf{w}\|$.

Because $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ are parallel, the angle θ between them is 0 or π. Hence $\cos(\theta) = \pm 1$, so the volume is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \|\mathbf{u}\|\|\mathbf{v} \times \mathbf{w}\| \cos(\theta) = \|\mathbf{u}\|\|(\mathbf{v} \times \mathbf{w})\|$. But the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{2}$ so $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \cos(\frac{\pi}{2}) = \|\mathbf{v}\|\|\mathbf{w}\|$. The result follows.

Exercise 4.3.13 Show that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is $\|\mathbf{u} \times \mathbf{v}\|^2$.

Exercise 4.3.14 Complete the proof of Theorem 4.3.3.

Exercise 4.3.15 Prove the following properties in Theorem 4.3.2.

a) Property 6

b) Property 7

c) Property 8

---

b. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$,

then $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \det \begin{bmatrix} \mathbf{i} & u_1 & v_1+w_1 \\ \mathbf{j} & u_2 & v_2+w_2 \\ \mathbf{k} & u_3 & v_3+w_3 \end{bmatrix}$

$= \det \begin{bmatrix} \mathbf{i} & u_1 & v_1 \\ \mathbf{j} & u_2 & v_2 \\ \mathbf{k} & u_3 & v_3 \end{bmatrix} + \det \begin{bmatrix} \mathbf{i} & u_1 & w_1 \\ \mathbf{j} & u_2 & w_2 \\ \mathbf{k} & u_3 & w_3 \end{bmatrix}$

$= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ where we used Exercise 4.3.21.

Exercise 4.3.16

a. Show that $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\mathbf{w} \times \mathbf{u})$ holds for all vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$.

b. Show that $\mathbf{v}-\mathbf{w}$ and $(\mathbf{u}\times\mathbf{v})+(\mathbf{v}\times\mathbf{w})+(\mathbf{w}\times\mathbf{u})$ are orthogonal.

---

b. $(\mathbf{v}-\mathbf{w})\cdot[(\mathbf{u}\times\mathbf{v})+(\mathbf{v}\times\mathbf{w})+(\mathbf{w}\times\mathbf{u})]=(\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}\times\mathbf{v})+(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w})+(\mathbf{v}-\mathbf{w})\cdot(\mathbf{w}\times\mathbf{u})=-\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})+0+\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})=0$.

Exercise 4.3.17 Show $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\times\mathbf{v})\mathbf{w}$. [*Hint*: First do it for $\mathbf{u}=\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; then write $\mathbf{u}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and use Theorem 4.3.2.]

Exercise 4.3.18 Prove the **Jacobi identity**:

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v})=\mathbf{0}$$

[*Hint*: The preceding exercise.]

Exercise 4.3.19 Show that

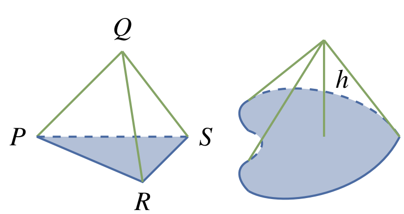
$$(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{z})=\det\begin{bmatrix}\mathbf{u}\cdot\mathbf{w} & \mathbf{u}\cdot\mathbf{z}\\ \mathbf{v}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{z}\end{bmatrix}$$

[*Hint*: Exercises 4.3.16 and 4.3.17.]

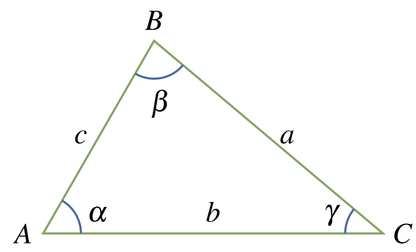
Exercise 4.3.20 Let P, Q, R, and S be four points, not all on one plane, as in the diagram. Show that the volume of the pyramid they determine is

$$\frac{1}{6}|\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})|.$$

[*Hint*: The volume of a cone with base area A and height h as in the diagram below right is $\frac{1}{3}Ah$.]



Exercise 4.3.21 Consider a triangle with vertices A, B, and C, as in the diagram below. Let α, β, and γ denote the angles at A, B, and C, respectively, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Write $\mathbf{u}=\overrightarrow{AB}$, $\mathbf{v}=\overrightarrow{BC}$, and $\mathbf{w}=\overrightarrow{CA}$.



a. Deduce that $\mathbf{u}+\mathbf{v}+\mathbf{w}=\mathbf{0}$.

b. Show that $\mathbf{u}\times\mathbf{v}=\mathbf{w}\times\mathbf{u}=\mathbf{v}\times\mathbf{w}$. [*Hint*: Compute $\mathbf{u}\times(\mathbf{u}+\mathbf{v}+\mathbf{w})$ and $\mathbf{v}\times(\mathbf{u}+\mathbf{v}+\mathbf{w})$.]

c. Deduce the **law of sines**:

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

Exercise 4.3.22 Show that the (shortest) distance between two planes $\mathbf{n}\cdot\mathbf{p}=d_1$ and $\mathbf{n}\cdot\mathbf{p}=d_2$ with $\mathbf{n}$ as normal is $\frac{|d_2-d_1|}{\|\mathbf{n}\|}$.

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be vectors of points in the planes, so $\mathbf{p}_1\cdot\mathbf{n}=d_1$ and $\mathbf{p}_2\cdot\mathbf{n}=d_2$. The distance is the length of the projection of $\mathbf{p}_2-\mathbf{p}_1$ along $\mathbf{n}$; that is $\frac{|(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{n}|}{\|\mathbf{n}\|}=\frac{|d_1-d_2|}{\|\mathbf{n}\|}$.

Exercise 4.3.23 Let A and B be points other than the origin, and let $\mathbf{a}$ and $\mathbf{b}$ be their vectors. If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, show that the plane through A, B, and the origin is given by

$$\left\{P(x,\ y,\ z)\ \middle|\ \begin{bmatrix}x\\y\\z\end{bmatrix}=s\mathbf{a}+t\mathbf{b}\text{ for some } s \text{ and } t\right\}$$

Exercise 4.3.24 Let A be a 2×3 matrix of rank 2 with rows $\mathbf{r}_1$ and $\mathbf{r}_2$. Show that

$$P=\{XA\mid X=[xy];x,\ y\text{ arbitrary}\}$$

is the plane through the origin with normal $\mathbf{r}_1\times\mathbf{r}_2$.

Exercise 4.3.25 Given the cube with vertices $P(x,\ y,\ z)$, where each of x, y, and z is either 0 or 2, consider the plane perpendicular to the diagonal through $P(0,\ 0,\ 0)$ and $P(2,\ 2,\ 2)$ and bisecting it.

a. Show that the plane meets six of the edges of the cube and bisects them.

b. Show that the six points in (a) are the vertices of a regular hexagon.

4.4 Linear Operators on $\mathbb{R}^3$

Recall that a transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ holds for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. In this case we showed (in Theorem 2.6.2) that there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and we say that T is the **matrix transformation induced** by A.

Definition 4.9 Linear Operator on $\mathbb{R}^n$

A linear transformation

$$T:\mathbb{R}^n \to \mathbb{R}^n$$

is called a ***linear operator*** *on* $\mathbb{R}^n$.

In Section 2.6 we investigated three important linear operators on $\mathbb{R}^2$: rotations about the origin, reflections in a line through the origin, and projections on this line.

In this section we investigate the analogous operators on $\mathbb{R}^3$: Rotations about a line through the origin, reflections in a plane through the origin, and projections onto a plane or line through the origin in $\mathbb{R}^3$. In every case we show that the operator is linear, and we find the matrices of all the reflections and projections.

To do this we must prove that these reflections, projections, and rotations are actually *linear* operators on $\mathbb{R}^3$. In the case of reflections and rotations, it is convenient to examine a more general situation. A transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is said to be **distance preserving** if the distance between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the distance between $\mathbf{v}$ and $\mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$; that is,

$$\|T(\mathbf{v}) - T(\mathbf{w})\| = \|\mathbf{v}-\mathbf{w}\| \text{ for all } \mathbf{v} \text{ and } \mathbf{w} \text{ in } \mathbb{R}^3 \tag{4.4}$$

Clearly reflections and rotations are distance preserving, and both carry $\mathbf{0}$ to $\mathbf{0}$, so the following theorem shows that they are both linear.

Theorem 4.4.1

If $T:\mathbb{R}^3 \to \mathbb{R}^3$ is distance preserving, and if $T(\mathbf{0}) = \mathbf{0}$, then T is linear.

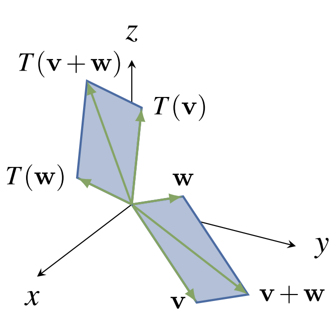


Figure 4.4.1

Proof. Since $T(\mathbf{0}) = \mathbf{0}$, taking $\mathbf{w} = \mathbf{0}$ in (4.4) shows that $\|T(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v}$ in $\mathbb{R}^3$, that is T preserves length. Also, $\|T(\mathbf{v}) - T(\mathbf{w})\|^2 = \|\mathbf{v}-\mathbf{w}\|^2$ by (4.4). Since $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2$ always holds, it follows that $T(\mathbf{v}) \cdot T(\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$. Hence (by Theorem 4.2.2) the angle between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the angle between $\mathbf{v}$ and $\mathbf{w}$ for all (nonzero) vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.

With this we can show that T is linear. Given nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, the vector $\mathbf{v}+\mathbf{w}$ is the diagonal of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. By the preceding paragraph, the effect of T is to carry this *entire parallelogram* to the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$, with diagonal $T(\mathbf{v}+\mathbf{w})$. But this diagonal is $T(\mathbf{v})+T(\mathbf{w})$ by the parallelogram law (see Figure 4.4.1).

In other words, $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v})+T(\mathbf{w})$. A similar argument shows that $T(a\mathbf{v}) = aT(\mathbf{v})$ for all scalars a, proving that T is indeed linear. □

Distance-preserving linear operators are called **isometries**, and we return to them in Section **??**.

Reflections and Projections

In Section 2.6 we studied the reflection $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ in the line $y = mx$ and projection $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ on the same line. We found (in Theorems 2.6.5 and 2.6.6) that they are both linear and

$$Q_m \text{ has matrix } \frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right] \quad \text{and} \quad P_m \text{ has matrix } \frac{1}{1+m^2}\left[\begin{array}{cc} 1 & m \\ m & m^2 \end{array}\right].$$

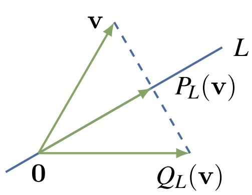


Figure 4.4.2

We now look at the analogues in $\mathbb{R}^3$.

Let L denote a line through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_L(\mathbf{v})$ of $\mathbf{v}$ in L and the projection $P_L(\mathbf{v})$ of $\mathbf{v}$ on L are defined in Figure 4.4.2. In the same figure, we see that

$$P_L(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_L(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_L(\mathbf{v}) + \mathbf{v}] \tag{4.5}$$

so the fact that Q_L is linear (by Theorem 4.4.1) shows that P_L is also linear.[13]

However, Theorem 4.2.4 gives us the matrix of P_L directly. In fact, if $\mathbf{d} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right] \neq \mathbf{0}$ is a direction vector for L, and we write $\mathbf{v} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$, then

$$P_L(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{ax+by+cz}{a^2+b^2+c^2}\left[\begin{array}{c} a \\ b \\ c \end{array}\right] = \frac{1}{a^2+b^2+c^2}\left[\begin{array}{ccc} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right]$$

as the reader can verify. Note that this shows directly that P_L is a matrix transformation and so gives another proof that it is linear.

Theorem 4.4.2

Let L denote the line through the origin in $\mathbb{R}^3$ with direction vector $\mathbf{d} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right] \neq \mathbf{0}$. Then

[13] Note that Theorem 4.4.1 does *not* apply to P_L since it does not preserve distance.

P_L and Q_L are both linear and

$$P_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$$Q_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}$$

Proof. It remains to find the matrix of Q_L. But (4.5) implies that $Q_L(\mathbf{v}) = 2P_L(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, so if $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we obtain (with some matrix arithmetic):

$$\begin{aligned} Q_L(\mathbf{v}) &= \left\{ \frac{2}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

as required. □

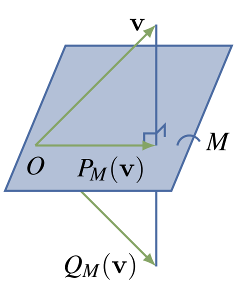


Figure 4.4.3

In $\mathbb{R}^3$ we can reflect in planes as well as lines. Let M denote a plane through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_M(\mathbf{v})$ of $\mathbf{v}$ in M and the projection $P_M(\mathbf{v})$ of $\mathbf{v}$ on M are defined in Figure 4.4.3. As above, we have

$$P_M(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_M(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_M(\mathbf{v}) + \mathbf{v}]$$

so the fact that Q_M is linear (again by Theorem 4.4.1) shows that P_M is also linear.

Again we can obtain the matrix directly. If $\mathbf{n}$ is a normal for the plane M, then Figure 4.4.3 shows that

$$P_M(\mathbf{v}) = \mathbf{v} - \text{proj}_{\mathbf{n}}\, \mathbf{v} = \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \text{ for all vectors } \mathbf{v}.$$

If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, a computation like the above gives

$$\begin{aligned} P_M(\mathbf{v}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{ax+by+cz}{a^2+b^2+c^2}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & b^2+c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

This proves the first part of

Theorem 4.4.3

Let M denote the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$. Then P_M and Q_M are both linear and

$$P_M \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}$$

$$Q_M \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

Proof. It remains to compute the matrix of Q_M. Since $Q_M(\mathbf{v}) = 2P_M(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, the computation is similar to the above and is left as an exercise for the reader. □

Rotations

In Section 2.6 we studied the rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise about the origin through the angle θ. Moreover, we showed in Theorem 2.6.4 that R_θ is linear and has matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. One extension of this is given in the following example.

Example 4.4.1

Let $R_{z,\ \theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation of $\mathbb{R}^3$ about the z axis through an angle θ from the positive x axis toward the positive y axis. Show that $R_{z,\ \theta}$ is linear and find its matrix.

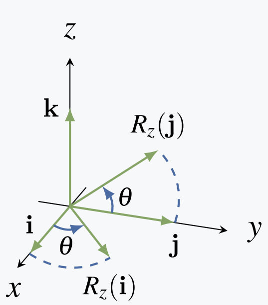


Figure 4.4.4

Solution. First R is distance preserving and so is linear by Theorem 4.4.1. Hence we apply Theorem 2.6.2 to obtain the matrix of $R_{z,\ \theta}$.

Let $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ denote the standard basis of $\mathbb{R}^3$; we must find $R_{z,\ \theta}(\mathbf{i})$, $R_{z,\ \theta}(\mathbf{j})$, and $R_{z,\ \theta}(\mathbf{k})$. Clearly $R_{z,\ \theta}(\mathbf{k}) = \mathbf{k}$. The effect of $R_{z,\ \theta}$ on the x-y plane is to rotate it counterclockwise through the angle θ. Hence Figure 4.4.4 gives

$$R_{z,\ \theta}(\mathbf{i}) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad R_{z,\ \theta}(\mathbf{j}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

so, by Theorem 2.6.2, $R_{z,\ \theta}$ has matrix

$$\left[\begin{array}{ccc} R_{z,\ \theta}(\mathbf{i}) & R_{z,\ \theta}(\mathbf{j}) & R_{z,\ \theta}(\mathbf{k}) \end{array}\right] = \left[\begin{array}{ccc} \cos\theta & -\sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Example 4.4.1 begs to be generalized. Given a line L through the origin in $\mathbb{R}^3$, every rotation about L through a fixed angle is clearly distance preserving, and so is a linear operator by Theorem 4.4.1. However, giving a precise description of the matrix of this rotation is not easy and will have to wait until more techniques are available.

Transformations of Areas and Volumes

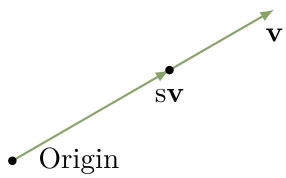


Figure 4.4.5

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Each vector in the same direction as $\mathbf{v}$ whose length is a fraction s of the length of $\mathbf{v}$ has the form $s\mathbf{v}$ (see Figure 4.4.5).

With this, scrutiny of Figure 4.4.6 shows that a vector $\mathbf{u}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ if and only if it has the form $\mathbf{u} = s\mathbf{v} + t\mathbf{w}$ where $0 \leq s \leq 1$ and $0 \leq t \leq 1$. But then, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, we have

$$T(s\mathbf{v} + t\mathbf{w}) = T(s\mathbf{v}) + T(t\mathbf{w}) = sT(\mathbf{v}) + tT(\mathbf{w})$$

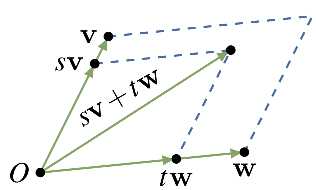


Figure 4.4.6

Hence $T(s\mathbf{v} + t\mathbf{w})$ is in the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$. Conversely, every vector in this parallelogram has the form $T(s\mathbf{v} + t\mathbf{w})$ where $s\mathbf{v} + t\mathbf{w}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. For this reason, the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$ is called the **image** of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. We record this discussion as:

Theorem 4.4.4

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ (or $\mathbb{R}^2 \to \mathbb{R}^2$) is a linear operator, the image of the parallelogram determined by vectors $\mathbf{v}$ and $\mathbf{w}$ is the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$.

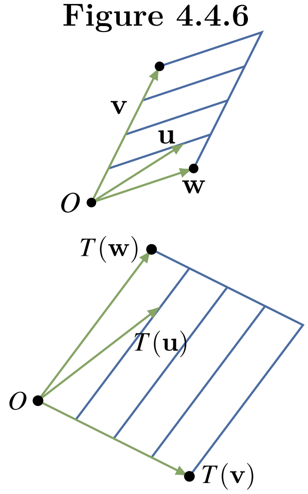


Figure 4.4.7

This result is illustrated in Figure 4.4.7, and was used in Examples 2.2.15 and 2.2.16 to reveal the effect of expansion and shear transformations.

We now describe the effect of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ on the parallelepiped determined by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ (see the discussion preceding Theorem 4.3.5). If T has matrix A, Theorem 4.4.4 shows that this parallelepiped is carried to the parallelepiped determined by $T(\mathbf{u}) = A\mathbf{u}$, $T(\mathbf{v}) = A\mathbf{v}$, and $T(\mathbf{w}) = A\mathbf{w}$. In particular, we want to discover how the volume changes, and it turns out to be closely related to the determinant of the matrix A.

Theorem 4.4.5

Let $\text{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ *denote the volume of the parallelepiped determined by three vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* $\mathbb{R}^3$, *and let* $\text{area}(\mathbf{p}, \mathbf{q})$ *denote the area of the parallelogram determined by two vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{R}^2$. *Then:*

1. *If* A *is a* 3×3 *matrix, then* $\text{vol}(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = |\det(A)| \cdot \text{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.
2. *If* A *is a* 2×2 *matrix, then* $\text{area}(A\mathbf{p}, A\mathbf{q}) = |\det(A)| \cdot \text{area}(\mathbf{p}, \mathbf{q})$.

Proof.

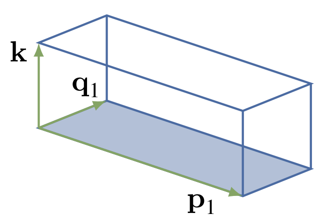
1. Let $\left[\begin{array}{ccc} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{array}\right]$ denote the 3×3 matrix with columns $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then

$$\text{vol}(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = |A\mathbf{u}\cdot(A\mathbf{v}\times A\mathbf{w})|$$

by Theorem 4.3.5. Now apply Theorem 4.3.1 twice to get

$$\begin{aligned} A\mathbf{u}\cdot(A\mathbf{v}\times A\mathbf{w}) &= \det\left[\begin{array}{ccc} A\mathbf{u} & A\mathbf{v} & A\mathbf{w} \end{array}\right] = \det\left(A\left[\begin{array}{ccc} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{array}\right]\right) \\ &= \det(A)\det\left[\begin{array}{ccc} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{array}\right] \\ &= \det(A)(\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})) \end{aligned}$$

where we used Definition 2.9 and the product theorem for determinants. Finally (1) follows from Theorem 4.3.5 by taking absolute values.



2. Given $\mathbf{p} = \left[\begin{array}{c} x \\ y \end{array}\right]$ in $\mathbb{R}^2$, $\mathbf{p}_1 = \left[\begin{array}{c} x \\ y \\ 0 \end{array}\right]$ in $\mathbb{R}^3$. By the diagram, $\text{area}(\mathbf{p}, \mathbf{q}) = \text{vol}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{k})$ where $\mathbf{k}$ is the (length 1) coordinate vector along the z axis. If A is a 2×2 matrix, write $A_1 = \left[\begin{array}{cc} A & 0 \\ 0 & 1 \end{array}\right]$ in block form, and observe that $(A\mathbf{v})_1 = (A_1\mathbf{v}_1)$ for all $\mathbf{v}$ in $\mathbb{R}^2$ and $A_1\mathbf{k} = \mathbf{k}$. Hence part (1) of this theorem shows

$$\begin{aligned} \text{area}(A\mathbf{p}, A\mathbf{q}) &= \text{vol}(A_1\mathbf{p}_1, A_1\mathbf{q}_1, A_1\mathbf{k}) \\ &= |\det(A_1)|\,\text{vol}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{k}) \\ &= |\det(A)|\,\text{area}(\mathbf{p}, \mathbf{q}) \end{aligned}$$

as required.

□

Define the **unit square** and **unit cube** to be the square and cube corresponding to the coordinate vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then Theorem 4.4.5 gives a geometrical meaning to the determinant of a matrix A:

- If A is a 2×2 matrix, then $|\det(A)|$ is the area of the image of the unit square under multiplication by A;
- If A is a 3×3 matrix, then $|\det(A)|$ is the volume of the image of the unit cube under multiplication by A.

These results, together with the importance of areas and volumes in geometry, were among the reasons for the initial development of determinants.

Exercises for 4.4

Exercise 4.4.1 In each case show that that T is either projection on a line, reflection in a line, or rotation through an angle, and find the line or angle.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-y \\ y-x \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -x-y \\ x-y \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$

e. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$

f. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}$

---

b. $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, projection on $y=-x$.

d. $A = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$, reflection in $y=2x$.

f. $A = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$, rotation through $\frac{\pi}{3}$.

Exercise 4.4.2 Determine the effect of the following transformations.

a. Rotation through $\frac{\pi}{2}$, followed by projection on the y axis, followed by reflection in the line $y=x$.

b. Projection on the line $y=x$ followed by projection on the line $y=-x$.

c. Projection on the x axis followed by reflection in the line $y=x$.

---

b. The zero transformation.

Exercise 4.4.3 In each case solve the problem by finding the matrix of the operator.

a. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ on the plane with equation $3x-5y+2z=0$.

b. Find the projection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ on the plane with equation $2x-y+4z=0$.

c. Find the reflection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ in the plane with equation $x - y + 3z = 0$.

d. Find the reflection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ in the plane with equation $2x + y - 5z = 0$.

e. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

f. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$.

g. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$.

h. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

---

b. $\frac{1}{21} \begin{bmatrix} 17 & 2 & -8 \\ 2 & 20 & 4 \\ -8 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

d. $\frac{1}{30} \begin{bmatrix} 22 & -4 & 20 \\ -4 & 28 & 10 \\ 20 & 10 & -20 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

f. $\frac{1}{25} \begin{bmatrix} 9 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$

h. $\frac{1}{11} \begin{bmatrix} -9 & 2 & -6 \\ 2 & -9 & -6 \\ -6 & -6 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$

Exercise 4.4.4

a. Find the rotation of $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{4}$.

b. Find the rotation of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{6}$.

---

b. $\frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Exercise 4.4.5 Find the matrix of the rotation in $\mathbb{R}^3$ about the x axis through the angle θ (from the positive y axis to the positive z axis).

Exercise 4.4.6 Find the matrix of the rotation about the y axis through the angle θ (from the positive x axis to the positive z axis).

$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

Exercise 4.4.7 If A is 3×3, show that the image of the line in $\mathbb{R}^3$ through $\mathbf{p}_0$ with direction vector $\mathbf{d}$ is the line through $A\mathbf{p}_0$ with direction vector $A\mathbf{d}$, assuming that $A\mathbf{d} \neq \mathbf{0}$. What happens if $A\mathbf{d} = \mathbf{0}$?

Exercise 4.4.8 If A is 3×3 and invertible, show that the image of the plane through the origin with normal $\mathbf{n}$ is the plane through the origin with normal $\mathbf{n}_1 = B\mathbf{n}$ where $B = (A^{-1})^T$. [*Hint*: Use the fact that $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$ to show that $\mathbf{n}_1 \cdot (A\mathbf{p}) = \mathbf{n} \cdot \mathbf{p}$ for each $\mathbf{p}$ in $\mathbb{R}^3$.]

Exercise 4.4.9 Let L be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix} \neq 0$.

a. If P_L denotes projection on L, show that P_L has matrix $\frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.

b. If Q_L denotes reflection in L, show that Q_L has matrix $\frac{1}{a^2+b^2}\begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$.

---

a. Write $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$P_L(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\right)\mathbf{d} = \frac{ax+by}{a^2+b^2}\begin{bmatrix} a \\ b \end{bmatrix}$$
$$= \frac{1}{a^2+b^2}\begin{bmatrix} a^2x+aby \\ abx+b^2y \end{bmatrix}$$
$$= \frac{1}{a^2+b^2}\begin{bmatrix} a^2+ab \\ ab+b^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Exercise 4.4.10 Let $\mathbf{n}$ be a nonzero vector in $\mathbb{R}^3$, let L be the line through the origin with direction vector $\mathbf{n}$, and let M be the plane through the origin with normal $\mathbf{n}$. Show that $P_L(\mathbf{v}) = Q_L(\mathbf{v}) + P_M(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^3$. [In this case, we say that $P_L = Q_L + P_M$.]

Exercise 4.4.11 If M is the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that Q_M has matrix

$$\frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

Supplementary Exercises for Chapter 4

Exercise 4.1 Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, and $a\mathbf{u} + b\mathbf{v} = a_1\mathbf{u} + b_1\mathbf{v}$, show that $a = a_1$ and $b = b_1$.

Exercise 4.2 Consider a triangle with vertices A, B, and C. Let E and F be the midpoints of sides AB and AC, respectively, and let the medians EC and FB meet at O. Write $\overrightarrow{EO} = s\overrightarrow{EC}$ and $\overrightarrow{FO} = t\overrightarrow{FB}$, where s and t are scalars. Show that $s = t = \frac{1}{3}$ by expressing $\overrightarrow{AO}$ two ways in the form $a\overrightarrow{EO} + b\overrightarrow{AC}$, and applying Exercise 4.1. Conclude that the medians of a triangle meet at the point on each that is one-third of the way from the midpoint to the vertex (and so are concurrent).

Exercise 4.3 A river flows at 1 km/h and a swimmer moves at 2 km/h (relative to the water). At what angle must he swim to go straight across? What is his resulting speed?

Exercise 4.4 A wind is blowing from the south at 75 knots, and an airplane flies heading east at 100 knots. Find the resulting velocity of the airplane.

---

125 knots in a direction θ degrees east of north, where $\cos\theta = 0.6$ ($\theta = 53°$ or 0.93 radians).

Exercise 4.5 An airplane pilot flies at 300 km/h in a direction 30° south of east. The wind is blowing from the south at 150 km/h.

a. Find the resulting direction and speed of the airplane.

b. Find the speed of the airplane if the wind is from the west (at 150 km/h).

Exercise 4.6 A rescue boat has a top speed of 13 knots. The captain wants to go due east as fast as possible in water with a current of 5 knots due south. Find the velocity vector $\mathbf{v} = (x,\ y)$ that she must achieve, assuming the x and y axes point east and north, respectively, and find her resulting speed.

(12, 5). Actual speed 12 knots.

Exercise 4.7 A boat goes 12 knots heading north. The current is 5 knots from the west. In what direction does the boat actually move and at what speed?

Exercise 4.8 Show that the distance from a point A (with vector $\mathbf{a}$) to the plane with vector equation $\mathbf{n} \cdot \mathbf{p} = d$ is $\frac{1}{\|n\|}|\mathbf{n} \cdot \mathbf{a} - d|$.

Exercise 4.9 If two distinct points lie in a plane, show that the line through these points is contained in the plane.

Exercise 4.10 The line through a vertex of a triangle, perpendicular to the opposite side, is called an **altitude** of the triangle. Show that the three altitudes of any triangle are concurrent. (The intersection of the altitudes is called the **orthocentre** of the triangle.) [*Hint*: If P is the intersection of two of the altitudes, show that the line through P and the remaining vertex is perpendicular to the remaining side.]

5. Vector Space $\mathbb{R}^n$

Contents

5.1 Subspaces and Spanning

In Section 2.2 we introduced the set $\mathbb{R}^n$ of all n-tuples (called *vectors*), and began our investigation of the matrix transformations $\mathbb{R}^n \to \mathbb{R}^m$ given by matrix multiplication by an $m \times n$ matrix. Particular attention was paid to the euclidean plane $\mathbb{R}^2$ where certain simple geometric transformations were seen to be matrix transformations. Then in Section 2.6 we introduced linear transformations, showed that they are all matrix transformations, and found the matrices of rotations and reflections in $\mathbb{R}^2$. We returned to this in Section 4.4 where we showed that projections, reflections, and rotations of $\mathbb{R}^2$ and $\mathbb{R}^3$ were all linear, and where we related areas and volumes to determinants.

In this chapter we investigate $\mathbb{R}^n$ in full generality, and introduce some of the most important concepts and methods in linear algebra. The n-tuples in $\mathbb{R}^n$ will continue to be denoted $\mathbf{x}$, $\mathbf{y}$, and so on, and will be written as rows or columns depending on the context.

Subspaces of $\mathbb{R}^n$

Definition 5.1 Subspace of $\mathbb{R}^n$

A set[1] U of vectors in $\mathbb{R}^n$ is called a **subspace** *of $\mathbb{R}^n$ if it satisfies the following properties:*

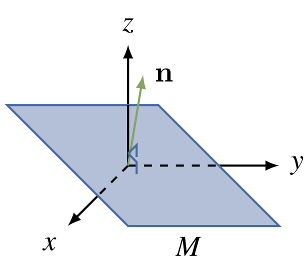
S1. The zero vector $\mathbf{0} \in U$.

S2. If $\mathbf{x} \in U$ and $\mathbf{y} \in U$, then $\mathbf{x} + \mathbf{y} \in U$.

S3. If $\mathbf{x} \in U$, then $a\mathbf{x} \in U$ for every real number a.

We say that the subset U is **closed under addition** if S2 holds, and that U is **closed under scalar multiplication** if S3 holds.

Clearly $\mathbb{R}^n$ is a subspace of itself, and this chapter is about these subspaces and their properties. The set $U = \{\mathbf{0}\}$, consisting of only the zero vector, is also a subspace because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for each a in $\mathbb{R}$; it is called the **zero subspace**. Any subspace of $\mathbb{R}^n$ other than $\{\mathbf{0}\}$ or $\mathbb{R}^n$ is called a **proper** subspace.



We saw in Section 4.2 that every plane M through the origin in $\mathbb{R}^3$ has equation $ax + by + cz = 0$ where a, b, and c are not all zero. Here $\mathbf{n} = \left[\begin{array}{c} a \\ b \\ c \end{array} \right]$ is a normal for the plane and

$$M = \{\mathbf{v} \text{ in } \mathbb{R}^3 \mid \mathbf{n} \cdot \mathbf{v} = 0\}$$

[1]We use the language of sets. Informally, a **set** X is a collection of objects, called the **elements** of the set. The fact that x is an element of X is denoted $x \in X$. Two sets X and Y are called equal (written $X = Y$) if they have the same elements. If every element of X is in the set Y, we say that X is a **subset** of Y, and write $X \subseteq Y$. Hence $X \subseteq Y$ and $Y \subseteq X$ both hold if and only if $X = Y$.

where $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{n} \cdot \mathbf{v}$ denotes the dot product introduced in Section 2.2 (see the diagram).[2] Then M is a subspace of $\mathbb{R}^3$. Indeed we show that M satisfies S1, S2, and S3 as follows:

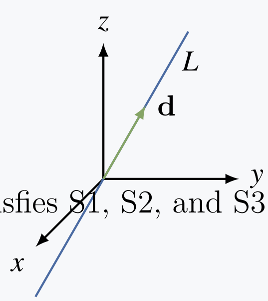
S1. $\mathbf{0} \in M$ *because* $\mathbf{n} \cdot \mathbf{0} = 0$;

S2. *If* $\mathbf{v} \in M$ *and* $\mathbf{v}_1 \in M$, *then* $\mathbf{n} \cdot (\mathbf{v} + \mathbf{v}_1) = \mathbf{n} \cdot \mathbf{v} + \mathbf{n} \cdot \mathbf{v}_1 = 0 + 0 = 0$, *so* $\mathbf{v} + \mathbf{v}_1 \in M$;

S3. *If* $\mathbf{v} \in M$, *then* $\mathbf{n} \cdot (a\mathbf{v}) = a(\mathbf{n} \cdot \mathbf{v}) = a(0) = 0$, *so* $a\mathbf{v} \in M$.

This proves the first part of

Example 5.1.1



Planes and lines through the origin in $\mathbb{R}^3$ are all subspaces of $\mathbb{R}^3$.

Solution. We dealt with planes above. If L is a line through the origin with direction vector $\mathbf{d}$, then $L = \{t\mathbf{d} \mid t \in \mathbb{R}\}$ (see the diagram). We leave it as an exercise to verify that L satisfies S1, S2, and S3.

Example 5.1.1 shows that lines through the origin in $\mathbb{R}^2$ are subspaces; in fact, they are the *only* proper subspaces of $\mathbb{R}^2$ (Exercise 5.1.24). Indeed, we shall see in Example 5.2.14 that lines and planes through the origin in $\mathbb{R}^3$ are the only proper subspaces of $\mathbb{R}^3$. Thus the geometry of lines and planes through the origin is captured by the subspace concept. (Note that *every* line or plane is just a translation of one of these.)

Subspaces can also be used to describe important features of an $m \times n$ matrix A. The **null space** of A, denoted $\text{null}\, A$, and the **image space** of A, denoted $\text{im}\, A$, are defined by

$$\text{null}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \quad \text{and} \quad \text{im}\, A = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$$

In the language of Chapter 2, $\text{null}\, A$ consists of all solutions $\mathbf{x}$ in $\mathbb{R}^n$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, and $\text{im}\, A$ is the set of all vectors $\mathbf{y}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{y}$ *has* a solution $\mathbf{x}$. Note that $\mathbf{x}$ is in $\text{null}\, A$ if it satisfies the *condition* $A\mathbf{x} = \mathbf{0}$, while $\text{im}\, A$ consists of vectors of the *form* $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. These two ways to describe subsets occur frequently.

[2]We are using set notation here. In general $\{q \mid p\}$ means the set of all objects q with property p.

Example 5.1.2

If A is an $m \times n$ matrix, then:

1. $\textbf{null}\, A$ is a subspace of $\mathbb{R}^n$.
2. $\textbf{im}\, A$ is a subspace of $\mathbb{R}^m$.

Solution.

1. The zero vector $\mathbf{0} \in \mathbb{R}^n$ lies in $\textbf{null}\, A$ because $A\mathbf{0} = \mathbf{0}$.[3] If $\mathbf{x}$ and $\mathbf{x}_1$ are in $\textbf{null}\, A$, then $\mathbf{x} + \mathbf{x}_1$ and $a\mathbf{x}$ are in $\textbf{null}\, A$ because they satisfy the required condition:

$$A(\mathbf{x}+\mathbf{x}_1) = A\mathbf{x} + A\mathbf{x}_1 = \mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad A(a\mathbf{x}) = a(A\mathbf{x}) = a\mathbf{0} = \mathbf{0}$$

   Hence $\textbf{null}\, A$ satisfies S1, S2, and S3, and so is a subspace of $\mathbb{R}^n$.

2. The zero vector $\mathbf{0} \in \mathbb{R}^m$ lies in $\textbf{im}\, A$ because $\mathbf{0} = A\mathbf{0}$. Suppose that $\mathbf{y}$ and $\mathbf{y}_1$ are in $\textbf{im}\, A$, say $\mathbf{y} = A\mathbf{x}$ and $\mathbf{y}_1 = A\mathbf{x}_1$ where $\mathbf{x}$ and $\mathbf{x}_1$ are in $\mathbb{R}^n$. Then

$$\mathbf{y} + \mathbf{y}_1 = A\mathbf{x} + A\mathbf{x}_1 = A(\mathbf{x}+\mathbf{x}_1) \quad \text{and} \quad a\mathbf{y} = a(A\mathbf{x}) = A(a\mathbf{x})$$

   show that $\mathbf{y} + \mathbf{y}_1$ and $a\mathbf{y}$ are both in $\textbf{im}\, A$ (they have the required form). Hence $\textbf{im}\, A$ is a subspace of $\mathbb{R}^m$.

There are other important subspaces associated with a matrix A that clarify basic properties of A. If A is an $n \times n$ matrix and λ is any number, let

$$E_\lambda(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

A vector $\mathbf{x}$ is in $E_\lambda(A)$ if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$, so Example 5.1.2 gives:

Example 5.1.3

$E_\lambda(A) = \textbf{null}\,(\lambda I - A)$ is a subspace of $\mathbb{R}^n$ for each $n \times n$ matrix A and number λ.

$E_\lambda(A)$ is called the **eigenspace** of A corresponding to λ. The reason for the name is that, in the terminology of Section 3.3, λ is an **eigenvalue** of A if $E_\lambda(A) \neq \{\mathbf{0}\}$. In this case the nonzero vectors in $E_\lambda(A)$ are called the **eigenvectors** of A corresponding to λ.

The reader should not get the impression that *every* subset of $\mathbb{R}^n$ is a subspace. For example:

$$U_1 = \left\{ \left[\begin{array}{c} x \\ y \end{array}\right] \middle| x \geq 0 \right\} \text{ satisfies S1 and S2, but not S3;}$$

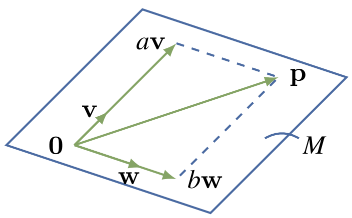
$$U_2 = \left\{ \left[\begin{array}{c} x \\ y \end{array}\right] \middle| x^2 = y^2 \right\} \text{ satisfies S1 and S3, but not S2;}$$

Hence neither U_1 nor U_2 is a subspace of $\mathbb{R}^2$. (However, see Exercise 5.1.20.)

[3] We are using $\mathbf{0}$ to represent the zero vector in both $\mathbb{R}^m$ and $\mathbb{R}^n$. This abuse of notation is common and causes no confusion once everybody knows what is going on.

Spanning Sets

Let $\mathbf{v}$ and $\mathbf{w}$ be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. The plane M through the origin containing these vectors is described in Section 4.2 by saying that $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is a *normal* for M, and that M consists of all vectors $\mathbf{p}$ such that $\mathbf{n} \cdot \mathbf{p} = 0$.[4] While this is a very useful way to look at planes, there is another approach that is at least as useful in $\mathbb{R}^3$ and, more importantly, works for all subspaces of $\mathbb{R}^n$ for any $n \geq 1$.



The idea is as follows: Observe that, by the diagram, a vector $\mathbf{p}$ is in M if and only if it has the form

$$\mathbf{p} = a\mathbf{v} + b\mathbf{w}$$

for certain real numbers a and b (we say that $\mathbf{p}$ is a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$). Hence we can describe M as

$$M = \{a\mathbf{x} + b\mathbf{w} \mid a,\ b \in \mathbb{R}\}.^5$$

and we say that $\{\mathbf{v},\ \mathbf{w}\}$ is a *spanning set* for M. It is this notion of a spanning set that provides a way to describe all subspaces of $\mathbb{R}^n$.

As in Section 1.3, given vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ in $\mathbb{R}^n$, a vector of the form

$$t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \quad \text{where the } t_i \text{ are scalars}$$

is called a **linear combination** of the $\mathbf{x}_i$, and t_i is called the **coefficient** of $\mathbf{x}_i$ in the linear combination.

Definition 5.2 Linear Combinations and Span in $\mathbb{R}^n$

*The set of all such linear combinations is called the **span** of the $\mathbf{x}_i$ and is denoted*

$$\text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\} = \{t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \mid t_i \text{ in } \mathbb{R}\}$$

*If $V = \text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$, we say that V is **spanned** by the vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$, and that the vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ **span** the space V.*

Here are two examples:

$$\text{span}\,\{\mathbf{x}\} = \{t\mathbf{x} \mid t \in \mathbb{R}\}$$

which we write as $\text{span}\,\{\mathbf{x}\} = \mathbb{R}\mathbf{x}$ for simplicity.

$$\text{span}\,\{\mathbf{x},\ \mathbf{y}\} = \{r\mathbf{x} + s\mathbf{y} \mid r,\ s \in \mathbb{R}\}$$

In particular, the above discussion shows that, if $\mathbf{v}$ and $\mathbf{w}$ are two nonzero, nonparallel vectors in $\mathbb{R}^3$, then

$$M = \text{span}\,\{\mathbf{v},\ \mathbf{w}\}$$

[4]The vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is nonzero because $\mathbf{v}$ and $\mathbf{w}$ are not parallel.

[5]In particular, this implies that any vector $\mathbf{p}$ orthogonal to $\mathbf{v} \times \mathbf{w}$ must be a linear combination $\mathbf{p} = a\mathbf{v} + b\mathbf{w}$ of $\mathbf{v}$ and $\mathbf{w}$ for some a and b. Can you prove this directly?

is the plane in $\mathbb{R}^3$ containing $\mathbf{v}$ and $\mathbf{w}$. Moreover, if $\mathbf{d}$ is any nonzero vector in $\mathbb{R}^3$ (or $\mathbb{R}^2$), then

$$L = \text{span}\{\mathbf{v}\} = \{t\mathbf{d} \mid t \in \mathbb{R}\} = \mathbb{R}\mathbf{d}$$

is the line with direction vector $\mathbf{d}$. Hence lines and planes can both be described in terms of spanning sets.

Example 5.1.4

Let $\mathbf{x} = (2,\ -1,\ 2,\ 1)$ and $\mathbf{y} = (3,\ 4,\ -1,\ 1)$ in $\mathbb{R}^4$. Determine whether $\mathbf{p} = (0,\ -11,\ 8,\ 1)$ or $\mathbf{q} = (2,\ 3,\ 1,\ 2)$ are in $U = \text{span}\{\mathbf{x},\ \mathbf{y}\}$.

Solution. The vector $\mathbf{p}$ is in U if and only if $\mathbf{p} = s\mathbf{x} + t\mathbf{y}$ for scalars s and t. Equating components gives equations

$$2s + 3t = 0, \quad -s + 4t = -11, \quad 2s - t = 8, \quad \text{and} \quad s + t = 1$$

This linear system has solution $s = 3$ and $t = -2$, so $\mathbf{p}$ is in U. On the other hand, asking that $\mathbf{q} = s\mathbf{x} + t\mathbf{y}$ leads to equations

$$2s + 3t = 2, \quad -s + 4t = 3, \quad 2s - t = 1, \quad \text{and} \quad s + t = 2$$

and this system has *no* solution. So $\mathbf{q}$ does *not* lie in U.

Theorem 5.1.1: Span Theorem

Let $U = \text{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ *in* $\mathbb{R}^n$. *Then:*

1. U *is a subspace of* $\mathbb{R}^n$ *containing each* $\mathbf{x}_i$.
2. *If* W *is a subspace of* $\mathbb{R}^n$ *and each* $\mathbf{x}_i \in W$, *then* $U \subseteq W$.

Proof.

1. The zero vector $\mathbf{0}$ is in U because $\mathbf{0} = 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k$ is a linear combination of the $\mathbf{x}_i$. If $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ and $\mathbf{y} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$ are in U, then $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ are in U because
$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (t_1 + s_1)\mathbf{x}_1 + (t_2 + s_2)\mathbf{x}_2 + \cdots + (t_k + s_k)\mathbf{x}_k, \quad \text{and} \\ a\mathbf{x} &= (at_1)\mathbf{x}_1 + (at_2)\mathbf{x}_2 + \cdots + (at_k)\mathbf{x}_k \end{aligned}$$
Finally each $\mathbf{x}_i$ is in U (for example, $\mathbf{x}_2 = 0\mathbf{x}_1 + 1\mathbf{x}_2 + \cdots + 0\mathbf{x}_k$) so S1, S2, and S3 are satisfied for U, proving (1).

2. Let $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ where the t_i are scalars and each $\mathbf{x}_i \in W$. Then each $t_i\mathbf{x}_i \in W$ because W satisfies S3. But then $\mathbf{x} \in W$ because W satisfies S2 (verify). This proves (2). □

Condition (2) in Theorem 5.1.1 can be expressed by saying that $\text{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is the *smallest* subspace of $\mathbb{R}^n$ that contains each $\mathbf{x}_i$. This is useful for showing that two subspaces U and W are equal, since this amounts to showing that both $U \subseteq W$ and $W \subseteq U$. Here is an example of how it is used.

Example 5.1.5

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, show that $\text{span}\{\mathbf{x}, \mathbf{y}\} = \text{span}\{\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\}$.

Solution. Since both $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are in $\text{span}\{\mathbf{x}, \mathbf{y}\}$, Theorem 5.1.1 gives

$$\text{span}\{\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\} \subseteq \text{span}\{\mathbf{x}, \mathbf{y}\}$$

But $\mathbf{x} = \frac{1}{2}(\mathbf{x}+\mathbf{y}) + \frac{1}{2}(\mathbf{x}-\mathbf{y})$ and $\mathbf{y} = \frac{1}{2}(\mathbf{x}+\mathbf{y}) - \frac{1}{2}(\mathbf{x}-\mathbf{y})$ are both in $\text{span}\{\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\}$, so

$$\text{span}\{\mathbf{x}, \mathbf{y}\} \subseteq \text{span}\{\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\}$$

again by Theorem 5.1.1. Thus $\text{span}\{\mathbf{x}, \mathbf{y}\} = \text{span}\{\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\}$, as desired.

It turns out that many important subspaces are best described by giving a spanning set. Here are three examples, beginning with an important spanning set for $\mathbb{R}^n$ itself. Column j of the $n \times n$ identity matrix I_n is denoted $\mathbf{e}_j$ and called the jth **coordinate vector** in $\mathbb{R}^n$, and the set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is called the **standard basis** of $\mathbb{R}^n$. If $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ is any vector in $\mathbb{R}^n$, then $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$, as the reader can verify. This proves:

Example 5.1.6

$\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the columns of I_n.

If A is an $m \times n$ matrix A, the next two examples show that it is a routine matter to find spanning sets for $\text{null}\, A$ and $\text{im}\, A$.

Example 5.1.7

Given an $m \times n$ matrix A, let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ denote the basic solutions to the system $A\mathbf{x} = \mathbf{0}$ given by the gaussian algorithm. Then

$$\text{null}\, A = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$$

Solution. If $\mathbf{x} \in \text{null}\, A$, then $A\mathbf{x} = \mathbf{0}$ so Theorem 1.3.2 shows that $\mathbf{x}$ is a linear combination of the basic solutions; that is, $\text{null}\, A \subseteq \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$. On the other hand, if $\mathbf{x}$ is in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$, then $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \dots + t_k\mathbf{x}_k$ for scalars t_i, so

$$A\mathbf{x} = t_1A\mathbf{x}_1 + t_2A\mathbf{x}_2 + \dots + t_kA\mathbf{x}_k = t_1\mathbf{0} + t_2\mathbf{0} + \dots + t_k\mathbf{0} = \mathbf{0}$$

This shows that $\mathbf{x} \in \text{null}\, A$, and hence that $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} \subseteq \text{null}\, A$. Thus we have equality.

Example 5.1.8

Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ denote the columns of the $m \times n$ matrix A. Then

$$\operatorname{im} A = \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$$

Solution. If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, observe that

$$\left[\begin{array}{cccc} A\mathbf{e}_1 & A\mathbf{e}_2 & \cdots & A\mathbf{e}_n \end{array}\right] = A\left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = AI_n = A = \left[\begin{array}{ccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots \mathbf{c}_n \end{array}\right].$$

Hence $\mathbf{c}_i = A\mathbf{e}_i$ is in $\operatorname{im} A$ for each i, so $\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \subseteq \operatorname{im} A$.

Conversely, let $\mathbf{y}$ be in $\operatorname{im} A$, say $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$, then

Definition 2.5 gives

$$\mathbf{y} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n \text{ is in } \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$$

This shows that $\operatorname{im} A \subseteq \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$, and the result follows.

Exercises for 5.1

We often write vectors in $\mathbb{R}^n$ as rows.

Exercise 5.1.1 In each case determine whether U is a subspace of $\mathbb{R}^3$. Support your answer.

a. $U = \{(1, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.

b. $U = \{(0, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.

c. $U = \{(r, s, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}, -r + 3s + 2t = 0\}$.

d. $U = \{(r, 3s, r-2) \mid r \text{ and } s \text{ in } \mathbb{R}\}$.

e. $U = \{(r, 0, s) \mid r^2 + s^2 = 0, r \text{ and } s \text{ in } \mathbb{R}\}$.

f. $U = \{(2r, -s^2, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}\}$.

---

b. Yes

d. No

f. No.

Exercise 5.1.2 In each case determine if $\mathbf{x}$ lies in $U = \operatorname{span}\{\mathbf{y}, \mathbf{z}\}$. If $\mathbf{x}$ is in U, write it as a linear combination of $\mathbf{y}$ and $\mathbf{z}$; if $\mathbf{x}$ is not in U, show why not.

a. $\mathbf{x} = (2, -1, 0, 1)$, $\mathbf{y} = (1, 0, 0, 1)$, and $\mathbf{z} = (0, 1, 0, 1)$.

b. $\mathbf{x} = (1, 2, 15, 11)$, $\mathbf{y} = (2, -1, 0, 2)$, and $\mathbf{z} = (1, -1, -3, 1)$.

c. $\mathbf{x} = (8, 3, -13, 20)$, $\mathbf{y} = (2, 1, -3, 5)$, and $\mathbf{z} = (-1, 0, 2, -3)$.

d. $\mathbf{x} = (2, 5, 8, 3)$, $\mathbf{y} = (2, -1, 0, 5)$, and $\mathbf{z} = (-1, 2, 2, -3)$.

---

b. No

d. Yes, $\mathbf{x} = 3\mathbf{y} + 4\mathbf{z}$.

Exercise 5.1.3 In each case determine if the given vectors span $\mathbb{R}^4$. Support your answer.

a. $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$

b. $\{(1, 3, -5, 0), (-2, 1, 0, 0), (0, 2, 1, -1), (1, -4, 5, 0)\}$.

---

b. No

Exercise 5.1.4 Is it possible that $\{(1, 2, 0), (2, 0, 3)\}$ can span the subspace $U = \{(r, s, 0) \mid r \text{ and } s \text{ in } \mathbb{R}\}$? Defend your answer.

Exercise 5.1.5 Give a spanning set for the zero subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$.

Exercise 5.1.6 Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Defend your answer.

Exercise 5.1.7 If $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x} + t\mathbf{z}, \mathbf{y}, \mathbf{z}\}$ for every t in $\mathbb{R}$.

Exercise 5.1.8 If $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}\}$.

Exercise 5.1.9 If $a \neq 0$ is a scalar, show that $\text{span}\{a\mathbf{x}\} = \text{span}\{\mathbf{x}\}$ for every vector $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 5.1.10 If $a_1, a_2, \dots, a_k$ are nonzero scalars, show that $\text{span}\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ for any vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

$\text{span}\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\} \subseteq \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ by Theorem 5.1.1 because, for each i, $a_i\mathbf{x}_i$ is in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$. Similarly, the fact that $\mathbf{x}_i = a_i^{-1}(a_i\mathbf{x}_i)$ is in $\text{span}\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\}$ for each i shows that $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} \subseteq \text{span}\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\}$, again by Theorem 5.1.1.

Exercise 5.1.11 If $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, determine all subspaces of $\text{span}\{\mathbf{x}\}$.

Exercise 5.1.12 Suppose that $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ where each $\mathbf{x}_i$ is in $\mathbb{R}^n$. If A is an $m \times n$ matrix and $A\mathbf{x}_i = \mathbf{0}$ for each i, show that $A\mathbf{y} = \mathbf{0}$ for every vector $\mathbf{y}$ in U.

If $\mathbf{y} = r_1\mathbf{x}_1 + \cdots + r_k\mathbf{x}_k$ then $A\mathbf{y} = r_1(A\mathbf{x}_1) + \cdots + r_k(A\mathbf{x}_k) = 0$.

Exercise 5.1.13 If A is an $m \times n$ matrix, show that, for each invertible $m \times m$ matrix U, $\text{null}(A) = \text{null}(UA)$.

Exercise 5.1.14 If A is an $m \times n$ matrix, show that, for each invertible $n \times n$ matrix V, $\text{im}(A) = \text{im}(AV)$.

Exercise 5.1.15 Let U be a subspace of $\mathbb{R}^n$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$.

a. If $a\mathbf{x}$ is in U where $a \neq 0$ is a number, show that $\mathbf{x}$ is in U.

b. If $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are in U where $\mathbf{y}$ is a vector in $\mathbb{R}^n$, show that $\mathbf{x}$ is in U.

---

b. $\mathbf{x} = (\mathbf{x} + \mathbf{y}) - \mathbf{y} = (\mathbf{x} + \mathbf{y}) + (-\mathbf{y})$ is in U because U is a subspace and both $\mathbf{x} + \mathbf{y}$ and $-\mathbf{y} = (-1)\mathbf{y}$ are in U.

Exercise 5.1.16 In each case either show that the statement is true or give an example showing that it is false.

a. If $U \neq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and $\mathbf{x} + \mathbf{y}$ is in U, then $\mathbf{x}$ and $\mathbf{y}$ are both in U.

b. If U is a subspace of $\mathbb{R}^n$ and $r\mathbf{x}$ is in U for all r in $\mathbb{R}$, then $\mathbf{x}$ is in U.

c. If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in U, then $-\mathbf{x}$ is also in U.

d. If $\mathbf{x}$ is in U and $U = \text{span}\{\mathbf{y}, \mathbf{z}\}$, then $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.

e. The empty set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

f. $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ is in $\text{span}\left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 2 \\ 0 \end{array}\right]\right\}$.

---

b. True. $\mathbf{x} = 1\mathbf{x}$ is in U.

d. True. Always $\text{span}\{\mathbf{y},\ \mathbf{z}\} \subseteq \text{span}\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ by Theorem 5.1.1. Since $\mathbf{x}$ is in $\text{span}\{\mathbf{x},\ \mathbf{y}\}$ we have $\text{span}\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\} \subseteq \text{span}\{\mathbf{y},\ \mathbf{z}\}$, again by Theorem 5.1.1.

f. False. $a\left[\begin{array}{c}1\\0\end{array}\right]+b\left[\begin{array}{c}2\\0\end{array}\right]=\left[\begin{array}{c}a+2b\\0\end{array}\right]$ cannot equal $\left[\begin{array}{c}0\\1\end{array}\right]$.

Exercise 5.1.17

a. If A and B are $m \times n$ matrices, show that $U = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = B\mathbf{x}\}$ is a subspace of $\mathbb{R}^n$.

b. What if A is $m \times n$, B is $k \times n$, and $m \neq k$?

Exercise 5.1.18 Suppose that $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ are vectors in $\mathbb{R}^n$. If $\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ where $a_1 \neq 0$, show that $\text{span}\{\mathbf{x}_1\, \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\} = \text{span}\{\mathbf{y}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$.

Exercise 5.1.19 If $U \neq \{\mathbf{0}\}$ is a subspace of $\mathbb{R}$, show that $U = \mathbb{R}$.

Exercise 5.1.20 Let U be a nonempty subset of $\mathbb{R}^n$. Show that U is a subspace if and only if S2 and S3 hold.

If U is a subspace, then S2 and S3 certainly hold. Conversely, assume that S2 and S3 hold for U. Since U is nonempty, choose $\mathbf{x}$ in U. Then $\mathbf{0} = 0\mathbf{x}$ is in U by S3, so S1 also holds. This means that U is a subspace.

Exercise 5.1.21 If S and T are nonempty sets of vectors in $\mathbb{R}^n$, and if $S \subseteq T$, show that $\text{span}\{S\} \subseteq \text{span}\{T\}$.

Exercise 5.1.22 Let U and W be subspaces of $\mathbb{R}^n$. Define their **intersection** $U \cap W$ and their **sum** $U + W$ as follows: $U \cap W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ belongs to both } U \text{ and } W\}$. $U + W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is a sum of a vector in } U$ and a vector in $W\}$.

a. Show that $U \cap W$ is a subspace of $\mathbb{R}^n$.

b. Show that $U + W$ is a subspace of $\mathbb{R}^n$.

---

b. The zero vector $\mathbf{0}$ is in $U + W$ because $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Let $\mathbf{p}$ and $\mathbf{q}$ be vectors in $U + W$, say $\mathbf{p} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{q} = \mathbf{x}_2 + \mathbf{y}_2$ where $\mathbf{x}_1$ and $\mathbf{x}_2$ are in U, and $\mathbf{y}_1$ and $\mathbf{y}_2$ are in W. Then $\mathbf{p} + \mathbf{q} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2)$ is in $U + W$ because $\mathbf{x}_1 + \mathbf{x}_2$ is in U and $\mathbf{y}_1 + \mathbf{y}_2$ is in W. Similarly, $a(\mathbf{p} + \mathbf{q}) = a\mathbf{p} + a\mathbf{q}$ is in $U + W$ for any scalar a because $a\mathbf{p}$ is in U and $a\mathbf{q}$ is in W. Hence $U + W$ is indeed a subspace of $\mathbb{R}^n$.

Exercise 5.1.23 Let P denote an invertible $n \times n$ matrix. If λ is a number, show that

$$E_\lambda(PAP^{-1}) = \{P\mathbf{x} \mid \mathbf{x} \text{ is in } E_\lambda(A)\}$$

for each $n \times n$ matrix A.

Exercise 5.1.24 Show that every proper subspace U of $\mathbb{R}^2$ is a line through the origin. [*Hint*: If $\mathbf{d}$ is a nonzero vector in U, let $L = \mathbb{R}\mathbf{d} = \{r\mathbf{d} \mid r \text{ in } \mathbb{R}\}$ denote the line with direction vector $\mathbf{d}$. If $\mathbf{u}$ is in U but not in L, argue geometrically that every vector $\mathbf{v}$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{d}$.]

5.2 Independence and Dimension

Some spanning sets are better than others. If $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a subspace of $\mathbb{R}^n$, then every vector in U can be written as a linear combination of the $\mathbf{x}_i$ in at least one way. Our interest here is in spanning sets where each vector in U has a *exactly one* representation as a linear combination of these vectors.

Linear Independence

Given $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ in $\mathbb{R}^n$, suppose that two linear combinations are equal:

$$r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$$

We are looking for a condition on the set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors that guarantees that this representation is *unique*; that is, $r_i = s_i$ for each i. Taking all terms to the left side gives

$$(r_1 - s_1)\mathbf{x}_1 + (r_2 - s_2)\mathbf{x}_2 + \cdots + (r_k - s_k)\mathbf{x}_k = \mathbf{0}$$

so the required condition is that this equation forces all the coefficients $r_i - s_i$ to be zero.

Definition 5.3 Linear Independence in $\mathbb{R}^n$

*With this in mind, we call a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors **linearly independent** (or simply **independent**) if it satisfies the following condition:*

$$\text{If } t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0} \text{ then } t_1 = t_2 = \cdots = t_k = 0$$

We record the result of the above discussion for reference.

Theorem 5.2.1

*If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is an independent set of vectors in $\mathbb{R}^n$, then every vector in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ has a **unique** representation as a linear combination of the $\mathbf{x}_i$.*

It is useful to state the definition of independence in different language. Let us say that a linear combination **vanishes** if it equals the zero vector, and call a linear combination **trivial** if every coefficient is zero. Then the definition of independence can be compactly stated as follows:

> A set of vectors is independent if and only if the only linear combination that vanishes is the trivial one.

Hence we have a procedure for checking that a set of vectors is independent:

Independence Test

To verify that a set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is independent, proceed as follows:

1. *Set a linear combination equal to zero: $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$.*
2. *Show that $t_i = 0$ for each i (that is, the linear combination is trivial).*

Of course, if some nontrivial linear combination vanishes, the vectors are not independent.

Example 5.2.1

Determine whether $\{(1,\ 0,\ -2,\ 5),\ (2,\ 1,\ 0,\ -1),\ (1,\ 1,\ 2,\ 1)\}$ is independent in $\mathbb{R}^4$.

Solution. Suppose a linear combination vanishes:

$$r(1,\ 0,\ -2,\ 5) + s(2,\ 1,\ 0,\ -1) + t(1,\ 1,\ 2,\ 1) = (0,\ 0,\ 0,\ 0)$$

Equating corresponding entries gives a system of four equations:

$$r + 2s + t = 0,\ s + t = 0,\ -2r + 2t = 0,\ \text{ and } 5r - s + t = 0$$

The only solution is the trivial one $r = s = t = 0$ (verify), so these vectors are independent by the independence test.

Example 5.2.2

Show that the standard basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ of $\mathbb{R}^n$ is independent.

Solution. The components of $t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + \cdots + t_n\mathbf{e}_n$ are $t_1,\ t_2,\ \dots,\ t_n$ (see the discussion preceding Example 5.1.6) So the linear combination vanishes if and only if each $t_i = 0$. Hence the independence test applies.

Example 5.2.3

If $\{\mathbf{x},\ \mathbf{y}\}$ is independent, show that $\{2\mathbf{x} + 3\mathbf{y},\ \mathbf{x} - 5\mathbf{y}\}$ is also independent.

Solution. If $s(2\mathbf{x} + 3\mathbf{y}) + t(\mathbf{x} - 5\mathbf{y}) = \mathbf{0}$, collect terms to get $(2s + t)\mathbf{x} + (3s - 5t)\mathbf{y} = \mathbf{0}$. Since $\{\mathbf{x},\ \mathbf{y}\}$ is independent this combination must be trivial; that is, $2s + t = 0$ and $3s - 5t = 0$. These equations have only the trivial solution $s = t = 0$, as required.

Example 5.2.4

Show that the zero vector in $\mathbb{R}^n$ does not belong to any independent set.

Solution. No set $\{\mathbf{0},\ \mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ of vectors is independent because we have a vanishing, nontrivial linear combination $1 \cdot \mathbf{0} + 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k = \mathbf{0}$.

Example 5.2.5

Given $\mathbf{x}$ in $\mathbb{R}^n$, show that $\{\mathbf{x}\}$ is independent if and only if $\mathbf{x} \neq \mathbf{0}$.

Solution. A vanishing linear combination from $\{\mathbf{x}\}$ takes the form $t\mathbf{x} = \mathbf{0}$, t in $\mathbb{R}$. This implies that $t = 0$ because $\mathbf{x} \neq \mathbf{0}$.

The next example will be needed later.

Example 5.2.6

Show that the nonzero rows of a row-echelon matrix R are independent.

Solution. We illustrate the case with 3 leading 1s; the general case is analogous. Suppose R has the form $R = \left[\begin{array}{cccccc} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$ where $*$ indicates a nonspecified number. Let R_1, R_2, and R_3 denote the nonzero rows of R. If $t_1R_1 + t_2R_2 + t_3R_3 = 0$ we show that $t_1 = 0$, then $t_2 = 0$, and finally $t_3 = 0$. The condition $t_1R_1 + t_2R_2 + t_3R_3 = 0$ becomes

$$(0,\ t_1,\ *,\ *,\ *,\ *) + (0,\ 0,\ 0,\ t_2,\ *,\ *) + (0,\ 0,\ 0,\ 0,\ t_3,\ *) = (0,\ 0,\ 0,\ 0,\ 0,\ 0)$$

Equating second entries show that $t_1 = 0$, so the condition becomes $t_2R_2 + t_3R_3 = 0$. Now the same argument shows that $t_2 = 0$. Finally, this gives $t_3R_3 = 0$ and we obtain $t_3 = 0$.

A set of vectors in $\mathbb{R}^n$ is called **linearly dependent** (or simply **dependent**) if it is *not* linearly independent, equivalently if some nontrivial linear combination vanishes.

Example 5.2.7

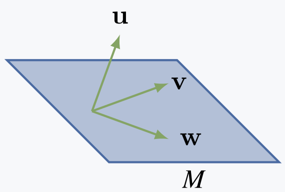
If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^3$, show that $\{\mathbf{v},\ \mathbf{w}\}$ is dependent if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.

Solution. If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then one is a scalar multiple of the other (Theorem 4.1.4), say $\mathbf{v} = a\mathbf{w}$ for some scalar a. Then the nontrivial linear combination $\mathbf{v} - a\mathbf{w} = \mathbf{0}$ vanishes, so $\{\mathbf{v},\ \mathbf{w}\}$ is dependent.
Conversely, if $\{\mathbf{v},\ \mathbf{w}\}$ is dependent, let $s\mathbf{v} + t\mathbf{w} = \mathbf{0}$ be nontrivial, say $s \neq 0$. Then $\mathbf{v} = -\frac{t}{s}\mathbf{w}$ so $\mathbf{v}$ and $\mathbf{w}$ are parallel (by Theorem 4.1.4). A similar argument works if $t \neq 0$.

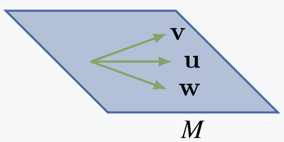
With this we can give a geometric description of what it means for a set $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ in $\mathbb{R}^3$ to be independent. Note that this requirement means that $\{\mathbf{v},\ \mathbf{w}\}$ is also independent ($a\mathbf{v} + b\mathbf{w} = \mathbf{0}$

means that $0\mathbf{u}+a\mathbf{v}+b\mathbf{w}=\mathbf{0}$), so $M=\text{span}\{\mathbf{v},\ \mathbf{w}\}$ is the plane containing $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{0}$ (see the discussion preceding Example 5.1.4). So we assume that $\{\mathbf{v},\ \mathbf{w}\}$ is independent in the following example.

Example 5.2.8



$\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ independent



$\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ not independent

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$ where $\{\mathbf{v},\ \mathbf{w}\}$ independent. Show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent if and only if $\mathbf{u}$ is not in the plane $M=\text{span}\{\mathbf{v},\ \mathbf{w}\}$. This is illustrated in the diagrams.

Solution. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, suppose $\mathbf{u}$ is in the plane $M=\text{span}\{\mathbf{v},\ \mathbf{w}\}$, say $\mathbf{u}=a\mathbf{v}+b\mathbf{w}$, where a and b are in $\mathbb{R}$. Then $1\mathbf{u}-a\mathbf{v}-b\mathbf{w}=\mathbf{0}$, contradicting the independence of $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$.
On the other hand, suppose that $\mathbf{u}$ is not in M; we must show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent. If $r\mathbf{u}+s\mathbf{v}+t\mathbf{w}=\mathbf{0}$ where r, s, and t are in $\mathbb{R}^3$, then $r=0$ since otherwise $\mathbf{u}=-\frac{s}{r}\mathbf{v}+\frac{-t}{r}\mathbf{w}$ is in M. But then $s\mathbf{v}+t\mathbf{w}=\mathbf{0}$, so $s=t=0$ by our assumption. This shows that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, as required.

By the inverse theorem, the following conditions are equivalent for an $n\times n$ matrix A:

1. *A is invertible.*
2. *If $A\mathbf{x}=\mathbf{0}$ where $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x}=\mathbf{0}$.*
3. *$A\mathbf{x}=\mathbf{b}$ has a solution $\mathbf{x}$ for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*

While condition 1 makes no sense if A is not square, conditions 2 and 3 are meaningful for any matrix A and, in fact, are related to independence and spanning. Indeed, if $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n$ are the columns of A, and if we write $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then

$$A\mathbf{x}=x_1\mathbf{c}_1+x_2\mathbf{c}_2+\cdots+x_n\mathbf{c}_n$$

by Definition 2.5. Hence the definitions of independence and spanning show, respectively, that condition 2 is equivalent to the independence of $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}$ and condition 3 is equivalent to the requirement that $\text{span}\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}=\mathbb{R}^m$. This discussion is summarized in the following theorem:

Theorem 5.2.2

If A is an $m\times n$ matrix, let $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}$ denote the columns of A.

1. *$\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}$ is independent in $\mathbb{R}^m$ if and only if $A\mathbf{x}=\mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, implies $\mathbf{x}=\mathbf{0}$.*

2. $\mathbb{R}^m = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ *if and only if* $A\mathbf{x} = \mathbf{b}$ *has a solution* $\mathbf{x}$ *for every vector* $\mathbf{b}$ *in* $\mathbb{R}^m$.

For a *square* matrix A, Theorem 5.2.2 characterizes the invertibility of A in terms of the spanning and independence of its columns (see the discussion preceding Theorem 5.2.2). It is important to be able to discuss these notions for *rows*. If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are $1 \times n$ rows, we define $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ to be the set of all linear combinations of the $\mathbf{x}_i$ (as matrices), and we say that $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is linearly independent if the only vanishing linear combination is the trivial one (that is, if $\{\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_k^T\}$ is independent in $\mathbb{R}^n$, as the reader can verify).[6]

Theorem 5.2.3

The following are equivalent for an $n \times n$ *matrix* A:

1. A *is invertible.*
2. *The columns of* A *are linearly independent.*
3. *The columns of* A *span* $\mathbb{R}^n$.
4. *The rows of* A *are linearly independent.*
5. *The rows of* A *span the set of all* $1 \times n$ *rows.*

Proof. Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ denote the columns of A.

(1) $\Leftrightarrow$ (2). By Theorem 2.4.5, A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$; this holds if and only if $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ is independent by Theorem 5.2.2.

(1) $\Leftrightarrow$ (3). Again by Theorem 2.4.5, A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every column B in $\mathbb{R}^n$; this holds if and only if $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} = \mathbb{R}^n$ by Theorem 5.2.2.

(1) $\Leftrightarrow$ (4). The matrix A is invertible if and only if A^T is invertible (by Corollary 2.4.1 to Theorem 2.4.4); this in turn holds if and only if A^T has independent columns (by (1) $\Leftrightarrow$ (2)); finally, this last statement holds if and only if A has independent rows (because the rows of A are the transposes of the columns of A^T).

(1) $\Leftrightarrow$ (5). The proof is similar to (1) $\Leftrightarrow$ (4). □

Example 5.2.9

Show that $S = \{(2, -2, 5), (-3, 1, 1), (2, 7, -4)\}$ is independent in $\mathbb{R}^3$.

Solution. Consider the matrix $A = \begin{bmatrix} 2 & -2 & 5 \\ -3 & 1 & 1 \\ 2 & 7 & -4 \end{bmatrix}$ with the vectors in S as its rows. A routine computation shows that $\det A = -117 \neq 0$, so A is invertible. Hence S is independent by Theorem 5.2.3. Note that Theorem 5.2.3 also shows that $\mathbb{R}^3 = \text{span}\, S$.

[6] It is best to view columns and rows as just two different *notations* for ordered n-tuples. This discussion will become redundant in Chapter 6 where we define the general notion of a vector space.

Dimension

It is common geometrical language to say that $\mathbb{R}^3$ is 3-dimensional, that planes are 2-dimensional and that lines are 1-dimensional. The next theorem is a basic tool for clarifying this idea of "dimension". Its importance is difficult to exaggerate.

Theorem 5.2.4: Fundamental Theorem

Let U be a subspace of $\mathbb{R}^n$. If U is spanned by m vectors, and if U contains k linearly independent vectors, then $k \leq m$.

This proof is given in Theorem 6.3.2 in much greater generality.

Definition 5.4 Basis of $\mathbb{R}^n$

If U is a subspace of $\mathbb{R}^n$, a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ of vectors in U is called a **basis** *of U if it satisfies the following two conditions:*

1. $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ *is linearly independent.*
2. $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$.

The most remarkable result about bases[7] is:

Theorem 5.2.5: Invariance Theorem

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$ are bases of a subspace U of $\mathbb{R}^n$, then $m = k$.

Proof. We have $k \leq m$ by the fundamental theorem because $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ spans U, and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$ is independent. Similarly, by interchanging $\mathbf{x}$'s and $\mathbf{y}$'s we get $m \leq k$. Hence $m = k$. □

The invariance theorem guarantees that there is no ambiguity in the following definition:

Definition 5.5 Dimension of a Subspace of $\mathbb{R}^n$

If U is a subspace of $\mathbb{R}^n$ and $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is any basis of U, the number, m, of vectors in the basis is called the **dimension** *of U, denoted*

$$\dim U = m$$

The importance of the invariance theorem is that the dimension of U can be determined by counting the number of vectors in *any* basis.[8]

[7] The plural of "basis" is "bases".

[8] We will show in Theorem 5.2.6 that every subspace of $\mathbb{R}^n$ does indeed *have* a basis.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ denote the standard basis of $\mathbb{R}^n$, that is the set of columns of the identity matrix. Then $\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ by Example 5.1.6, and $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is independent by Example 5.2.2. Hence it is indeed a basis of $\mathbb{R}^n$ in the present terminology, and we have

Example 5.2.10

$\dim(\mathbb{R}^n) = n$ and $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis.

This agrees with our geometric sense that $\mathbb{R}^2$ is two-dimensional and $\mathbb{R}^3$ is three-dimensional. It also says that $\mathbb{R}^1 = \mathbb{R}$ is one-dimensional, and $\{1\}$ is a basis. Returning to subspaces of $\mathbb{R}^n$, we define

$$\dim\{\mathbf{0}\} = 0$$

This amounts to saying $\{\mathbf{0}\}$ has a basis containing *no* vectors. This makes sense because $\mathbf{0}$ cannot belong to *any* independent set (Example 5.2.4).

Example 5.2.11

Let $U = \left\{ \left[\begin{array}{c} r \\ s \\ r \end{array} \right] \mid r, s \text{ in } \mathbb{R} \right\}$. Show that U is a subspace of $\mathbb{R}^3$, find a basis, and calculate $\dim U$.

Solution. Clearly, $\left[\begin{array}{c} r \\ s \\ r \end{array} \right] = r\mathbf{u} + s\mathbf{v}$ where $\mathbf{u} = \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right]$ and $\mathbf{v} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right]$. It follows that $U = \text{span}\{\mathbf{u}, \mathbf{v}\}$, and hence that U is a subspace of $\mathbb{R}^3$. Moreover, if $r\mathbf{u} + s\mathbf{v} = \mathbf{0}$, then $\left[\begin{array}{c} r \\ s \\ r \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$ so $r = s = 0$. Hence $\{\mathbf{u}, \mathbf{v}\}$ is independent, and so a **basis** of U. This means $\dim U = 2$.

Example 5.2.12

Let $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be a basis of $\mathbb{R}^n$. If A is an invertible $n \times n$ matrix, then $D = \{A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_n\}$ is also a basis of $\mathbb{R}^n$.

Solution. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then $A^{-1}\mathbf{x}$ is in $\mathbb{R}^n$ so, since B is a basis, we have $A^{-1}\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_n\mathbf{x}_n$ for t_i in $\mathbb{R}$. Left multiplication by A gives $\mathbf{x} = t_1(A\mathbf{x}_1) + t_2(A\mathbf{x}_2) + \cdots + t_n(A\mathbf{x}_n)$, and it follows that D spans $\mathbb{R}^n$. To show independence, let $s_1(A\mathbf{x}_1) + s_2(A\mathbf{x}_2) + \cdots + s_n(A\mathbf{x}_n) = \mathbf{0}$, where the s_i are in $\mathbb{R}$. Then $A(s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n) = \mathbf{0}$ so left multiplication by A^{-1} gives $s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n = \mathbf{0}$. Now the independence of B shows that each $s_i = 0$, and so proves the independence of D. Hence D is a basis of $\mathbb{R}^n$.

While we have found bases in many subspaces of $\mathbb{R}^n$, we have not yet shown that *every* subspace *has* a basis. This is part of the next theorem, the proof of which is deferred to Section 6.4 (Theorem 6.4.1) where it will be proved in more generality.

Theorem 5.2.6

Let $U \neq \{\mathbf{0}\}$ be a subspace of $\mathbb{R}^n$. Then:

1. *U has a basis and $\dim U \leq n$.*
2. *Any independent set in U can be enlarged (by adding vectors from the standard basis) to a basis of U.*
3. *Any spanning set for U can be cut down (by deleting vectors) to a basis of U.*

Example 5.2.13

Find a basis of $\mathbb{R}^4$ containing $S = \{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u} = (0, 1, 2, 3)$ and $\mathbf{v} = (2, -1, 0, 1)$.

Solution. By Theorem 5.2.6 we can find such a basis by adding vectors from the standard basis of $\mathbb{R}^4$ to S. If we try $\mathbf{e}_1 = (1, 0, 0, 0)$, we find easily that $\{\mathbf{e}_1, \mathbf{u}, \mathbf{v}\}$ is independent. Now add another vector from the standard basis, say $\mathbf{e}_2$.
Again we find that $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{u}, \mathbf{v}\}$ is independent. Since B has $4 = \dim \mathbb{R}^4$ vectors, then B must span $\mathbb{R}^4$ by Theorem 5.2.7 below (or simply verify it directly). Hence B is a basis of $\mathbb{R}^4$.

Theorem 5.2.6 has a number of useful consequences. Here is the first.

Theorem 5.2.7

Let U be a subspace of $\mathbb{R}^n$ where $\dim U = m$ and let $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ be a set of m vectors in U. Then B is independent if and only if B spans U.

Proof. Suppose B is independent. If B does not span U then, by Theorem 5.2.6, B can be enlarged to a basis of U containing more than m vectors. This contradicts the invariance theorem because $\dim U = m$, so B spans U. Conversely, if B spans U but is not independent, then B can be cut down to a basis of U containing fewer than m vectors, again a contradiction. So B is independent, as required. □

As we saw in Example 5.2.13, Theorem 5.2.7 is a "labour-saving" result. It asserts that, given a subspace U of dimension m and a set B of exactly m vectors in U, to prove that B is a basis of U it suffices to show either that B spans U or that B is independent. It is not necessary to verify both properties.

Theorem 5.2.8

Let $U \subseteq W$ be subspaces of $\mathbb{R}^n$. Then:

1. $\dim U \leq \dim W$.
2. *If* $\dim U = \dim W$, *then* $U = W$.

Proof. Write $\dim W = k$, and let B be a basis of U.

1. If $\dim U > k$, then B is an independent set in W containing more than k vectors, contradicting the fundamental theorem. So $\dim U \leq k = \dim W$.

2. If $\dim U = k$, then B is an independent set in W containing $k = \dim W$ vectors, so B spans W by Theorem 5.2.7. Hence $W = \text{span}\, B = U$, proving (2). □

It follows from Theorem 5.2.8 that if U is a subspace of $\mathbb{R}^n$, then $\dim U$ is one of the integers $0, 1, 2, \dots, n$, and that:

$$\begin{array}{lll} \dim U = 0 & \text{if and only if} & U = \{\mathbf{0}\}, \\ \dim U = n & \text{if and only if} & U = \mathbb{R}^n \end{array}$$

The other subspaces of $\mathbb{R}^n$ are called **proper**. The following example uses Theorem 5.2.8 to show that the proper subspaces of $\mathbb{R}^2$ are the lines through the origin, while the proper subspaces of $\mathbb{R}^3$ are the lines and planes through the origin.

Example 5.2.14

1. If U is a subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\dim U = 1$ if and only if U is a line through the origin.

2. If U is a subspace of $\mathbb{R}^3$, then $\dim U = 2$ if and only if U is a plane through the origin.

Proof.

1. Since $\dim U = 1$, let $\{\mathbf{u}\}$ be a basis of U. Then $U = \text{span}\{\mathbf{u}\} = \{t\mathbf{u} \mid t \text{ in } \mathbb{R}\}$, so U is the line through the origin with direction vector $\mathbf{u}$. Conversely each line L with direction vector $\mathbf{d} \neq \mathbf{0}$ has the form $L = \{t\mathbf{d} \mid t \text{ in } \mathbb{R}\}$. Hence $\{\mathbf{d}\}$ is a basis of U, so U has dimension 1.

2. If $U \subseteq \mathbb{R}^3$ has dimension 2, let $\{\mathbf{v}, \mathbf{w}\}$ be a basis of U. Then $\mathbf{v}$ and $\mathbf{w}$ are not parallel (by Example 5.2.7) so $\mathbf{n} = \mathbf{v} \times \mathbf{w} \neq \mathbf{0}$. Let $P = \{\mathbf{x} \text{ in } \mathbb{R}^3 \mid \mathbf{n} \cdot \mathbf{x} = 0\}$ denote the plane through the origin with normal $\mathbf{n}$. Then P is a subspace of $\mathbb{R}^3$ (Example 5.1.1) and both $\mathbf{v}$ and $\mathbf{w}$ lie in P (they are orthogonal to $\mathbf{n}$), so $U = \text{span}\{\mathbf{v}, \mathbf{w}\} \subseteq P$ by Theorem 5.1.1. Hence

   $$U \subseteq P \subseteq \mathbb{R}^3$$

   Since $\dim U = 2$ and $\dim(\mathbb{R}^3) = 3$, it follows from Theorem 5.2.8 that $\dim P = 2$ or 3, whence $P = U$ or $\mathbb{R}^3$. But $P \neq \mathbb{R}^3$ (for example, $\mathbf{n}$ is not in P) and so $U = P$ is a plane through the origin.

   Conversely, if U is a plane through the origin, then $\dim U = 0$, 1, 2, or 3 by Theorem 5.2.8. But $\dim U \neq 0$ or 3 because $U \neq \{\mathbf{0}\}$ and $U \neq \mathbb{R}^3$, and $\dim U \neq 1$ by (1). So $\dim U = 2$. □

Note that this proof shows that if $\mathbf{v}$ and $\mathbf{w}$ are nonzero, nonparallel vectors in $\mathbb{R}^3$, then $\text{span}\{\mathbf{v}, \mathbf{w}\}$ is the plane with normal $\mathbf{n} = \mathbf{v} \times \mathbf{w}$. We gave a geometrical verification of this fact in Section 5.1.

Exercises for 5.2

In Exercises 5.2.1-5.2.6 we write vectors $\mathbb{R}^n$ as rows.

Exercise 5.2.1 Which of the following subsets are independent? Support your answer.

a. $\{(1,\ -1,\ 0),\ (3,\ 2,\ -1),\ (3,\ 5,\ -2)\}$ in $\mathbb{R}^3$

b. $\{(1,\ 1,\ 1),\ (1,\ -1,\ 1),\ (0,\ 0,\ 1)\}$ in $\mathbb{R}^3$

c. $\{(1,\ -1,\ 1,\ -1),\ (2,\ 0,\ 1,\ 0),\ (0,\ -2,\ 1,\ -2)\}$ in $\mathbb{R}^4$

d. $\{(1,\ 1,\ 0,\ 0),\ (1,\ 0,\ 1,\ 0),\ (0,\ 0,\ 1,\ 1),\ (0,\ 1,\ 0,\ 1)\}$ in $\mathbb{R}^4$

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b. Yes. If $r\begin{bmatrix}1\\1\\1\end{bmatrix}+s\begin{bmatrix}1\\1\\1\end{bmatrix}+t\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, then $r+s=0$, $r-s=0$, and $r+s+t=0$. These equations give $r=s=t=0$.

d. No. Indeed: $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}-\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\\1\end{bmatrix}-\begin{bmatrix}0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

Exercise 5.2.2 Let $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}\}$ be an independent set in $\mathbb{R}^n$. Which of the following sets is independent? Support your answer.

a. $\{\mathbf{x}-\mathbf{y},\ \mathbf{y}-\mathbf{z},\ \mathbf{z}-\mathbf{x}\}$

b. $\{\mathbf{x}+\mathbf{y},\ \mathbf{y}+\mathbf{z},\ \mathbf{z}+\mathbf{x}\}$

c. $\{\mathbf{x}-\mathbf{y},\ \mathbf{y}-\mathbf{z},\ \mathbf{z}-\mathbf{w},\ \mathbf{w}-\mathbf{x}\}$

d. $\{\mathbf{x}+\mathbf{y},\ \mathbf{y}+\mathbf{z},\ \mathbf{z}+\mathbf{w},\ \mathbf{w}+\mathbf{x}\}$

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b. Yes. If $r(\mathbf{x}+\mathbf{y})+s(\mathbf{y}+\mathbf{z})+t(\mathbf{z}+\mathbf{x})=\mathbf{0}$, then $(r+t)\mathbf{x}+(r+s)\mathbf{y}+(s+t)\mathbf{z}=\mathbf{0}$. Since $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent, this implies that $r+t=0$, $r+s=0$, and $s+t=0$. The only solution is $r=s=t=0$.

d. No. In fact, $(\mathbf{x}+\mathbf{y})-(\mathbf{y}+\mathbf{z})+(\mathbf{z}+\mathbf{w})-(\mathbf{w}+\mathbf{x})=\mathbf{0}$.

Exercise 5.2.3 Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.

a. $\text{span}\,\{(1,\ -1,\ 2,\ 0),\ (2,\ 3,\ 0,\ 3),\ (1,\ 9,\ -6,\ 6)\}$

b. $\text{span}\,\{(2,\ 1,\ 0,\ -1),\ (-1,\ 1,\ 1,\ 1),\ (2,\ 7,\ 4,\ 1)\}$

c. $\text{span}\,\{(-1,\ 2,\ 1,\ 0),\ (2,\ 0,\ 3,\ -1),\ (4,\ 4,\ 11,\ -3),\ (3,\ -2,\ 2,\ -1)\}$

d. $\text{span}\,\{(-2,\ 0,\ 3,\ 1),\ (1,\ 2,\ -1,\ 0),\ (-2,\ 8,\ 5,\ 3),\ (-1,\ 2,\ 2,\ 1)\}$

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b. $\left\{\begin{bmatrix}2\\1\\0\\-1\end{bmatrix},\ \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}$; dimension 2.

d. $\left\{\begin{bmatrix}-2\\0\\3\\1\end{bmatrix},\ \begin{bmatrix}1\\2\\-1\\0\end{bmatrix}\right\}$; dimension 2.

Exercise 5.2.4 Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.

a. $U=\left\{\begin{bmatrix}a\\a+b\\a-b\\b\end{bmatrix}\middle|\ a \text{ and } b \text{ in } \mathbb{R}\right\}$

b. $U=\left\{\begin{bmatrix}a+b\\a-b\\b\\a\end{bmatrix}\middle|\ a \text{ and } b \text{ in } \mathbb{R}\right\}$

c. $U = \left\{ \begin{bmatrix} a \\ b \\ c+a \\ c \end{bmatrix} \middle| a,\ b,\ \text{and } c \text{ in } \mathbb{R} \right\}$

d. $U = \left\{ \begin{bmatrix} a-b \\ b+c \\ a \\ b+c \end{bmatrix} \middle| a,\ b,\ \text{and } c \text{ in } \mathbb{R} \right\}$

e. $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| a+b-c+d=0 \text{ in } \mathbb{R} \right\}$

f. $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| a+b=c+d \text{ in } \mathbb{R} \right\}$

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b. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$; dimension 2.

d. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$; dimension 3.

f. $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$; dimension 3.

Exercise 5.2.5 Suppose that $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}\}$ is a basis of $\mathbb{R}^4$. Show that:

a. $\{\mathbf{x}+a\mathbf{w},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}\}$ is also a basis of $\mathbb{R}^4$ for any choice of the scalar a.

b. $\{\mathbf{x}+\mathbf{w},\ \mathbf{y}+\mathbf{w},\ \mathbf{z}+\mathbf{w},\ \mathbf{w}\}$ is also a basis of $\mathbb{R}^4$.

c. $\{\mathbf{x},\ \mathbf{x}+\mathbf{y},\ \mathbf{x}+\mathbf{y}+\mathbf{z},\ \mathbf{x}+\mathbf{y}+\mathbf{z}+\mathbf{w}\}$ is also a basis of $\mathbb{R}^4$.

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b. If $r(\mathbf{x}+\mathbf{w})+s(\mathbf{y}+\mathbf{w})+t(\mathbf{z}+\mathbf{w})+u(\mathbf{w})=\mathbf{0}$, then $r\mathbf{x}+s\mathbf{y}+t\mathbf{z}+(r+s+t+u)\mathbf{w}=\mathbf{0}$, so $r=0$, $s=0$, $t=0$, and $r+s+t+u=0$. The only solution is $r=s=t=u=0$, so the set is independent. Since $\dim \mathbb{R}^4 = 4$, the set is a basis by Theorem 5.2.7.

Exercise 5.2.6 Use Theorem 5.2.3 to determine if the following sets of vectors are a basis of the indicated space.

a. $\{(3,\ -1),\ (2,\ 2)\}$ in $\mathbb{R}^2$

b. $\{(1,\ 1,\ -1),\ (1,\ -1,\ 1),\ (0,\ 0,\ 1)\}$ in $\mathbb{R}^3$

c. $\{(-1,\ 1,\ -1),\ (1,\ -1,\ 2),\ (0,\ 0,\ 1)\}$ in $\mathbb{R}^3$

d. $\{(5,\ 2,\ -1),\ (1,\ 0,\ 1),\ (3,\ -1,\ 0)\}$ in $\mathbb{R}^3$

e. $\{(2,\ 1,\ -1,\ 3),\ (1,\ 1,\ 0,\ 2),\ (0,\ 1,\ 0,\ -3),\ (-1,\ 2,\ 3,\ 1)\}$ in $\mathbb{R}^4$

f. $\{(1,\ 0,\ -2,\ 5),\ (4,\ 4,\ -3,\ 2),\ (0,\ 1,\ 0,\ -3),\ (1,\ 3,\ 3,\ -10)\}$ in $\mathbb{R}^4$

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b. Yes

d. Yes

f. No.

Exercise 5.2.7 In each case show that the statement is true or give an example showing that it is false.

a. If $\{\mathbf{x},\ \mathbf{y}\}$ is independent, then $\{\mathbf{x},\ \mathbf{y},\ \mathbf{x}+\mathbf{y}\}$ is independent.

b. If $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent, then $\{\mathbf{y},\ \mathbf{z}\}$ is independent.

c. If $\{\mathbf{y},\ \mathbf{z}\}$ is dependent, then $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is dependent for any $\mathbf{x}$.

d. If all of $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ are nonzero, then $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is independent.

e. If one of $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ is zero, then $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is dependent.

f. If $a\mathbf{x}+b\mathbf{y}+c\mathbf{z}=\mathbf{0}$, then $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent.

g. If $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent, then $a\mathbf{x}+b\mathbf{y}+c\mathbf{z}=\mathbf{0}$ for some a, b, and c in $\mathbb{R}$.

h. If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is dependent, then $t_1\mathbf{x}_1+t_2\mathbf{x}_2+\cdots+t_k\mathbf{x}_k=\mathbf{0}$ for some numbers t_i in $\mathbb{R}$ not all zero.

i. If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is independent, then $t_1\mathbf{x}_1+t_2\mathbf{x}_2+\cdots+t_k\mathbf{x}_k=\mathbf{0}$ for some t_i in $\mathbb{R}$.

j. Every non-empty subset of a linearly independent set is again linearly independent.

k. Every set containing a spanning set is again a spanning set.

---

b. T. If $r\mathbf{y}+s\mathbf{z}=\mathbf{0}$, then $0\mathbf{x}+r\mathbf{y}+s\mathbf{z}=\mathbf{0}$ so $r=s=0$ because $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is independent.

d. F. If $\mathbf{x}\neq\mathbf{0}$, take $k=2$, $\mathbf{x}_1=\mathbf{x}$ and $\mathbf{x}_2=-\mathbf{x}$.

f. F. If $\mathbf{y}=-\mathbf{x}$ and $\mathbf{z}=\mathbf{0}$, then $1\mathbf{x}+1\mathbf{y}+1\mathbf{z}=\mathbf{0}$.

h. T. This is a nontrivial, vanishing linear combination, so the $\mathbf{x}_i$ cannot be independent.

Exercise 5.2.8 If A is an $n\times n$ matrix, show that $\det A=0$ if and only if some column of A is a linear combination of the other columns.

Exercise 5.2.9 Let $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ be a linearly independent set in $\mathbb{R}^4$. Show that $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{e}_k\}$ is a basis of $\mathbb{R}^4$ for some $\mathbf{e}_k$ in the standard basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_4\}$.

Exercise 5.2.10 If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4,\ \mathbf{x}_5,\ \mathbf{x}_6\}$ is an independent set of vectors, show that the subset $\{\mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_5\}$ is also independent.

If $r\mathbf{x}_2+s\mathbf{x}_3+t\mathbf{x}_5=\mathbf{0}$ then $0\mathbf{x}_1+r\mathbf{x}_2+s\mathbf{x}_3+0\mathbf{x}_4+t\mathbf{x}_5+0\mathbf{x}_6=\mathbf{0}$ so $r=s=t=0$.

Exercise 5.2.11 Let A be any $m\times n$ matrix, and let $\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3,\ \dots,\ \mathbf{b}_k$ be columns in $\mathbb{R}^m$ such that the system $A\mathbf{x}=\mathbf{b}_i$ has a solution $\mathbf{x}_i$ for each i. If $\{\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3,\ \dots,\ \mathbf{b}_k\}$ is independent in $\mathbb{R}^m$, show that $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \dots,\ \mathbf{x}_k\}$ is independent in $\mathbb{R}^n$.

Exercise 5.2.12 If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \dots,\ \mathbf{x}_k\}$ is independent, show $\{\mathbf{x}_1,\ \mathbf{x}_1+\mathbf{x}_2,\ \mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3,\ \dots,\ \mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\}$ is also independent.

If $t_1\mathbf{x}_1+t_2(\mathbf{x}_1+\mathbf{x}_2)+\cdots+t_k(\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k)=\mathbf{0}$, then $(t_1+t_2+\cdots+t_k)\mathbf{x}_1+(t_2+\cdots+t_k)\mathbf{x}_2+\cdots+(t_{k-1}+t_k)\mathbf{x}_{k-1}+(t_k)\mathbf{x}_k=\mathbf{0}$. Hence all these coefficients are zero, so we obtain successively $t_k=0$, $t_{k-1}=0$, $\dots$, $t_2=0$, $t_1=0$.

Exercise 5.2.13 If $\{\mathbf{y},\ \mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \dots,\ \mathbf{x}_k\}$ is independent, show that $\{\mathbf{y}+\mathbf{x}_1,\ \mathbf{y}+\mathbf{x}_2,\ \mathbf{y}+\mathbf{x}_3,\ \dots,\ \mathbf{y}+\mathbf{x}_k\}$ is also independent.

Exercise 5.2.14 If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is independent in $\mathbb{R}^n$, and if $\mathbf{y}$ is not in $\operatorname{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$, show that $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k,\ \mathbf{y}\}$ is independent.

Exercise 5.2.15 If A and B are matrices and the columns of AB are independent, show that the columns of B are independent.

Exercise 5.2.16 Suppose that $\{\mathbf{x},\ \mathbf{y}\}$ is a basis of $\mathbb{R}^2$, and let $A=\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$.

a. If A is invertible, show that $\{a\mathbf{x}+b\mathbf{y},\ c\mathbf{x}+d\mathbf{y}\}$ is a basis of $\mathbb{R}^2$.

b. If $\{a\mathbf{x}+b\mathbf{y},\ c\mathbf{x}+d\mathbf{y}\}$ is a basis of $\mathbb{R}^2$, show that A is invertible.

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b. We show A^T is invertible (then A is invertible). Let $A^T\mathbf{x}=\mathbf{0}$ where $\mathbf{x}=[s\ \ t]^T$. This means $as+ct=0$ and $bs+dt=0$, so $s(a\mathbf{x}+b\mathbf{y})+t(c\mathbf{x}+d\mathbf{y})=(sa+tc)\mathbf{x}+(sb+td)\mathbf{y}=\mathbf{0}$. Hence $s=t=0$ by hypothesis.

Exercise 5.2.17 Let A denote an $m\times n$ matrix.

a. Show that $\operatorname{null} A=\operatorname{null}(UA)$ for every invertible $m\times m$ matrix U.

b. Show that $\dim(\operatorname{null} A)=\dim(\operatorname{null}(AV))$ for every invertible $n\times n$ matrix V. [*Hint*: If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is a basis of $\operatorname{null} A$, show that $\{V^{-1}\mathbf{x}_1,\ V^{-1}\mathbf{x}_2,\ \dots,\ V^{-1}\mathbf{x}_k\}$ is a basis of $\operatorname{null}(AV)$.]

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b. Each $V^{-1}\mathbf{x}_i$ is in $\operatorname{null}(AV)$ because $AV(V^{-1}\mathbf{x}_i)=A\mathbf{x}_i=\mathbf{0}$. The set

$\{V^{-1}\mathbf{x}_1, \dots, V^{-1}\mathbf{x}_k\}$ is independent as V^{-1} is invertible. If $\mathbf{y}$ is in $\text{null}(AV)$, then $V\mathbf{y}$ is in $\text{null}(A)$ so let $V\mathbf{y} = t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k$ where each t_k is in $\mathbb{R}$. Thus $\mathbf{y} = t_1V^{-1}\mathbf{x}_1 + \cdots + t_kV^{-1}\mathbf{x}_k$ is in $\text{span}\{V^{-1}\mathbf{x}_1, \dots, V^{-1}\mathbf{x}_k\}$.

Exercise 5.2.18 Let A denote an $m \times n$ matrix.

a. Show that $\text{im}\, A = \text{im}\,(AV)$ for every invertible $n \times n$ matrix V.

b. Show that $\dim(\text{im}\, A) = \dim(\text{im}\,(UA))$ for every invertible $m \times m$ matrix U. [*Hint*: If $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$ is a basis of $\text{im}\,(UA)$, show that $\{U^{-1}\mathbf{y}_1, U^{-1}\mathbf{y}_2, \dots, U^{-1}\mathbf{y}_k\}$ is a basis of $\text{im}\, A$.]

Exercise 5.2.19 Let U and W denote subspaces of $\mathbb{R}^n$, and assume that $U \subseteq W$. If $\dim U = n - 1$, show that either $W = U$ or $W = \mathbb{R}^n$.

Exercise 5.2.20 Let U and W denote subspaces of $\mathbb{R}^n$, and assume that $U \subseteq W$. If $\dim W = 1$, show that either $U = \{\mathbf{0}\}$ or $U = W$.

We have $\{\mathbf{0}\} \subseteq U \subseteq W$ where $\dim\{\mathbf{0}\} = 0$ and $\dim W = 1$. Hence $\dim U = 0$ or $\dim U = 1$ by Theorem 5.2.8, that is $U = 0$ or $U = W$, again by Theorem 5.2.8.

5.3 Orthogonality

Length and orthogonality are basic concepts in geometry and, in $\mathbb{R}^2$ and $\mathbb{R}^3$, they both can be defined using the dot product. In this section we extend the dot product to vectors in $\mathbb{R}^n$, and so endow $\mathbb{R}^n$ with euclidean geometry. We then introduce the idea of an orthogonal basis—one of the most useful concepts in linear algebra, and begin exploring some of its applications.

Dot Product, Length, and Distance

If $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ are two n-tuples in $\mathbb{R}^n$, recall that their **dot product** was defined in Section 2.2 as follows:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Observe that if $\mathbf{x}$ and $\mathbf{y}$ are written as columns then $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$ is a matrix product (and $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}\mathbf{y}^T$ if they are written as rows). Here $\mathbf{x} \cdot \mathbf{y}$ is a 1×1 matrix, which we take to be a number.

Definition 5.6 Length in $\mathbb{R}^n$

As in $\mathbb{R}^3$, the ***length*** $\|\mathbf{x}\|$ *of the vector is defined by*

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Where $\sqrt{(\quad)}$ indicates the positive square root.

A vector $\mathbf{x}$ of length 1 is called a **unit vector**. If $\mathbf{x} \neq \mathbf{0}$, then $\|\mathbf{x}\| \neq 0$ and it follows easily that $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a unit vector (see Theorem 5.3.6 below), a fact that we shall use later.

Example 5.3.1

If $\mathbf{x} = (1, -1, -3, 1)$ and $\mathbf{y} = (2, 1, 1, 0)$ in $\mathbb{R}^4$, then $\mathbf{x} \cdot \mathbf{y} = 2 - 1 - 3 + 0 = -2$ and $\|\mathbf{x}\| = \sqrt{1+1+9+1} = \sqrt{12} = 2\sqrt{3}$. Hence $\frac{1}{2\sqrt{3}}\mathbf{x}$ is a unit vector; similarly $\frac{1}{\sqrt{6}}\mathbf{y}$ is a unit vector.

These definitions agree with those in $\mathbb{R}^2$ and $\mathbb{R}^3$, and many properties carry over to $\mathbb{R}^n$:

Theorem 5.3.1

Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ denote vectors in $\mathbb{R}^n$. Then:

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
2. $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.
3. $(a\mathbf{x}) \cdot \mathbf{y} = a(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (a\mathbf{y})$ *for all scalars a.*

4. $\|\mathbf{x}\|^2 = \mathbf{x}\cdot\mathbf{x}$.

5. $\|\mathbf{x}\| \geq 0$, *and* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

6. $\|a\mathbf{x}\| = |a|\|\mathbf{x}\|$ *for all scalars* a.

Proof. (1), (2), and (3) follow from matrix arithmetic because $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y}$; (4) is clear from the definition; and (6) is a routine verification since $|a| = \sqrt{a^2}$. If $\mathbf{x} = (x_1, x_2, \dots, x_n)$, then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ so $\|\mathbf{x}\| = 0$ if and only if $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$. Since each x_i is a real number this happens if and only if $x_i = 0$ for each i; that is, if and only if $\mathbf{x} = \mathbf{0}$. This proves (5). □

Because of Theorem 5.3.1, computations with dot products in $\mathbb{R}^n$ are similar to those in $\mathbb{R}^3$. In particular, the dot product

$$(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_m)\cdot(\mathbf{y}_1 + \mathbf{y}_2 + \cdots + \mathbf{y}_k)$$

equals the sum of mk terms, $\mathbf{x}_i\cdot\mathbf{y}_j$, one for each choice of i and j. For example:

$$\begin{aligned}(3\mathbf{x} - 4\mathbf{y})\cdot(7\mathbf{x} + 2\mathbf{y}) &= 21(\mathbf{x}\cdot\mathbf{x}) + 6(\mathbf{x}\cdot\mathbf{y}) - 28(\mathbf{y}\cdot\mathbf{x}) - 8(\mathbf{y}\cdot\mathbf{y}) \\ &= 21\|\mathbf{x}\|^2 - 22(\mathbf{x}\cdot\mathbf{y}) - 8\|\mathbf{y}\|^2\end{aligned}$$

holds for all vectors $\mathbf{x}$ and $\mathbf{y}$.

Example 5.3.2

Show that $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2$ for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

Solution. Using Theorem 5.3.1 several times:

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y})\cdot(\mathbf{x} + \mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\ &= \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2\end{aligned}$$

Example 5.3.3

Suppose that $\mathbb{R}^n = \text{span}\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k\}$ for some vectors $\mathbf{f}_i$. If $\mathbf{x}\cdot\mathbf{f}_i = 0$ for each i where $\mathbf{x}$ is in $\mathbb{R}^n$, show that $\mathbf{x} = \mathbf{0}$.

Solution. We show $\mathbf{x} = \mathbf{0}$ by showing that $\|\mathbf{x}\| = 0$ and using (5) of Theorem 5.3.1. Since the $\mathbf{f}_i$ span $\mathbb{R}^n$, write $\mathbf{x} = t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + \cdots + t_k\mathbf{f}_k$ where the t_i are in $\mathbb{R}$. Then

$$\begin{aligned}\|\mathbf{x}\|^2 &= \mathbf{x}\cdot\mathbf{x} = \mathbf{x}\cdot(t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + \cdots + t_k\mathbf{f}_k) \\ &= t_1(\mathbf{x}\cdot\mathbf{f}_1) + t_2(\mathbf{x}\cdot\mathbf{f}_2) + \cdots + t_k(\mathbf{x}\cdot\mathbf{f}_k) \\ &= t_1(0) + t_2(0) + \cdots + t_k(0) \\ &= 0\end{aligned}$$

We saw in Section 4.2 that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, then $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Since $|\cos\theta| \leq 1$ for any angle θ, this shows that $|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$. In this form the result holds in $\mathbb{R}^n$.

Theorem 5.3.2: Cauchy Inequality[9]

If $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$, *then*

$$|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$

Moreover $|\mathbf{x}\cdot\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ *if and only if one of* $\mathbf{x}$ *and* $\mathbf{y}$ *is a multiple of the other.*

Proof. The inequality holds if $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ (in fact it is equality). Otherwise, write $\|\mathbf{x}\| = a > 0$ and $\|\mathbf{y}\| = b > 0$ for convenience. A computation like that preceding Example 5.3.2 gives

$$\|b\mathbf{x} - a\mathbf{y}\|^2 = 2ab(ab - \mathbf{x}\cdot\mathbf{y}) \text{ and } \|b\mathbf{x} + a\mathbf{y}\|^2 = 2ab(ab + \mathbf{x}\cdot\mathbf{y}) \tag{5.1}$$

It follows that $ab - \mathbf{x}\cdot\mathbf{y} \geq 0$ and $ab + \mathbf{x}\cdot\mathbf{y} \geq 0$, and hence that $-ab \leq \mathbf{x}\cdot\mathbf{y} \leq ab$. Hence $|\mathbf{x}\cdot\mathbf{y}| \leq ab = \|\mathbf{x}\|\|\mathbf{y}\|$, proving the Cauchy inequality.

If equality holds, then $|\mathbf{x}\cdot\mathbf{y}| = ab$, so $\mathbf{x}\cdot\mathbf{y} = ab$ or $\mathbf{x}\cdot\mathbf{y} = -ab$. Hence Equation 5.1 shows that $b\mathbf{x} - a\mathbf{y} = 0$ or $b\mathbf{x} + a\mathbf{y} = 0$, so one of $\mathbf{x}$ and $\mathbf{y}$ is a multiple of the other (even if $a = 0$ or $b = 0$). □

The Cauchy inequality is equivalent to $(\mathbf{x}\cdot\mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2$. In $\mathbb{R}^5$ this becomes

$$(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5)^2 \leq (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2)$$

for all x_i and y_i in $\mathbb{R}$.

There is an important consequence of the Cauchy inequality. Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, use Example 5.3.2 and the fact that $\mathbf{x}\cdot\mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$ to compute

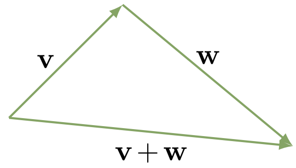
$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x} + \mathbf{y}\|)^2$$

Taking positive square roots gives:

Corollary 5.3.1: Triangle Inequality

If $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$, *then* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

[9] Augustin Louis Cauchy (1789–1857) was born in Paris and became a professor at the École Polytechnique at the age of 26. He was one of the great mathematicians, producing more than 700 papers, and is best remembered for his work in analysis in which he established new standards of rigour and founded the theory of functions of a complex variable. He was a devout Catholic with a long-term interest in charitable work, and he was a royalist, following King Charles X into exile in Prague after he was deposed in 1830. Theorem 5.3.2 first appeared in his 1812 memoir on determinants.

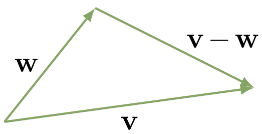


The reason for the name comes from the observation that in $\mathbb{R}^3$ the inequality asserts that the sum of the lengths of two sides of a triangle is not less than the length of the third side. This is illustrated in the diagram.

Definition 5.7 Distance in $\mathbb{R}^n$

If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^n$, we define the ***distance*** *$d(\mathbf{x},\ \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$ by*

$$d(\mathbf{x},\ \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$



The motivation again comes from $\mathbb{R}^3$ as is clear in the diagram. This distance function has all the intuitive properties of distance in $\mathbb{R}^3$, including another version of the triangle inequality.

Theorem 5.3.3

If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are three vectors in $\mathbb{R}^n$ we have:

1. $d(\mathbf{x},\ \mathbf{y}) \geq 0$ *for all $\mathbf{x}$ and $\mathbf{y}$.*
2. $d(\mathbf{x},\ \mathbf{y}) = 0$ *if and only if $\mathbf{x} = \mathbf{y}$.*
3. $d(\mathbf{x},\ \mathbf{y}) = d(\mathbf{y},\ \mathbf{x})$ *for all $\mathbf{x}$ and $\mathbf{y}$.*
4. $d(\mathbf{x},\ \mathbf{z}) \leq d(\mathbf{x},\ \mathbf{y}) + d(\mathbf{y},\ \mathbf{z})$*for all $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.* *Triangle inequality.*

Proof. (1) and (2) restate part (5) of Theorem 5.3.1 because $d(\mathbf{x},\ \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, and (3) follows because $\|\mathbf{u}\| = \|-\mathbf{u}\|$ for every vector $\mathbf{u}$ in $\mathbb{R}^n$. To prove (4) use the Corollary to Theorem 5.3.2:

$$\begin{aligned} d(\mathbf{x},\ \mathbf{z}) = \|\mathbf{x} - \mathbf{z}\| &= \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\| \\ &\leq \|(\mathbf{x} - \mathbf{y})\| + \|(\mathbf{y} - \mathbf{z})\| = d(\mathbf{x},\ \mathbf{y}) + d(\mathbf{y},\ \mathbf{z}) \end{aligned}$$

□

Orthogonal Sets and the Expansion Theorem

Definition 5.8 Orthogonal and Orthonormal Sets

We say that two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are ***orthogonal*** *if $\mathbf{x} \cdot \mathbf{y} = 0$, extending the terminology in $\mathbb{R}^3$ (See Theorem 4.2.3). More generally, a set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is called an* ***orthogonal set*** *if*

$$\mathbf{x}_i \cdot \mathbf{x}_j = 0 \text{ for all } i \neq j \quad \text{and} \quad \mathbf{x}_i \neq \mathbf{0} \text{ for all } i$$[10]

Note that $\{\mathbf{x}\}$ is an orthogonal set if $\mathbf{x} \neq \mathbf{0}$. A set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is called

orthonormal *if it is orthogonal and, in addition, each* $\mathbf{x}_i$ *is a unit vector:*

$$\|\mathbf{x}_i\| = 1 \text{ for each } i.$$

Example 5.3.4

The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is an orthonormal set in $\mathbb{R}^n$.

The routine verification is left to the reader, as is the proof of:

Example 5.3.5

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is orthogonal, so also is $\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\}$ for any nonzero scalars a_i.

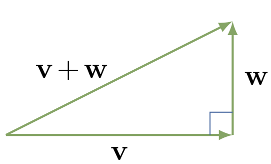
If $\mathbf{x} \neq \mathbf{0}$, it follows from item (6) of Theorem 5.3.1 that $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a unit vector, that is it has length 1.

Definition 5.9 Normalizing an Orthogonal Set

Hence if $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *is an orthogonal set, then* $\{\frac{1}{\|\mathbf{x}_1\|}\mathbf{x}_1, \frac{1}{\|\mathbf{x}_2\|}\mathbf{x}_2, \cdots, \frac{1}{\|\mathbf{x}_k\|}\mathbf{x}_k\}$ *is an orthonormal set, and we say that it is the result of* ***normalizing*** *the orthogonal set* $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$.

Example 5.3.6

If $\mathbf{f}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{f}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{f}_4 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 1 \end{bmatrix}$ then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ is an orthogonal set in $\mathbb{R}^4$ as is easily verified. After normalizing, the corresponding orthonormal set is $\{\frac{1}{2}\mathbf{f}_1, \frac{1}{\sqrt{6}}\mathbf{f}_2, \frac{1}{\sqrt{2}}\mathbf{f}_3, \frac{1}{2\sqrt{3}}\mathbf{f}_4\}$



The most important result about orthogonality is Pythagoras' theorem. Given orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, it asserts that

$$\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

as in the diagram. In this form the result holds for any orthogonal set in $\mathbb{R}^n$.

[10]The reason for insisting that orthogonal sets consist of *nonzero* vectors is that we will be primarily concerned with orthogonal bases.

Theorem 5.3.4: Pythagoras' Theorem

If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ *is an orthogonal set in* $\mathbb{R}^n$*, then*

$$\|\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\|^2 = \|\mathbf{x}_1\|^2+\|\mathbf{x}_2\|^2+\cdots+\|\mathbf{x}_k\|^2.$$

Proof. The fact that $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ whenever $i \neq j$ gives

$$\begin{aligned}\|\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k\|^2 &= (\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k)\cdot(\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_k)\\ &= (\mathbf{x}_1\cdot\mathbf{x}_1+\mathbf{x}_2\cdot\mathbf{x}_2+\cdots+\mathbf{x}_k\cdot\mathbf{x}_k)+\sum_{i\neq j}\mathbf{x}_i\cdot\mathbf{x}_j\\ &= \|\mathbf{x}_1\|^2+\|\mathbf{x}_2\|^2+\cdots+\|\mathbf{x}_k\|^2+0\end{aligned}$$

This is what we wanted. □

If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, nonzero vectors in $\mathbb{R}^3$, then they are certainly not parallel, and so are linearly independent Example 5.2.7. The next theorem gives a far-reaching extension of this observation.

Theorem 5.3.5

Every orthogonal set in $\mathbb{R}^n$ is linearly independent.

Proof. Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ be an orthogonal set in $\mathbb{R}^n$ and suppose a linear combination vanishes, say: $t_1\mathbf{x}_1+t_2\mathbf{x}_2+\cdots+t_k\mathbf{x}_k=\mathbf{0}$. Then

$$\begin{aligned}0=\mathbf{x}_1\cdot\mathbf{0}=\mathbf{x}_1\cdot(t_1\mathbf{x}_1+t_2\mathbf{x}_2+\cdots+t_k\mathbf{x}_k) &\\ = t_1(\mathbf{x}_1\cdot\mathbf{x}_1)+t_2(\mathbf{x}_1\cdot\mathbf{x}_2)+\cdots+t_k(\mathbf{x}_1\cdot\mathbf{x}_k) &\\ = t_1\|\mathbf{x}_1\|^2+t_2(0)+\cdots+t_k(0) &\\ = t_1\|\mathbf{x}_1\|^2 &\end{aligned}$$

Since $\|\mathbf{x}_1\|^2\neq 0$, this implies that $t_1=0$. Similarly $t_i=0$ for each i. □

Theorem 5.3.5 suggests considering orthogonal bases for $\mathbb{R}^n$, that is orthogonal sets that span $\mathbb{R}^n$. These turn out to be the best bases in the sense that, when expanding a vector as a linear combination of the basis vectors, there are explicit formulas for the coefficients.

Theorem 5.3.6: Expansion Theorem

Let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ be an orthogonal basis of a subspace U of $\mathbb{R}^n$. If $\mathbf{x}$ is any vector in U, we have

$$\mathbf{x}=\left(\frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\right)\mathbf{f}_1+\left(\frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\right)\mathbf{f}_1+\cdots+\left(\frac{\mathbf{x}\cdot\mathbf{f}_m}{\|\mathbf{f}_m\|^2}\right)\mathbf{f}_m$$

Proof. Since $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ spans U, we have $\mathbf{x}=t_1\mathbf{f}_1+t_2\mathbf{f}_2+\cdots+t_m\mathbf{f}_m$ where the t_i are scalars. To find t_1 we take the dot product of both sides with $\mathbf{f}_1$:

$$\begin{aligned}\mathbf{x}\cdot\mathbf{f}_1 &= (t_1\mathbf{f}_1+t_2\mathbf{f}_2+\cdots+t_m\mathbf{f}_m)\cdot\mathbf{f}_1\\ &= t_1(\mathbf{f}_1\cdot\mathbf{f}_1)+t_2(\mathbf{f}_2\cdot\mathbf{f}_1)+\cdots+t_m(\mathbf{f}_m\cdot\mathbf{f}_1)\\ &= t_1\|\mathbf{f}_1\|^2+t_2(0)+\cdots+t_m(0)\\ &= t_1\|\mathbf{f}_1\|^2\end{aligned}$$

Since $\mathbf{f}_1 \neq \mathbf{0}$, this gives $t_1 = \frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}$. Similarly, $t_i = \frac{\mathbf{x}\cdot\mathbf{f}_i}{\|\mathbf{f}_i\|^2}$ for each i. □

The expansion in Theorem 5.3.6 of $\mathbf{x}$ as a linear combination of the orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is called the **Fourier expansion** of $\mathbf{x}$, and the coefficients $t_1 = \frac{\mathbf{x}\cdot\mathbf{f}_i}{\|\mathbf{f}_i\|^2}$ are called the **Fourier coefficients**. Note that if $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is actually orthonormal, then $t_i = \mathbf{x}\cdot\mathbf{f}_i$ for each i. We will have a great deal more to say about this in Section **??**.

Example 5.3.7

Expand $\mathbf{x} = (a, b, c, d)$ as a linear combination of the orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ of $\mathbb{R}^4$ given in Example 5.3.6.

Solution. We have $\mathbf{f}_1 = (1, 1, 1, -1)$, $\mathbf{f}_2 = (1, 0, 1, 2)$, $\mathbf{f}_3 = (-1, 0, 1, 0)$, and $\mathbf{f}_4 = (-1, 3, -1, 1)$ so the Fourier coefficients are

$$t_1 = \frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2} = \frac{1}{4}(a+b+c+d) \qquad t_3 = \frac{\mathbf{x}\cdot\mathbf{f}_3}{\|\mathbf{f}_3\|^2} = \frac{1}{2}(-a+c)$$

$$t_2 = \frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2} = \frac{1}{6}(a+c+2d) \qquad t_4 = \frac{\mathbf{x}\cdot\mathbf{f}_4}{\|\mathbf{f}_4\|^2} = \frac{1}{12}(-a+3b-c+d)$$

The reader can verify that indeed $\mathbf{x} = t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + t_3\mathbf{f}_3 + t_4\mathbf{f}_4$.

A natural question arises here: Does every subspace U of $\mathbb{R}^n$ *have* an orthogonal basis? The answer is "yes"; in fact, there is a systematic procedure, called the Gram-Schmidt algorithm, for turning any basis of U into an orthogonal one. This leads to a definition of the projection onto a subspace U that generalizes the projection along a vector used in $\mathbb{R}^2$ and $\mathbb{R}^3$. All this is discussed in Section 8.1.

Exercises for 5.3

We often write vectors in $\mathbb{R}^n$ as row n-tuples.

Exercise 5.3.1 Obtain orthonormal bases of $\mathbb{R}^3$ by normalizing the following.

a. $\{(1, -1, 2), (0, 2, 1), (5, 1, -2)\}$

b. $\{(1, 1, 1), (4, 1, -5), (2, -3, 1)\}$

---

b.
$$\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{42}}\begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}, \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Exercise 5.3.2 In each case, show that the set of vectors is orthogonal in $\mathbb{R}^4$.

a. $\{(1, -1, 2, 5), (4, 1, 1, -1), (-7, 28, 5, 5)\}$

b. $\{(2, -1, 4, 5), (0, -1, 1, -1), (0, 3, 2, -1)\}$

Exercise 5.3.3 In each case, show that B is an orthogonal basis of $\mathbb{R}^3$ and use Theorem 5.3.6 to expand $\mathbf{x} = (a, b, c)$ as a linear combination of the basis vectors.

a. $B = \{(1, -1, 3), (-2, 1, 1), (4, 7, 1)\}$

b. $B = \{(1, 0, -1), (1, 4, 1), (2, -1, 2)\}$

c. $B = \{(1, 2, 3), (-1, -1, 1), (5, -4, 1)\}$

d. $B = \{(1,\ 1,\ 1),\ (1,\ -1,\ 0),\ (1,\ 1,\ -2)\}$

---

b. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2}(a-c)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{18}(a+4b+c)\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \frac{1}{9}(2a-b+2c)\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$

d. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{3}(a+b+c)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}(a-b)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{6}(a+b-2c)\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$

Exercise 5.3.4 In each case, write $\mathbf{x}$ as a linear combination of the orthogonal basis of the subspace U.

a. $\mathbf{x} = (13,\ -20,\ 15)$; $U = \text{span}\{(1,\ -2,\ 3),\ (-1,\ 1,\ 1)\}$

b. $\mathbf{x} = (14,\ 1,\ -8,\ 5)$;
$U = \text{span}\{(2,\ -1,\ 0,\ 3),\ (2,\ 1,\ -2,\ -1)\}$

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b. $\begin{bmatrix} 14 \\ 1 \\ -8 \\ 5 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}.$

Exercise 5.3.5 In each case, find all $(a,\ b,\ c,\ d)$ in $\mathbb{R}^4$ such that the given set is orthogonal.

a. $\{(1,\ 2,\ 1,\ 0),\ (1,\ -1,\ 1,\ 3),\ (2,\ -1,\ 0,\ -1),\ (a,\ b,\ c,\ d)\}$

b. $\{(1,\ 0,\ -1,\ 1),\ (2,\ 1,\ 1,\ -1),\ (1,\ -3,\ 1,\ 0),\ (a,\ b,\ c,\ d)\}$

---

b. $t\begin{bmatrix} -1 \\ 3 \\ 10 \\ 11 \end{bmatrix}$, in $\mathbb{R}$

Exercise 5.3.6 If $\|\mathbf{x}\| = 3$, $\|\mathbf{y}\| = 1$, and $\mathbf{x} \cdot \mathbf{y} = -2$, compute:

a) $\|3\mathbf{x} - 5\mathbf{y}\|$

b) $\|2\mathbf{x} + 7\mathbf{y}\|$

c) $(3\mathbf{x} - \mathbf{y}) \cdot (2\mathbf{y} - \mathbf{x})$

d) $(\mathbf{x} - 2\mathbf{y}) \cdot (3\mathbf{x} + 5\mathbf{y})$

---

b. $\sqrt{29}$

d. 19

Exercise 5.3.7 In each case either show that the statement is true or give an example showing that it is false.

a. Every independent set in $\mathbb{R}^n$ is orthogonal.

b. If $\{\mathbf{x},\ \mathbf{y}\}$ is an orthogonal set in $\mathbb{R}^n$, then $\{\mathbf{x},\ \mathbf{x}+\mathbf{y}\}$ is also orthogonal.

c. If $\{\mathbf{x},\ \mathbf{y}\}$ and $\{\mathbf{z},\ \mathbf{w}\}$ are both orthogonal in $\mathbb{R}^n$, then $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}\}$ is also orthogonal.

d. If $\{\mathbf{x}_1,\ \mathbf{x}_2\}$ and $\{\mathbf{y}_1,\ \mathbf{y}_2,\ \mathbf{y}_3\}$ are both orthogonal and $\mathbf{x}_i \cdot \mathbf{y}_j = 0$ for all i and j, then $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{y}_1,\ \mathbf{y}_2,\ \mathbf{y}_3\}$ is orthogonal.

e. If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_n\}$ is orthogonal in $\mathbb{R}^n$, then $\mathbb{R}^n = \text{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_n\}$.

f. If $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, then $\{\mathbf{x}\}$ is an orthogonal set.

---

b. F. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

d. T. Every $\mathbf{x}_i \cdot \mathbf{y}_j = 0$ by assumption, every $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ if $i \neq j$ because the $\mathbf{x}_i$ are orthogonal, and every $\mathbf{y}_i \cdot \mathbf{y}_j = 0$ if $i \neq j$ because the $\mathbf{y}_i$ are orthogonal. As all the vectors are nonzero, this does it.

f. T. Every pair of *distinct* vectors in the set $\{\mathbf{x}\}$ has dot product zero (there are no such pairs).

Exercise 5.3.8 Let $\mathbf{v}$ denote a nonzero vector in $\mathbb{R}^n$.

a. Show that $P = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \mathbf{x}\cdot\mathbf{v} = 0\}$ is a subspace of $\mathbb{R}^n$.

b. Show that $\mathbb{R}\mathbf{v} = \{t\mathbf{v} \mid t \text{ in } \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$.

c. Describe P and $\mathbb{R}\mathbf{v}$ geometrically when $n = 3$.

Exercise 5.3.9 If A is an $m \times n$ matrix with orthonormal columns, show that $A^TA = I_n$. [*Hint*: If $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A, show that column j of A^TA has entries $\mathbf{c}_1\cdot\mathbf{c}_j, \mathbf{c}_2\cdot\mathbf{c}_j, \dots, \mathbf{c}_n\cdot\mathbf{c}_j$].

Let $\mathbf{c}_1, \dots, \mathbf{c}_n$ be the columns of A. Then row i of A^T is $\mathbf{c}_i^T$, so the (i, j)-entry of A^TA is $\mathbf{c}_i^T\mathbf{c}_j = \mathbf{c}_i\cdot\mathbf{c}_j = 0, 1$ according as $i \neq j, i = j$. So $A^TA = I$.

Exercise 5.3.10 Use the Cauchy inequality to show that $\sqrt{xy} \leq \frac{1}{2}(x+y)$ for all $x \geq 0$ and $y \geq 0$. Here $\sqrt{xy}$ and $\frac{1}{2}(x+y)$ are called, respectively, the *geometric mean* and *arithmetic mean* of x and y. [*Hint*: Use $\mathbf{x} = \left[\begin{array}{c}\sqrt{x}\\ \sqrt{y}\end{array}\right]$ *and* $\mathbf{y} = \left[\begin{array}{c}\sqrt{y}\\ \sqrt{x}\end{array}\right]$.]

Exercise 5.3.11 Use the Cauchy inequality to prove that:

a. $r_1 + r_2 + \cdots + r_n \leq n(r_1^2 + r_2^2 + \cdots + r_n^2)$ for all r_i in $\mathbb{R}$ and all $n \geq 1$.

b. $r_1r_2 + r_1r_3 + r_2r_3 \leq r_1^2 + r_2^2 + r_3^2$ for all r_1, r_2, and r_3 in $\mathbb{R}$. [*Hint*: See part (a).]

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b. Take $n = 3$ in **(a)**, expand, and simplify.

Exercise 5.3.12

a. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal in $\mathbb{R}^n$ if and only if $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$.

b. Show that $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are orthogonal in $\mathbb{R}^n$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

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b. We have $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2$. Hence $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = 0$ if and only if $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2$; if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$—where we used the fact that $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{y}\| \geq 0$.

Exercise 5.3.13

a. Show that $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\mathbf{x}$ is orthogonal to $\mathbf{y}$.

b. If $\mathbf{x} = \left[\begin{array}{c}1\\1\end{array}\right]$, $\mathbf{y} = \left[\begin{array}{c}1\\0\end{array}\right]$ and $\mathbf{z} = \left[\begin{array}{r}-2\\3\end{array}\right]$, show that $\|\mathbf{x}+\mathbf{y}+\mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{z}\|^2$ but $\mathbf{x}\cdot\mathbf{y} \neq 0$, $\mathbf{x}\cdot\mathbf{z} \neq 0$, and $\mathbf{y}\cdot\mathbf{z} \neq 0$.

Exercise 5.3.14

a. Show that $\mathbf{x}\cdot\mathbf{y} = \frac{1}{4}[\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2]$ for all $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$.

b. Show that $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \frac{1}{2}\left[\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2\right]$ for all $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$.

Exercise 5.3.15 If A is $n \times n$, show that every eigenvalue of A^TA is nonnegative. [*Hint*: Compute $\|A\mathbf{x}\|^2$ where $\mathbf{x}$ is an eigenvector.]

If $A^TA\mathbf{x} = \lambda\mathbf{x}$, then $\|A\mathbf{x}\|^2 = (A\mathbf{x})\cdot(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda\|\mathbf{x}\|^2$.

Exercise 5.3.16 If $\mathbb{R}^n = \text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ and $\mathbf{x}\cdot\mathbf{x}_i = 0$ for all i, show that $\mathbf{x} = 0$. [*Hint*: Show $\|\mathbf{x}\| = 0$.]

Exercise 5.3.17 If $\mathbb{R}^n = \text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ and $\mathbf{x}\cdot\mathbf{x}_i = \mathbf{y}\cdot\mathbf{x}_i$ for all i, show that $\mathbf{x} = \mathbf{y}$. [*Hint*: Exercise 5.3.16]

Exercise 5.3.18 Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, show that

$$\mathbf{x}\cdot\mathbf{y} = \frac{(\mathbf{x}\cdot\mathbf{e}_1)(\mathbf{y}\cdot\mathbf{e}_1)}{\|\mathbf{e}_1\|^2} + \cdots + \frac{(\mathbf{x}\cdot\mathbf{e}_n)(\mathbf{y}\cdot\mathbf{e}_n)}{\|\mathbf{e}_n\|^2}$$

5.4 Rank of a Matrix

In this section we use the concept of dimension to clarify the definition of the rank of a matrix given in Section 1.2, and to study its properties. This requires that we deal with rows and columns in the same way. While it has been our custom to write the n-tuples in $\mathbb{R}^n$ as columns, in this section we will frequently write them as rows. Subspaces, independence, spanning, and dimension are defined for rows using matrix operations, just as for columns. If A is an $m \times n$ matrix, we define:

Definition 5.10 Column and Row Space of a Matrix

*The **column space**, $\text{col } A$, of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.*
*The **row space**, $\text{row } A$, of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.*

Much of what we do in this section involves these subspaces. We begin with:

Lemma 5.4.1

Let A and B denote $m \times n$ matrices.

1. *If $A \to B$ by elementary row operations, then $\text{row } A = \text{row } B$.*
2. *If $A \to B$ by elementary column operations, then $\text{col } A = \text{col } B$.*

Proof. We prove (1); the proof of (2) is analogous. It is enough to do it in the case when $A \to B$ by a single row operation. Let $R_1, R_2, \dots, R_m$ denote the rows of A. The row operation $A \to B$ either interchanges two rows, multiplies a row by a nonzero constant, or adds a multiple of a row to a different row. We leave the first two cases to the reader. In the last case, suppose that a times row p is added to row q where $p < q$. Then the rows of B are $R_1, \dots, R_p, \dots, R_q + aR_p, \dots, R_m$, and Theorem 5.1.1 shows that

$$\text{span}\{R_1, \dots, R_p, \dots, R_q, \dots, R_m\} = \text{span}\{R_1, \dots, R_p, \dots, R_q + aR_p, \dots, R_m\}$$

That is, $\text{row } A = \text{row } B$. □

If A is any matrix, we can carry $A \to R$ by elementary row operations where R is a row-echelon matrix. Hence $\text{row } A = \text{row } R$ by Lemma 5.4.1; so the first part of the following result is of interest.

Lemma 5.4.2

If R is a row-echelon matrix, then

1. *The nonzero rows of R are a basis of $\text{row } R$.*
2. *The columns of R containing leading ones are a basis of $\text{col } R$.*

Proof. The rows of R are independent by Example 5.2.6, and they span $\text{row } R$ by definition. This proves (1).

Let $\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}$ denote the columns of R containing leading 1s. Then $\{\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}\}$ is independent because the leading 1s are in different rows (and have zeros below and to the left of them). Let U denote the subspace of all columns in $\mathbb{R}^m$ in which the last $m-r$ entries are zero. Then $\dim U = r$ (it is just $\mathbb{R}^r$ with extra zeros). Hence the independent set $\{\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}\}$ is a basis of U by Theorem 5.2.7. Since each $\mathbf{c}_{j_i}$ is in $\text{col}\, R$, it follows that $\text{col}\, R = U$, proving (2). □

With Lemma 5.4.2 we can fill a gap in the definition of the rank of a matrix given in Chapter 1. Let A be any matrix and suppose A is carried to some row-echelon matrix R by row operations. Note that R is not unique. In Section 1.2 we defined the **rank** of A, denoted $\text{rank}\, A$, to be the number of leading 1s in R, that is the number of nonzero rows of R. The fact that this number does not depend on the choice of R was not proved in Section 1.2. However part 1 of Lemma 5.4.2 shows that

$$\text{rank}\, A = \dim(\text{row}\, A)$$

and hence that $\text{rank}\, A$ is independent of R.

Lemma 5.4.2 can be used to find bases of subspaces of $\mathbb{R}^n$ (written as rows). Here is an example.

Example 5.4.1

Find a basis of $U = \text{span}\{(1,\ 1,\ 2,\ 3),\ (2,\ 4,\ 1,\ 0),\ (1,\ 5,\ -4,\ -9)\}$.

Solution. U is the row space of $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 1 & 0 \\ 1 & 5 & -4 & -9 \end{bmatrix}$. This matrix has row-echelon form $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -\frac{3}{2} & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so $\{(1,\ 1,\ 2,\ 3),\ (0,\ 1,\ -\frac{3}{2},\ -3)\}$ is basis of U by Lemma 5.4.2.
Note that $\{(1,\ 1,\ 2,\ 3),\ (0,\ 2,\ -3,\ -6)\}$ is another basis that avoids fractions.

Lemmas 5.4.1 and 5.4.2 are enough to prove the following fundamental theorem.

Theorem 5.4.1: Rank Theorem

Let A denote any $m \times n$ matrix of rank r. Then

$$\dim(\text{col}\, A) = \dim(\text{row}\, A) = r$$

Moreover, if A is carried to a row-echelon matrix R by row operations, then

1. *The r nonzero rows of R are a basis of* $\text{row}\, A$.
2. *If the leading 1s lie in columns $j_1, j_2, \dots, j_r$ of R, then columns $j_1, j_2, \dots, j_r$ of A are a basis of* $\text{col}\, A$.

Proof. We have $\text{row}\, A = \text{row}\, R$ by Lemma 5.4.1, so (1) follows from Lemma 5.4.2. Moreover, $R = UA$ for some invertible matrix U by Theorem 2.5.1. Now write $A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{bmatrix}$ where $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A. Then

$$R = UA = U\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} U\mathbf{c}_1 & U\mathbf{c}_2 & \cdots & U\mathbf{c}_n \end{bmatrix}$$

Thus, in the notation of (2), the set $B = \{U\mathbf{c}_{j_1}, U\mathbf{c}_{j_2}, \dots, U\mathbf{c}_{j_r}\}$ is a basis of $\text{col}\, R$ by Lemma 5.4.2. So, to prove (2) and the fact that $\dim(\text{col}\, A) = r$, it is enough to show that $D = \{\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}\}$ is a basis of $\text{col}\, A$. First, D is linearly independent because U is invertible (verify), so we show that, for each j, column $\mathbf{c}_j$ is a linear combination of the $\mathbf{c}_{j_i}$. But $U\mathbf{c}_j$ is column j of R, and so is a linear combination of the $U\mathbf{c}_{j_i}$, say $U\mathbf{c}_j = a_1 U\mathbf{c}_{j_1} + a_2 U\mathbf{c}_{j_2} + \cdots + a_r U\mathbf{c}_{j_r}$ where each a_i is a real number.

Since U is invertible, it follows that $\mathbf{c}_j = a_1\mathbf{c}_{j_1} + a_2\mathbf{c}_{j_2} + \cdots + a_r\mathbf{c}_{j_r}$ and the proof is complete. □

Example 5.4.2

Compute the rank of $A = \left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{array}\right]$ and find bases for $\text{row}\, A$ and $\text{col}\, A$.

Solution. The reduction of A to row-echelon form is as follows:

$$\left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $\text{rank}\, A = 2$, and $\left\{\left[\begin{array}{cccc} 1 & 2 & 2 & -1 \end{array}\right], \left[\begin{array}{cccc} 0 & 0 & 1 & -3 \end{array}\right]\right\}$ is a basis of $\text{row}\, A$ by Lemma 5.4.2. Since the leading 1s are in columns 1 and 3 of the row-echelon matrix, Theorem 5.4.1 shows that columns 1 and 3 of A are a basis $\left\{\left[\begin{array}{c} 1 \\ 3 \\ 1 \end{array}\right], \left[\begin{array}{c} 2 \\ 5 \\ 1 \end{array}\right]\right\}$ of $\text{col}\, A$.

Theorem 5.4.1 has several important consequences. The first, Corollary 5.4.1 below, follows because the rows of A are independent (respectively span $\text{row}\, A$) if and only if their transposes are independent (respectively span $\text{col}\, A$).

Corollary 5.4.1

If A is any matrix, then $\text{rank}\, A = \text{rank}\,(A^T)$.

If A is an $m \times n$ matrix, we have $\text{col}\, A \subseteq \mathbb{R}^m$ and $\text{row}\, A \subseteq \mathbb{R}^n$. Hence Theorem 5.2.8 shows that $\dim(\text{col}\, A) \leq \dim(\mathbb{R}^m) = m$ and $\dim(\text{row}\, A) \leq \dim(\mathbb{R}^n) = n$. Thus Theorem 5.4.1 gives:

Corollary 5.4.2

If A is an $m \times n$ matrix, then $\text{rank}\, A \leq m$ and $\text{rank}\, A \leq n$.

Corollary 5.4.3

$\text{rank}\, A = \text{rank}\,(UA) = \text{rank}\,(AV)$ *whenever U and V are invertible.*

Proof. Lemma 5.4.1 gives $\text{rank } A = \text{rank } (UA)$. Using this and Corollary 5.4.1 we get

$$\text{rank } (AV) = \text{rank } (AV)^T = \text{rank } (V^T A^T) = \text{rank } (A^T) = \text{rank } A$$

□

The next corollary requires a preliminary lemma.

Lemma 5.4.3

Let A, U, and V be matrices of sizes $m \times n$, $p \times m$, and $n \times q$ respectively.

1. $\text{col } (AV) \subseteq \text{col } A$, *with equality if* $VV' = I_n$ *for some* V'.
2. $\text{row } (UA) \subseteq \text{row } A$, *with equality if* $U'U = I_m$ *for some* U'.

Proof. For (1), write $V = \begin{bmatrix} \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q \end{bmatrix}$ where $\mathbf{v}_j$ is column j of V. Then we have $AV = \begin{bmatrix} A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_q \end{bmatrix}$, and each $A\mathbf{v}_j$ is in $\text{col } A$ by Definition 2.4. It follows that $\text{col } (AV) \subseteq \text{col } A$. If $VV' = I_n$, we obtain $\text{col } A = \text{col } [(AV)V'] \subseteq \text{col } (AV)$ in the same way. This proves (1).

As to (2), we have $\text{col } \left[(UA)^T \right] = \text{col } (A^T U^T) \subseteq \text{col } (A^T)$ by (1), from which $\text{row } (UA) \subseteq \text{row } A$. If $U'U = I_m$, this is equality as in the proof of (1). □

Corollary 5.4.4

If A is $m \times n$ and B is $n \times m$, then $\text{rank } AB \leq \text{rank } A$ *and* $\text{rank } AB \leq \text{rank } B$.

Proof. By Lemma 5.4.3, $\text{col } (AB) \subseteq \text{col } A$ and $\text{row } (BA) \subseteq \text{row } A$, so Theorem 5.4.1 applies. □

In Section 5.1 we discussed two other subspaces associated with an $m \times n$ matrix A: the null space $\text{null } (A)$ and the image space $\text{im } (A)$

$$\text{null } (A) = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \text{ and } \text{im } (A) = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Using rank, there are simple ways to find bases of these spaces. If A has rank r, we have $\text{im } (A) = \text{col } (A)$ by Example 5.1.8, so $\dim [\text{im } (A)] = \dim [\text{col } (A)] = r$. Hence Theorem 5.4.1 provides a method of finding a basis of $\text{im } (A)$. This is recorded as part (2) of the following theorem.

Theorem 5.4.2

Let A denote an $m \times n$ matrix of rank r. Then

1. *The $n - r$ basic solutions to the system $A\mathbf{x} = \mathbf{0}$ provided by the gaussian algorithm are a basis of* $\text{null } (A)$, *so* $\dim [\text{null } (A)] = n - r$.
2. *Theorem 5.4.1 provides a basis of* $\text{im } (A) = \text{col } (A)$, *and* $\dim [\text{im } (A)] = r$.

Proof. It remains to prove (1). We already know (Theorem 2.2.1) that $\text{null } (A)$ is spanned by the $n - r$ basic solutions of $A\mathbf{x} = \mathbf{0}$. Hence using Theorem 5.2.7, it suffices to show that $\dim [\text{null } (A)] = n - r$. So let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis of $\text{null } (A)$, and extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n\}$

of $\mathbb{R}^n$ (by Theorem 5.2.6). It is enough to show that $\{A\mathbf{x}_{k+1}, \dots, A\mathbf{x}_n\}$ is a basis of $\mathbf{im}(A)$; then $n-k=r$ by the above and so $k=n-r$ as required.

Spanning. Choose $A\mathbf{x}$ in $\mathbf{im}(A)$, $\mathbf{x}$ in $\mathbb{R}^n$, and write $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + a_{k+1}\mathbf{x}_{k+1} + \cdots + a_n\mathbf{x}_n$ where the a_i are in $\mathbb{R}$. Then $A\mathbf{x} = a_{k+1}A\mathbf{x}_{k+1} + \cdots + a_nA\mathbf{x}_n$ because $\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \subseteq \mathbf{null}(A)$.

Independence. Let $t_{k+1}A\mathbf{x}_{k+1} + \cdots + t_nA\mathbf{x}_n = \mathbf{0}$, t_i in $\mathbb{R}$. Then $t_{k+1}\mathbf{x}_{k+1} + \cdots + t_n\mathbf{x}_n$ is in $\mathbf{null}\, A$, so $t_{k+1}\mathbf{x}_{k+1} + \cdots + t_n\mathbf{x}_n = t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k$ for some $t_1, \dots, t_k$ in $\mathbb{R}$. But then the independence of the $\mathbf{x}_i$ shows that $t_i = 0$ for every i. □

Example 5.4.3

If $A = \left[\begin{array}{rrrr} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{array} \right]$, find bases of $\mathbf{null}(A)$ and $\mathbf{im}(A)$, and so find their dimensions.

Solution. If $\mathbf{x}$ is in $\mathbf{null}(A)$, then $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x}$ is given by solving the system $A\mathbf{x} = \mathbf{0}$. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 \end{array} \right] \to \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Hence $r = \mathbf{rank}(A) = 2$. Here, $\mathbf{im}(A) = \mathbf{col}(A)$ has basis $\left\{ \left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array} \right], \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array} \right] \right\}$ by Theorem 5.4.1 because the leading 1s are in columns 1 and 3. In particular, $\dim[\mathbf{im}(A)] = 2 = r$ as in Theorem 5.4.2.

Turning to $\mathbf{null}(A)$, we use gaussian elimination. The leading variables are x_1 and x_3, so the nonleading variables become parameters: $x_2 = s$ and $x_4 = t$. It follows from the reduced matrix that $x_1 = 2s + t$ and $x_3 = -2t$, so the general solution is

$$\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] = \left[\begin{array}{c} 2s+t \\ s \\ -2t \\ t \end{array} \right] = s\mathbf{x}_1 + t\mathbf{x}_2 \text{ where } \mathbf{x}_1 = \left[\begin{array}{r} 2 \\ 1 \\ 0 \\ 0 \end{array} \right], \text{ and } \mathbf{x}_2 = \left[\begin{array}{r} 1 \\ 0 \\ -2 \\ 1 \end{array} \right].$$

Hence $\mathbf{null}(A)$. But $\mathbf{x}_1$ and $\mathbf{x}_2$ *are* solutions (basic), so

$$\mathbf{null}(A) = \mathbf{span}\{\mathbf{x}_1, \mathbf{x}_2\}$$

However Theorem 5.4.2 asserts that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a *basis* of $\mathbf{null}(A)$. (In fact it is easy to verify directly that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is independent in this case.) In particular, $\dim[\mathbf{null}(A)] = 2 = n - r$, as Theorem 5.4.2 asserts.

Let A be an $m \times n$ matrix. Corollary 5.4.2 of Theorem 5.4.1 asserts that $\mathbf{rank}\, A \leq m$ and $\mathbf{rank}\, A \leq n$, and it is natural to ask when these extreme cases arise. If $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A, Theorem 5.2.2 shows that $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ spans $\mathbb{R}^m$ if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$, and that $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ is independent if and only if $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, implies $\mathbf{x} = \mathbf{0}$. The next two useful theorems improve on both these results, and relate them to when the rank of A is n or m.

Theorem 5.4.3

The following are equivalent for an $m \times n$ matrix A:

1. $\text{rank}\, A = n$.
2. *The rows of A span $\mathbb{R}^n$.*
3. *The columns of A are linearly independent in $\mathbb{R}^m$.*
4. *The $n \times n$ matrix A^TA is invertible.*
5. *$CA = I_n$ for some $n \times m$ matrix C.*
6. *If $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, then $\mathbf{x} = \mathbf{0}$.*

Proof. (1) $\Rightarrow$ (2). We have $\text{row}\, A \subseteq \mathbb{R}^n$, and $\dim(\text{row}\, A) = n$ by (1), so $\text{row}\, A = \mathbb{R}^n$ by Theorem 5.2.8. This is (2).

(2) $\Rightarrow$ (3). By (2), $\text{row}\, A = \mathbb{R}^n$, so $\text{rank}\, A = n$. This means $\dim(\text{col}\, A) = n$. Since the n columns of A span $\text{col}\, A$, they are independent by Theorem 5.2.7.

(3) $\Rightarrow$ (4). If $(A^TA)\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, we show that $\mathbf{x} = \mathbf{0}$ (Theorem 2.4.5). We have

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T \mathbf{0} = \mathbf{0}$$

Hence $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ by (3) and Theorem 5.2.2.

(4) $\Rightarrow$ (5). Given (4), take $C = (A^TA)^{-1}A^T$.

(5) $\Rightarrow$ (6). If $A\mathbf{x} = \mathbf{0}$, then left multiplication by C (from (5)) gives $\mathbf{x} = \mathbf{0}$.

(6) $\Rightarrow$ (1). Given (6), the columns of A are independent by Theorem 5.2.2. Hence $\dim(\text{col}\, A) = n$, and (1) follows. □

Theorem 5.4.4

The following are equivalent for an $m \times n$ matrix A:

1. $\text{rank}\, A = m$.
2. *The columns of A span $\mathbb{R}^m$.*
3. *The rows of A are linearly independent in $\mathbb{R}^n$.*
4. *The $m \times m$ matrix AA^T is invertible.*
5. *$AC = I_m$ for some $n \times m$ matrix C.*
6. *The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.*

Proof. (1) $\Rightarrow$ (2). By (1), $\dim(\text{col}\, A = m$, so $\text{col}\, A = \mathbb{R}^m$ by Theorem 5.2.8.

(2) $\Rightarrow$ (3). By (2), $\text{col}\, A = \mathbb{R}^m$, so $\text{rank}\, A = m$. This means $\dim(\text{row}\, A) = m$. Since the m rows of A span $\text{row}\, A$, they are independent by Theorem 5.2.7.

(3) $\Rightarrow$ (4). We have $\text{rank}\, A = m$ by (3), so the $n \times m$ matrix A^T has rank m. Hence applying Theorem 5.4.3 to A^T in place of A shows that $(A^T)^T A^T$ is invertible, proving (4).

(4) $\Rightarrow$ (5). Given (4), take $C = A^T(AA_T)^{-1}$ in (5).

(5) $\Rightarrow$ (6). Comparing columns in $AC = I_m$ gives $A\mathbf{c}_j = \mathbf{e}_j$ for each j, where $\mathbf{c}_j$ and $\mathbf{e}_j$ denote column j of C and I_m respectively. Given $\mathbf{b}$ in $\mathbb{R}^m$, write $\mathbf{b} = \sum_{j=1}^m r_j\mathbf{e}_j$, r_j in $\mathbb{R}$. Then $A\mathbf{x} = \mathbf{b}$ holds with $\mathbf{x} = \sum_{j=1}^m r_j\mathbf{c}_j$ as the reader can verify.

(6) $\Rightarrow$ (1). Given (6), the columns of A span $\mathbb{R}^m$ by Theorem 5.2.2. Thus $\text{col}\, A = \mathbb{R}^m$ and (1) follows. □

Example 5.4.4

Show that $\begin{bmatrix} 3 & x+y+z \\ x+y+z & x^2+y^2+z^2 \end{bmatrix}$ is invertible if x, y, and z are not all equal.

Solution. The given matrix has the form A^TA where $A = \begin{bmatrix} 1 & x \\ 1 & y \\ 1 & z \end{bmatrix}$ has independent columns because x, y, and z are not all equal (verify). Hence Theorem 5.4.3 applies.

Theorem 5.4.3 and Theorem 5.4.4 relate several important properties of an $m \times n$ matrix A to the invertibility of the square, symmetric matrices A^TA and AA^T. In fact, even if the columns of A are not independent or do not span $\mathbb{R}^m$, the matrices A^TA and AA^T are both symmetric and, as such, have real eigenvalues as we shall see. We return to this in Chapter 7.

Exercises for 5.4

Exercise 5.4.1 In each case find bases for the row and column spaces of A and determine the rank of A.

a) $\begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ -3 & -6 & 3 & -2 \end{bmatrix}$

---

b.

$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}; 2$

d. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}; 2$

Exercise 5.4.2 In each case find a basis of the subspace U.

a. $U = \text{span}\{(1, -1, 0, 3), (2, 1, 5, 1), (4, -2, 5, 7)\}$

b. $U = \text{span}\{(1, -1, 2, 5, 1), (3, 1, 4, 2, 7), (1, 1, 0, 0, 0), (5, 1, 6, 7, 8)\}$

c. $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

d.
$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -10 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \right\}$

---

b. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 6 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Exercise 5.4.3

a. Can a 3×4 matrix have independent columns? Independent rows? Explain.

b. If A is 4×3 and $\text{rank}\, A = 2$, can A have independent columns? Independent rows? Explain.

c. If A is an $m \times n$ matrix and $\text{rank}\, A = m$, show that $m \leq n$.

d. Can a nonsquare matrix have its rows independent and its columns independent? Explain.

e. Can the null space of a 3×6 matrix have dimension 2? Explain.

f. Suppose that A is 5×4 and $\text{null}\,(A) = \mathbb{R}\mathbf{x}$ for some column $\mathbf{x} \neq \mathbf{0}$. Can $\dim\,(\text{im}\, A) = 2$?

---

b. No; no

d. No

f. Otherwise, if A is $m \times n$, we have $m = \dim\,(\text{row}\, A) = \text{rank}\, A = \dim\,(\text{col}\, A) = n$

Exercise 5.4.4 If A is $m \times n$ show that

$$\text{col}\,(A) = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Let $A = \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_n \end{bmatrix}$. Then $\text{col}\, A = \text{span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\} = \{x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n \mid x_i \text{ in } \mathbb{R}\} = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$.

Exercise 5.4.5 If A is $m \times n$ and B is $n \times m$, show that $AB = 0$ if and only if $\text{col}\, B \subseteq \text{null}\, A$.

Exercise 5.4.6 Show that the rank does not change when an elementary row or column operation is performed on a matrix.

Exercise 5.4.7 In each case find a basis of the null space of A. Then compute $\text{rank}\, A$ and verify (1) of Theorem 5.4.2.

a. $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & 5 & 5 & 2 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -4 & -2 \end{bmatrix}$

---

b. The basis is $\left\{ \begin{bmatrix} 6 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ so the dimension is 2. Have $\text{rank}\, A = 3$ and $n - 3 = 2$.

Exercise 5.4.8 Let $A = \mathbf{c}\mathbf{r}$ where $\mathbf{c} \neq \mathbf{0}$ is a column in $\mathbb{R}^m$ and $\mathbf{r} \neq \mathbf{0}$ is a row in $\mathbb{R}^n$.

a. Show that $\text{col}\, A = \text{span}\,\{\mathbf{c}\}$ and $\text{row}\, A = \text{span}\,\{\mathbf{r}\}$.

b. Find $\dim\,(\text{null}\, A)$.

c. Show that $\text{null}\, A = \text{null}\, \mathbf{r}$.

---

b. $n-1$

Exercise 5.4.9 Let A be $m \times n$ with columns $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n$.

a. If $\{\mathbf{c}_1,\ \dots,\ \mathbf{c}_n\}$ is independent, show $\text{null}\, A = \{\mathbf{0}\}$.

b. If $\text{null}\, A = \{\mathbf{0}\}$, show that $\{\mathbf{c}_1,\ \dots,\ \mathbf{c}_n\}$ is independent.

---

b. If $r_1\mathbf{c}_1 + \cdots + r_n\mathbf{c}_n = \mathbf{0}$, let $\mathbf{x} = [r_1,\ \dots,\ r_n]^T$. Then $C\mathbf{x} = r_1\mathbf{c}_1 + \cdots + r_n\mathbf{c}_n = \mathbf{0}$, so $\mathbf{x}$ is in $\text{null}\, A = 0$. Hence each $r_i = 0$.

Exercise 5.4.10 Let A be an $n \times n$ matrix.

a. Show that $A^2 = 0$ if and only if $\text{col}\, A \subseteq \text{null}\, A$.

b. Conclude that if $A^2 = 0$, then $\text{rank}\, A \leq \frac{n}{2}$.

c. Find a matrix A for which $\text{col}\, A = \text{null}\, A$.

---

b. Write $r = \text{rank}\, A$. Then **(a)** gives $r = \dim\,(\text{col}\, A \leq \dim\,(\text{null}\, A) = n - r$.

Exercise 5.4.11 Let B be $m \times n$ and let AB be $k \times n$. If $\text{rank}\, B = \text{rank}\,(AB)$, show that $\text{null}\, B = \text{null}\,(AB)$. [*Hint*: Theorem 5.4.1.]

Exercise 5.4.12 Give a careful argument why $\text{rank}\,(A^T) = \text{rank}\, A$.

We have $\text{rank}\,(A) = \dim\,[\text{col}\,(A)]$ and $\text{rank}\,(A^T) = \dim\,[\text{row}\,(A^T)]$. Let $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_k\}$ be a basis of $\text{col}\,(A)$; it suffices to show that $\{\mathbf{c}_1^T,\ \mathbf{c}_2^T,\ \dots,\ \mathbf{c}_k^T\}$ is a basis of $\text{row}\,(A^T)$. But if $t_1\mathbf{c}_1^T + t_2\mathbf{c}_2^T + \cdots + t_k\mathbf{c}_k^T = \mathbf{0}$, t_j in $\mathbb{R}$, then (taking transposes) $t_1\mathbf{c}_1 + t_2\mathbf{c}_2 + \cdots + t_k\mathbf{c}_k = \mathbf{0}$ so each $t_j = 0$. Hence $\{\mathbf{c}_1^T,\ \mathbf{c}_2^T,\ \dots,\ \mathbf{c}_k^T\}$ is independent. Given $\mathbf{v}$ in $\text{row}\,(A^T)$ then $\mathbf{v}^T$ is in $\text{col}\,(A)$; say $\mathbf{v}^T = s_1\mathbf{c}_1 + s_2\mathbf{c}_2 + \cdots + s_k\mathbf{c}_k$, s_j in $\mathbb{R}$: Hence $\mathbf{v} = s_1\mathbf{c}_1^T + s_2\mathbf{c}_2^T + \cdots + s_k\mathbf{c}_k^T$, so $\{\mathbf{c}_1^T,\ \mathbf{c}_2^T,\ \dots,\ \mathbf{c}_k^T\}$ spans $\text{row}\,(A^T)$, as required.

Exercise 5.4.13 Let A be an $m \times n$ matrix with columns $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n$. If $\text{rank}\, A = n$, show that $\{A^T\mathbf{c}_1,\ A^T\mathbf{c}_2,\ \dots,\ A^T\mathbf{c}_n\}$ is a basis of $\mathbb{R}^n$.

Exercise 5.4.14 If A is $m \times n$ and $\mathbf{b}$ is $m \times 1$, show that $\mathbf{b}$ lies in the column space of A if and only if $\text{rank}\,[A\ \mathbf{b}] = \text{rank}\, A$.

Exercise 5.4.15

a. Show that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\text{rank}\, A = \text{rank}\,[A\ \mathbf{b}]$. [*Hint*: Exercises 5.4.12 and 5.4.14.]

b. If $A\mathbf{x} = \mathbf{b}$ has no solution, show that $\text{rank}\,[A\ \mathbf{b}] = 1 + \text{rank}\, A$.

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b. Let $\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_r\}$ be a basis of $\text{col}\,(A)$. Then $\mathbf{b}$ is *not* in $\text{col}\,(A)$, so $\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_r,\ \mathbf{b}\}$ is linearly independent. Show that $\text{col}\,[A\ \mathbf{b}] = \text{span}\,\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_r,\ \mathbf{b}\}$.

Exercise 5.4.16 Let X be a $k \times m$ matrix. If I is the $m \times m$ identity matrix, show that $I + X^TX$ is invertible. [*Hint*: $I + X^TX = A^TA$ where $A = \left[\begin{array}{c} I \\ X \end{array}\right]$ in block form.]

Exercise 5.4.17 If A is $m \times n$ of rank r, show that A can be factored as $A = PQ$ where P is $m \times r$ with r independent columns, and Q is $r \times n$ with r independent rows. [*Hint*: Let $UAV = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]$ by Theorem 2.5.3, and write $U^{-1} = \left[\begin{array}{cc} U_1 & U_2 \\ U_3 & U_4 \end{array}\right]$ and $V^{-1} = \left[\begin{array}{cc} V_1 & V_2 \\ V_3 & V_4 \end{array}\right]$ in block form, where U_1 and V_1 are $r \times r$.]

Exercise 5.4.18

a. Show that if A and B have independent columns, so does AB.

b. Show that if A and B have independent rows, so does AB.

Exercise 5.4.19 A matrix obtained from A by deleting rows and columns is called a **submatrix** of A. If A has an invertible $k \times k$ submatrix, show that $\text{rank} A \geq k$. [*Hint*: Show that row and column operations carry $A \to \left[\begin{array}{cc} I_k & P \\ 0 & Q \end{array} \right]$ in block form.] *Remark*: It can be shown that $\text{rank} A$ is the largest integer r such that A has an invertible $r \times r$ submatrix.

5.5 Similarity and Diagonalization

In Section 3.3 we studied diagonalization of a square matrix A, and found important applications (for example to linear dynamical systems). We can now utilize the concepts of subspace, basis, and dimension to clarify the diagonalization process, reveal some new results, and prove some theorems which could not be demonstrated in Section 3.3.

Before proceeding, we introduce a notion that simplifies the discussion of diagonalization, and is used throughout the book.

Similar Matrices

Definition 5.11 Similar Matrices

*If A and B are $n \times n$ matrices, we say that A and B are **similar**, and write $A \sim B$, if $B = P^{-1}AP$ for some invertible matrix P.*

Note that $A \sim B$ if and only if $B = QAQ^{-1}$ where Q is invertible (write $P^{-1} = Q$). The language of similarity is used throughout linear algebra. For example, a matrix A is diagonalizable if and only if it is similar to a diagonal matrix.

If $A \sim B$, then necessarily $B \sim A$. To see why, suppose that $B = P^{-1}AP$. Then $A = PBP^{-1} = Q^{-1}BQ$ where $Q = P^{-1}$ is invertible. This proves the second of the following properties of similarity (the others are left as an exercise):

1. $A \sim A$ *for all square matrices* A.
2. *If* $A \sim B$, *then* $B \sim A$. (5.2)
3. *If* $A \sim B$ *and* $B \sim A$, *then* $A \sim C$.

These properties are often expressed by saying that the similarity relation $\sim$ is an **equivalence relation** on the set of $n \times n$ matrices. Here is an example showing how these properties are used.

Example 5.5.1

If A is similar to B and either A or B is diagonalizable, show that the other is also diagonalizable.

Solution. We have $A \sim B$. Suppose that A is diagonalizable, say $A \sim D$ where D is diagonal. Since $B \sim A$ by (2) of (5.2), we have $B \sim A$ and $A \sim D$. Hence $B \sim D$ by (3) of (5.2), so B is diagonalizable too. An analogous argument works if we assume instead that B is diagonalizable.

Similarity is compatible with inverses, transposes, and powers:

$$\text{If } A \sim B \text{ then} \quad A^{-1} \sim B^{-1}, \quad A^T \sim B^T, \quad \text{and} \quad A^k \sim B^k \text{ for all integers } k \geq 1.$$

The proofs are routine matrix computations using Theorem 3.3.1. Thus, for example, if A is diagonalizable, so also are A^T, A^{-1} (if it exists), and A^k (for each $k \geq 1$). Indeed, if $A \sim D$ where D is a diagonal matrix, we obtain $A^T \sim D^T$, $A^{-1} \sim D^{-1}$, and $A^k \sim D^k$, and each of the matrices D^T, D^{-1}, and D^k is diagonal.

We pause to introduce a simple matrix function that will be referred to later.

Definition 5.12 Trace of a Matrix

The ***trace*** $\text{tr}\, A$ *of an* $n \times n$ *matrix* A *is defined to be the sum of the main diagonal elements of* A.

In other words:

$$\text{If } A = \left[a_{ij}\right], \quad \text{then } \text{tr}\, A = a_{11} + a_{22} + \cdots + a_{nn}.$$

It is evident that $\text{tr}\,(A+B) = \text{tr}\, A + \text{tr}\, B$ and that $\text{tr}\,(cA) = c\, \text{tr}\, A$ holds for all $n \times n$ matrices A and B and all scalars c. The following fact is more surprising.

Lemma 5.5.1

Let A *and* B *be* $n \times n$ *matrices. Then* $\text{tr}\,(AB) = \text{tr}\,(BA)$.

Proof. Write $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$. For each i, the $(i,\ i)$-entry d_i of the matrix AB is given as follows: $d_i = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} = \sum_j a_{ij}b_{ji}$. Hence

$$\text{tr}\,(AB) = d_1 + d_2 + \cdots + d_n = \sum_i d_i = \sum_i \left(\sum_j a_{ij}b_{ji} \right)$$

Similarly we have $\text{tr}\,(BA) = \sum_i (\sum_j b_{ij}a_{ji})$. Since these two double sums are the same, Lemma 5.5.1 is proved. □

As the name indicates, similar matrices share many properties, some of which are collected in the next theorem for reference.

Theorem 5.5.1

If A *and* B *are similar* $n \times n$ *matrices, then* A *and* B *have the same determinant, rank, trace, characteristic polynomial, and eigenvalues.*

Proof. Let $B = P^{-1}AP$ for some invertible matrix P. Then we have

$$\det B = \det\,(P^{-1}) \det A \det P = \det A \text{ because } \det\,(P^{-1}) = 1/\det P$$

Similarly, $\text{rank}\, B = \text{rank}\,(P^{-1}AP) = \text{rank}\, A$ by Corollary 5.4.3. Next Lemma 5.5.1 gives

$$\text{tr}\,(P^{-1}AP) = \text{tr}\, \left[P^{-1}(AP)\right] = \text{tr}\, \left[(AP)P^{-1}\right] = \text{tr}\, A$$

As to the characteristic polynomial,

$$\begin{aligned} c_B(x) = \det(xI - B) &= \det\{x(P^{-1}IP) - P^{-1}AP\} \\ &= \det\{P^{-1}(xI - A)P\} \\ &= \det(xI - A) \\ &= c_A(x) \end{aligned}$$

Finally, this shows that A and B have the same eigenvalues because the eigenvalues of a matrix are the roots of its characteristic polynomial. □

Example 5.5.2

Sharing the five properties in Theorem 5.5.1 does not guarantee that two matrices are similar. The matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have the same determinant, rank, trace, characteristic polynomial, and eigenvalues, but they are not similar because $P^{-1}IP = I$ for any invertible matrix P.

Diagonalization Revisited

Recall that a square matrix A is **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix, that is if A is similar to a diagonal matrix D. Unfortunately, not all matrices are diagonalizable, for example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (see Example 3.3.10). Determining whether A is diagonalizable is closely related to the eigenvalues and eigenvectors of A. Recall that a number λ is called an **eigenvalue** of A if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero column $\mathbf{x}$ in $\mathbb{R}^n$, and any such nonzero vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ (or simply a λ-eigenvector of A). The eigenvalues and eigenvectors of A are closely related to the **characteristic polynomial** $c_A(x)$ of A, defined by

$$c_A(x) = \det(xI - A)$$

If A is $n \times n$ this is a polynomial of degree n, and its relationship to the eigenvalues is given in the following theorem (a repeat of Theorem 3.3.2).

Theorem 5.5.2

Let A be an $n \times n$ matrix.

1. *The eigenvalues λ of A are the roots of the characteristic polynomial $c_A(x)$ of A.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nonzero solutions to the homogeneous system*

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations with $\lambda I - A$ as coefficient matrix.

Example 5.5.3

Show that the eigenvalues of a triangular matrix are the main diagonal entries.

Solution. Assume that A is triangular. Then the matrix $xI - A$ is also triangular and has diagonal entries $(x - a_{11})$, $(x - a_{22})$, $\dots$, $(x - a_{nn})$ where $A = \left[a_{ij}\right]$. Hence Theorem 3.1.4 gives

$$c_A(x) = (x - a_{11})(x - a_{22}) \cdots (x - a_{nn})$$

and the result follows because the eigenvalues are the roots of $c_A(x)$.

Theorem 3.3.4 asserts (in part) that an $n \times n$ matrix A is diagonalizable if and only if it has n eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ such that the matrix $P = \left[\begin{array}{ccc} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{array} \right]$ with the $\mathbf{x}_i$ as columns is invertible. This is equivalent to requiring that $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. Hence we can restate Theorem 3.3.4 as follows:

Theorem 5.5.3

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if $\mathbb{R}^n$ has a basis $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ consisting of eigenvectors of A.*
2. *When this is the case, the matrix $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right]$ is invertible and $P^{-1}AP = \text{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$ where, for each i, λ_i is the eigenvalue of A corresponding to $\mathbf{x}_i$.*

The next result is a basic tool for determining when a matrix is diagonalizable. It reveals an important connection between eigenvalues and linear independence: Eigenvectors corresponding to distinct eigenvalues are necessarily linearly independent.

Theorem 5.5.4

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ of an $n \times n$ matrix A. Then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a linearly independent set.

Proof. We use induction on k. If $k = 1$, then $\{\mathbf{x}_1\}$ is independent because $\mathbf{x}_1 \neq \mathbf{0}$. In general, suppose the theorem is true for some $k \geq 1$. Given eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k+1}\}$, suppose a linear combination vanishes:

$$t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_{k+1}\mathbf{x}_{k+1} = \mathbf{0} \tag{5.3}$$

We must show that each $t_i = 0$. Left multiply (5.3) by A and use the fact that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ to get

$$t_1\lambda_1\mathbf{x}_1 + t_2\lambda_2\mathbf{x}_2 + \cdots + t_{k+1}\lambda_{k+1}\mathbf{x}_{k+1} = \mathbf{0} \tag{5.4}$$

If we multiply (5.3) by λ_1 and subtract the result from (5.4), the first terms cancel and we obtain

$$t_2(\lambda_2 - \lambda_1)\mathbf{x}_2 + t_3(\lambda_3 - \lambda_1)\mathbf{x}_3 + \cdots + t_{k+1}(\lambda_{k+1} - \lambda_1)\mathbf{x}_{k+1} = \mathbf{0}$$

Since $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{k+1}$ correspond to distinct eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_{k+1}$, the set $\{\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{k+1}\}$ is independent by the induction hypothesis. Hence,

$$t_2(\lambda_2 - \lambda_1) = 0, \quad t_3(\lambda_3 - \lambda_1) = 0, \quad \ldots, \quad t_{k+1}(\lambda_{k+1} - \lambda_1) = 0$$

and so $t_2 = t_3 = \cdots = t_{k+1} = 0$ because the λ_i are distinct. Hence (5.3) becomes $t_1\mathbf{x}_1 = \mathbf{0}$, which implies that $t_1 = 0$ because $\mathbf{x}_1 \neq \mathbf{0}$. This is what we wanted. □

Theorem 5.5.4 will be applied several times; we begin by using it to give a useful condition for when a matrix is diagonalizable.

Theorem 5.5.5

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Proof. Choose one eigenvector for each of the n distinct eigenvalues. Then these eigenvectors are independent by Theorem 5.5.4, and so are a basis of $\mathbb{R}^n$ by Theorem 5.2.7. Now use Theorem 5.5.3. □

Example 5.5.4

Show that $A = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{array} \right]$ is diagonalizable.

Solution. A routine computation shows that $c_A(x) = (x-1)(x-3)(x+1)$ and so has distinct eigenvalues 1, 3, and -1. Hence Theorem 5.5.5 applies.

However, a matrix can have multiple eigenvalues as we saw in Section 3.3. To deal with this situation, we prove an important lemma which formalizes a technique that is basic to diagonalization, and which will be used three times below.

Lemma 5.5.2

Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *be a linearly independent set of eigenvectors of an* $n \times n$ *matrix* A*, extend it to a basis* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \dots, \mathbf{x}_n\}$ *of* $\mathbb{R}^n$*, and let*

$$P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right]$$

be the (invertible) $n \times n$ *matrix with the* $\mathbf{x}_i$ *as its columns. If* $\lambda_1, \lambda_2, \dots, \lambda_k$ *are the (not necessarily distinct) eigenvalues of* A *corresponding to* $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ *respectively, then* $P^{-1}AP$ *has block form*

$$P^{-1}AP = \left[\begin{array}{cc} \text{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_k) & B \\ 0 & A_1 \end{array} \right]$$

where B *has size* $k \times (n-k)$ *and* A_1 *has size* $(n-k) \times (n-k)$*.*

Proof. If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, then

$$\begin{aligned} \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{array} \right] = I_n = P^{-1}P = P^{-1} \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right] \\ = \left[\begin{array}{cccc} P^{-1}\mathbf{x}_1 & P^{-1}\mathbf{x}_2 & \cdots & P^{-1}\mathbf{x}_n \end{array} \right] \end{aligned}$$

Comparing columns, we have $P^{-1}\mathbf{x}_i = \mathbf{e}_i$ for each $1 \leq i \leq n$. On the other hand, observe that

$$P^{-1}AP = P^{-1}A \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right] = \left[\begin{array}{cccc} (P^{-1}A)\mathbf{x}_1 & (P^{-1}A)\mathbf{x}_2 & \cdots & (P^{-1}A)\mathbf{x}_n \end{array} \right]$$

Hence, if $1 \leq i \leq k$, column i of $P^{-1}AP$ is

$$(P^{-1}A)\mathbf{x}_i = P^{-1}(\lambda_i \mathbf{x}_i) = \lambda_i(P^{-1}\mathbf{x}_i) = \lambda_i \mathbf{e}_i$$

This describes the first k columns of $P^{-1}AP$, and Lemma 5.5.2 follows. □

Note that Lemma 5.5.2 (with $k = n$) shows that an $n \times n$ matrix A is diagonalizable if $\mathbb{R}^n$ has a basis of eigenvectors of A, as in (1) of Theorem 5.5.3.

Definition 5.13 Eigenspace of a Matrix

If λ *is an eigenvalue of an* $n \times n$ *matrix* A*, define the* ***eigenspace*** *of* A *corresponding to* λ *by*

$$E_\lambda(A) = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \lambda \mathbf{x}\}$$

This is a subspace of $\mathbb{R}^n$ and the eigenvectors corresponding to λ are just the nonzero vectors in $E_\lambda(A)$. In fact $E_\lambda(A)$ is the null space of the matrix $(\lambda I - A)$:

$$E_\lambda(A) = \{\mathbf{x} \mid (\lambda I - A)\mathbf{x} = \mathbf{0}\} = \text{null}\,(\lambda I - A)$$

Hence, by Theorem 5.4.2, the basic solutions of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ given by the gaussian algorithm form a basis for $E_\lambda(A)$. In particular

$$\dim E_\lambda(A) \text{ is the number of basic solutions } \mathbf{x} \text{ of } (\lambda I - A)\mathbf{x} = \mathbf{0} \tag{5.5}$$

Now recall (Definition 3.7) that the **multiplicity**[11] of an eigenvalue λ of A is the number of times λ occurs as a root of the characteristic polynomial $c_A(x)$ of A. In other words, the multiplicity of λ is the largest integer $m \geq 1$ such that

$$c_A(x) = (x-\lambda)^m g(x)$$

for some polynomial $g(x)$. Because of (5.5), the assertion (without proof) in Theorem 3.3.5 can be stated as follows: A square matrix is diagonalizable if and only if the multiplicity of each eigenvalue λ equals $\dim[E_\lambda(A)]$. We are going to prove this, and the proof requires the following result which is valid for *any* square matrix, diagonalizable or not.

Lemma 5.5.3

Let λ be an eigenvalue of multiplicity m of a square matrix A. Then $\dim[E_\lambda(A)] \leq m$.

Proof. Write $\dim[E_\lambda(A)] = d$. It suffices to show that $c_A(x) = (x-\lambda)^d g(x)$ for some polynomial $g(x)$, because m is the highest power of $(x-\lambda)$ that divides $c_A(x)$. To this end, let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\}$ be a basis of $E_\lambda(A)$. Then Lemma 5.5.2 shows that an invertible $n\times n$ matrix P exists such that

$$P^{-1}AP = \left[\begin{array}{cc} \lambda I_d & B \\ 0 & A_1 \end{array}\right]$$

in block form, where I_d denotes the $d\times d$ identity matrix. Now write $A' = P^{-1}AP$ and observe that $c_{A'}(x) = c_A(x)$ by Theorem 5.5.1. But Theorem 3.1.5 gives

$$\begin{aligned} c_A(x) = c_{A'}(x) = \det(xI_n - A') &= \det\left[\begin{array}{cc} (x-\lambda)I_d & -B \\ 0 & xI_{n-d}-A_1 \end{array}\right] \\ &= \det[(x-\lambda)I_d]\det[(xI_{n-d}-A_1)] \\ &= (x-\lambda)^d g(x) \end{aligned}$$

where $g(x) = cA_1(x)$. This is what we wanted. □

It is impossible to ignore the question when equality holds in Lemma 5.5.3 for each eigenvalue λ. It turns out that this characterizes the diagonalizable $n\times n$ matrices A for which $c_A(x)$ **factors completely** over $\mathbb{R}$. By this we mean that $c_A(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$, where the λ_i are *real* numbers (not necessarily distinct); in other words, every eigenvalue of A is real. This need not happen (consider $A = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$), and we investigate the general case below.

Theorem 5.5.6

The following are equivalent for a square matrix A for which $c_A(x)$ factors completely.

1. *A is diagonalizable.*
2. *$\dim[E_\lambda(A)]$ equals the multiplicity of λ for every eigenvalue λ of the matrix A.*

[11] This is often called the *algebraic* multiplicity of λ.

Proof. Let A be $n \times n$ and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of A. For each i, let m_i denote the multiplicity of λ_i and write $d_i = \dim \left[E_{\lambda_i}(A)\right]$. Then

$$c_A(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\dots(x-\lambda_k)^{m_k}$$

so $m_1 + \dots + m_k = n$ because $c_A(x)$ has degree n. Moreover, $d_i \le m_i$ for each i by Lemma 5.5.3.

(1) $\Rightarrow$ (2). By (1), $\mathbb{R}^n$ has a basis of n eigenvectors of A, so let t_i of them lie in $E_{\lambda_i}(A)$ for each i. Since the subspace spanned by these t_i eigenvectors has dimension t_i, we have $t_i \le d_i$ for each i by Theorem 5.2.4. Hence

$$n = t_1 + \dots + t_k \le d_1 + \dots + d_k \le m_1 + \dots + m_k = n$$

It follows that $d_1 + \dots + d_k = m_1 + \dots + m_k$ so, since $d_i \le m_i$ for each i, we must have $d_i = m_i$. This is (2).

(2) $\Rightarrow$ (1). Let B_i denote a basis of $E_{\lambda_i}(A)$ for each i, and let $B = B_1 \cup \dots \cup B_k$. Since each B_i contains m_i vectors by (2), and since the B_i are pairwise disjoint (the λ_i are distinct), it follows that B contains n vectors. So it suffices to show that B is linearly independent (then B is a basis of $\mathbb{R}^n$). Suppose a linear combination of the vectors in B vanishes, and let $\mathbf{y}_i$ denote the sum of all terms that come from B_i. Then $\mathbf{y}_i$ lies in $E_{\lambda_i}(A)$, so the nonzero $\mathbf{y}_i$ are independent by Theorem 5.5.4 (as the λ_i are distinct). Since the sum of the $\mathbf{y}_i$ is zero, it follows that $\mathbf{y}_i = \mathbf{0}$ for each i. Hence all coefficients of terms in $\mathbf{y}_i$ are zero (because B_i is independent). Since this holds for each i, it shows that B is independent. □

Example 5.5.5

If $A = \left[\begin{array}{rrr} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{array}\right]$ and $B = \left[\begin{array}{rrr} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{array}\right]$ show that A is diagonalizable but B is not.

Solution. We have $c_A(x) = (x+3)^2(x-1)$ so the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 1$. The corresponding eigenspaces are $E_{\lambda_1}(A) = \text{span}\,\{\mathbf{x}_1, \mathbf{x}_2\}$ and $E_{\lambda_2}(A) = \text{span}\,\{\mathbf{x}_3\}$ where

$$\mathbf{x}_1 = \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right], \mathbf{x}_2 = \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array}\right], \mathbf{x}_3 = \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right]$$

as the reader can verify. Since $\{\mathbf{x}_1, \mathbf{x}_2\}$ is independent, we have $\dim(E_{\lambda_1}(A)) = 2$ which is the multiplicity of λ_1. Similarly, $\dim(E_{\lambda_2}(A)) = 1$ equals the multiplicity of λ_2. Hence A is diagonalizable by Theorem 5.5.6, and a diagonalizing matrix is $P = \left[\begin{array}{ccc} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{array}\right]$.

Turning to B, $c_B(x) = (x+1)^2(x-3)$ so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. The corresponding eigenspaces are $E_{\lambda_1}(B) = \text{span}\,\{\mathbf{y}_1\}$ and $E_{\lambda_2}(B) = \text{span}\,\{\mathbf{y}_2\}$ where

$$\mathbf{y}_1 = \left[\begin{array}{r} -1 \\ 2 \\ 1 \end{array}\right], \mathbf{y}_2 = \left[\begin{array}{r} 5 \\ 6 \\ -1 \end{array}\right]$$

Here $\dim(E_{\lambda_1}(B)) = 1$ is *smaller* than the multiplicity of λ_1, so the matrix B is *not* diagonalizable, again by Theorem 5.5.6. The fact that $\dim(E_{\lambda_1}(B)) = 1$ means that there is no possibility of finding *three* linearly independent eigenvectors.

Complex Eigenvalues

All the matrices we have considered have had real eigenvalues. But this need not be the case: The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has characteristic polynomial $c_A(x) = x^2 + 1$ which has no real roots. Nonetheless, this matrix is diagonalizable; the only difference is that we must use a larger set of scalars, the complex numbers. The basic properties of these numbers are outlined in Appendix **??**.

Indeed, nearly everything we have done for real matrices can be done for complex matrices. The methods are the same; the only difference is that the arithmetic is carried out with complex numbers rather than real ones. For example, the gaussian algorithm works in exactly the same way to solve systems of linear equations with complex coefficients, matrix multiplication is defined the same way, and the matrix inversion algorithm works in the same way.

But the complex numbers are better than the real numbers in one respect: While there are polynomials like $x^2 + 1$ with real coefficients that have no real root, this problem does not arise with the complex numbers: *Every* nonconstant polynomial with complex coefficients has a complex root, and hence factors completely as a product of linear factors. This fact is known as the fundamental theorem of algebra.[12]

Example 5.5.6

Diagonalize the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution. The characteristic polynomial of A is

$$c_A(x) = \det(xI - A) = x^2 + 1 = (x - i)(x + i)$$

where $i^2 = -1$. Hence the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Hence A is diagonalizable by the complex version of Theorem 5.5.5, and the complex version of Theorem 5.5.3 shows that $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ is invertible and $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Of course, this can be checked directly.

We shall return to complex linear algebra in Section **??**.

[12] This was a famous open problem in 1799 when Gauss solved it at the age of 22 in his Ph.D. dissertation.

Symmetric Matrices[13]

On the other hand, many of the applications of linear algebra involve a real matrix A and, while A will have complex eigenvalues by the fundamental theorem of algebra, it is always of interest to know when the eigenvalues are, in fact, real. While this can happen in a variety of ways, it turns out to hold whenever A is symmetric. This important theorem will be used extensively later. Surprisingly, the theory of *complex* eigenvalues can be used to prove this useful result about *real* eigenvalues.

Let $\overline{z}$ denote the conjugate of a complex number z. If A is a complex matrix, the **conjugate matrix** $\overline{A}$ is defined to be the matrix obtained from A by conjugating every entry. Thus, if $A = \left[z_{ij}\right]$, then $\overline{A} = \left[\overline{z}_{ij}\right]$. For example,

$$\text{If } A = \begin{bmatrix} -i+2 & 5 \\ i & 3+4i \end{bmatrix} \text{ then } \overline{A} = \begin{bmatrix} i+2 & 5 \\ -i & 3-4i \end{bmatrix}$$

Recall that $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$ hold for all complex numbers z and w. It follows that if A and B are two complex matrices, then

$$\overline{A+B} = \overline{A} + \overline{B}, \quad \overline{AB} = \overline{A}\,\overline{B} \quad \text{and } \overline{\lambda A} = \overline{\lambda}\,\overline{A}$$

hold for all complex scalars λ. These facts are used in the proof of the following theorem.

Theorem 5.5.7

Let A be a symmetric real matrix. If λ is any complex eigenvalue of A, then λ is real.[14]

Proof. Observe that $\overline{A} = A$ because A is real. If λ is an eigenvalue of A, we show that λ is real by showing that $\overline{\lambda} = \lambda$. Let $\mathbf{x}$ be a (possibly complex) eigenvector corresponding to λ, so that $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \lambda\mathbf{x}$. Define $c = \mathbf{x}^T\overline{\mathbf{x}}$.

If we write $\mathbf{x} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ where the z_i are complex numbers, we have

$$c = \mathbf{x}^T\overline{\mathbf{x}} = z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n} = |\overline{z_1}|^2 + |\overline{z_2}|^2 + \cdots + |\overline{z_n}|^2$$

Thus c is a real number, and $c > 0$ because at least one of the $z_i \neq 0$ (as $\mathbf{x} \neq \mathbf{0}$). We show that $\overline{\lambda} = \lambda$ by verifying that $\lambda c = \overline{\lambda} c$. We have

$$\lambda c = \lambda(\mathbf{x}^T\overline{\mathbf{x}}) = (\lambda\mathbf{x})^T\overline{\mathbf{x}} = (A\mathbf{x})^T\overline{\mathbf{x}} = \mathbf{x}^T A^T \overline{\mathbf{x}}$$

At this point we use the hypothesis that A is symmetric and real. This means $A^T = A = \overline{A}$ so we continue the calculation:

[13]This discussion uses complex conjugation and absolute value. These topics are discussed in Appendix ??.

[14]This theorem was first proved in 1829 by the great French mathematician Augustin Louis Cauchy (1789–1857).

$$\begin{aligned}\lambda c = \mathbf{x}^T A^T \overline{\mathbf{x}} = \mathbf{x}^T (\overline{A}\ \overline{\mathbf{x}}) = \mathbf{x}^T (\overline{A\mathbf{x}}) &= \mathbf{x}^T (\overline{\lambda \mathbf{x}}) \\ &= \mathbf{x}^T (\overline{\lambda}\ \overline{\mathbf{x}}) \\ &= \overline{\lambda} \mathbf{x}^T \overline{\mathbf{x}} \\ &= \overline{\lambda} c\end{aligned}$$

as required. □

The technique in the proof of Theorem 5.5.7 will be used again when we return to complex linear algebra in Section **??**.

Example 5.5.7

Verify Theorem 5.5.7 for every real, symmetric 2×2 matrix A.

Solution. If $A = \left[\begin{array}{cc} a & b \\ b & c \end{array}\right]$ we have $c_A(x) = x^2 - (a+c)x + (ac - b^2)$, so the eigenvalues are given by $\lambda = \frac{1}{2}[(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}]$. But here

$$(a+c)^2 - 4(ac-b^2) = (a-c)^2 + 4b^2 \geq 0$$

for any choice of a, b, and c. Hence, the eigenvalues are real numbers.

Exercises for 5.5

Exercise 5.5.1 By computing the trace, determinant, and rank, show that A and B are *not* similar in each case.

a. $A = \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right]$, $B = \left[\begin{array}{rr} 1 & 1 \\ -1 & 1 \end{array}\right]$

b. $A = \left[\begin{array}{rr} 3 & 1 \\ 2 & -1 \end{array}\right]$, $B = \left[\begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array}\right]$

c. $A = \left[\begin{array}{rr} 2 & 1 \\ 1 & -1 \end{array}\right]$, $B = \left[\begin{array}{rr} 3 & 0 \\ 1 & -1 \end{array}\right]$

d. $A = \left[\begin{array}{rr} 3 & 1 \\ -1 & 2 \end{array}\right]$, $B = \left[\begin{array}{rr} 2 & -1 \\ 3 & 2 \end{array}\right]$

e. $A = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$, $B = \left[\begin{array}{rrr} 1 & -2 & 1 \\ -2 & 4 & -2 \\ -3 & 6 & -3 \end{array}\right]$

f. $A = \left[\begin{array}{rrr} 1 & 2 & -3 \\ 1 & -1 & 2 \\ 0 & 3 & -5 \end{array}\right]$, $B = \left[\begin{array}{rrr} -2 & 1 & 3 \\ 6 & -3 & -9 \\ 0 & 0 & 0 \end{array}\right]$

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b. traces $= 2$, ranks $= 2$, but $\det A = -5$, $\det B = -1$

d. ranks $= 2$, determinants $= 7$, but $\operatorname{tr} A = 5$, $\operatorname{tr} B = 4$

f. traces $= -5$, determinants $= 0$, but $\operatorname{rank} A = 2$, $\operatorname{rank} B = 1$

Exercise 5.5.2 Show that $\left[\begin{array}{rrrr} 1 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 4 & 3 & 0 & 0 \end{array}\right]$ and

$\begin{bmatrix} 1 & -1 & 3 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 4 & 1 \\ 5 & -1 & -1 & -4 \end{bmatrix}$ are *not* similar.

Exercise 5.5.3 If $A \sim B$, show that:

a) $A^T \sim B^T$ b) $A^{-1} \sim B^{-1}$

c) $rA \sim rB$ for r in $\mathbb{R}$ d) $A^n \sim B^n$ for $n \geq 1$

---

b. If $B = P^{-1}AP$, then $B^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$.

Exercise 5.5.4 In each case, decide whether the matrix A is diagonalizable. If so, find P such that $P^{-1}AP$ is diagonal.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 & 6 \\ 0 & -3 & 0 \\ 5 & 0 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$ d) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

b. Yes, $P = \begin{bmatrix} -1 & 0 & 6 \\ 0 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

d. No, $c_A(x) = (x+1)(x-4)^2$ so $\lambda = 4$ has multiplicity 2. But $\dim(E_4) = 1$ so Theorem 5.5.6 applies.

Exercise 5.5.5 If A is invertible, show that AB is similar to BA for all B.

Exercise 5.5.6 Show that the only matrix similar to a scalar matrix $A = rI$, r in $\mathbb{R}$, is A itself.

Exercise 5.5.7 Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. If $B = P^{-1}AP$ is similar to A, show that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ.

Exercise 5.5.8 If $A \sim B$ and A has any of the following properties, show that B has the same property.

a. Idempotent, that is $A^2 = A$.

b. Nilpotent, that is $A^k = 0$ for some $k \geq 1$.

c. Invertible.

b. If $B = P^{-1}AP$ and $A^k = 0$, then $B^k = (P^{-1}AP)^k = P^{-1}A^kP = P^{-1}0P = 0$.

Exercise 5.5.9 Let A denote an $n \times n$ upper triangular matrix.

a. If all the main diagonal entries of A are distinct, show that A is diagonalizable.

b. If all the main diagonal entries of A are equal, show that A is diagonalizable only if it is *already* diagonal.

c. Show that $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonalizable but that $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

b. The eigenvalues of A are all equal (they are the diagonal elements), so if $P^{-1}AP = D$ is diagonal, then $D = \lambda I$. Hence $A = P^{-1}(\lambda I)P = \lambda I$.

Exercise 5.5.10 Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (including multiplicities). Show that:

a. $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$

b. $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

b. A is similar to $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ so (Theorem 5.5.1) $\operatorname{tr} A = \operatorname{tr} D = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Exercise 5.5.11 Given a polynomial $p(x) = r_0 + r_1x + \cdots + r_nx^n$ and a square matrix A, the matrix $p(A) = r_0I + r_1A + \cdots + r_nA^n$ is called the **evaluation** of $p(x)$ at A. Let $B = P^{-1}AP$. Show that $p(B) = P^{-1}p(A)P$ for all polynomials $p(x)$.

Exercise 5.5.12 Let P be an invertible $n \times n$ matrix. If A is any $n \times n$ matrix, write $T_P(A) = P^{-1}AP$. Verify that:

a) $T_P(I) = I$

b) $T_P(AB) = T_P(A)T_P(B)$

c) $T_P(A+B) = T_P(A) + T_P(B)$

d) $T_P(rA) = rT_P(A)$

e) $T_P(A^k) = [T_P(A)]^k$ for $k \geq 1$

f) If A is invertible, $T_P(A^{-1}) = [T_P(A)]^{-1}$.

g) If Q is invertible, $T_Q[T_P(A)] = T_{PQ}(A)$.

b. $T_P(A)T_P(B) = (P^{-1}AP)(P^{-1}BP) = P^{-1}(AB)P = T_P(AB)$.

Exercise 5.5.13

a. Show that two diagonalizable matrices are similar if and only if they have the same eigenvalues with the same multiplicities.

b. If A is diagonalizable, show that $A \sim A^T$.

c. Show that $A \sim A^T$ if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b. If A is diagonalizable, so is A^T, and they have the same eigenvalues. Use **(a)**.

Exercise 5.5.14 If A is 2×2 and diagonalizable, show that $C(A) = \{X \mid XA = AX\}$ has dimension 2 or 4. [*Hint*: If $P^{-1}AP = D$, show that X is in $C(A)$ if and only if $P^{-1}XP$ is in $C(D)$.]

Exercise 5.5.15 If A is diagonalizable and $p(x)$ is a polynomial such that $p(\lambda) = 0$ for all eigenvalues λ of A, show that $p(A) = 0$ (see Example 3.3.9). In particular, show $c_A(A) = 0$. [*Remark*: $c_A(A) = 0$ for *all* square matrices A—this is the Cayley-Hamilton theorem, see Theorem **??**.]

Exercise 5.5.16 Let A be $n \times n$ with n distinct real eigenvalues. If $AC = CA$, show that C is diagonalizable.

Exercise 5.5.17 Let $A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ and

$$B = \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}.$$

a. Show that $x^3 - (a^2+b^2+c^2)x - 2abc$ has real roots by considering A.

b. Show that $a^2+b^2+c^2 \geq ab+ac+bc$ by considering B.

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b. $c_B(x) = [x-(a+b+c)][x^2-k]$ where $k = a^2+b^2+c^2-[ab+ac+bc]$. Use Theorem 5.5.7.

Exercise 5.5.18 Assume the 2×2 matrix A is similar to an upper triangular matrix. If $\text{tr}\, A = 0 = \text{tr}\, A^2$, show that $A^2 = 0$.

Exercise 5.5.19 Show that A is similar to A^T for all 2×2 matrices A. [*Hint*: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $c = 0$ treat the cases $b = 0$ and $b \neq 0$ separately. If $c \neq 0$, reduce to the case $c = 1$ using Exercise 5.5.12(d).]

Exercise 5.5.20 Refer to Section **??** on linear recurrences. Assume that the sequence $x_0, x_1, x_2, \dots$ satisfies

$$x_{n+k} = r_0x_n + r_1x_{n+1} + \cdots + r_{k-1}x_{n+k-1}$$

for all $n \geq 0$. Define

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ r_0 & r_1 & r_2 & \cdots & r_{k-1} \end{bmatrix}, \ V_n = \begin{bmatrix} x_n \\ x_{n+1} \\ \vdots \\ x_{n+k-1} \end{bmatrix}.$$

Then show that:

a. $V_n = A^nV_0$ for all n.

b. $c_A(x) = x^k - r_{k-1}x^{k-1} - \cdots - r_1x - r_0$

c. If λ is an eigenvalue of A, the eigenspace E_λ has dimension 1, and $\mathbf{x} = (1, \lambda, \lambda^2, \dots, \lambda^{k-1})^T$ is an eigenvector. [*Hint*: Use $c_A(\lambda) = 0$ to show that $E_\lambda = \mathbb{R}\mathbf{x}$.]

d. A is diagonalizable if and only if the eigenvalues of A are distinct. [*Hint*: See part (c) and Theorem 5.5.4.]

e. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct real eigenvalues, there exist constants $t_1, t_2, \dots, t_k$ such that $x_n = t_1\lambda_1^n + \cdots + t_k\lambda_k^n$ holds for all n. [*Hint*: If D is diagonal with $\lambda_1, \lambda_2, \dots, \lambda_k$ as the main diagonal entries, show that $A^n = PD^nP^{-1}$ has entries that are linear combinations of $\lambda_1^n, \lambda_2^n, \dots, \lambda_k^n$.]

Exercise 5.5.21 Suppose A is 2×2 and $A^2 = 0$. If $\text{tr}\, A \neq 0$ show that $A = 0$.

Supplementary Exercises for Chapter 5

Exercise 5.1 In each case either show that the statement is true or give an example showing that it is false. Throughout, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\dots$, $\mathbf{x}_n$ denote vectors in $\mathbb{R}^n$.

a. If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}+\mathbf{y}$ is in U, then $\mathbf{x}$ and $\mathbf{y}$ are both in U.

b. If U is a subspace of $\mathbb{R}^n$ and $r\mathbf{x}$ is in U, then $\mathbf{x}$ is in U.

c. If U is a nonempty set and $s\mathbf{x}+t\mathbf{y}$ is in U for any s and t whenever $\mathbf{x}$ and $\mathbf{y}$ are in U, then U is a subspace.

d. If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in U, then $-\mathbf{x}$ is in U.

e. If $\{\mathbf{x}, \mathbf{y}\}$ is independent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}\}$ is independent.

f. If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, then $\{\mathbf{x}, \mathbf{y}\}$ is independent.

g. If $\{\mathbf{x}, \mathbf{y}\}$ is not independent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is not independent.

h. If all of $\mathbf{x}_1$, $\mathbf{x}_2$, $\dots$, $\mathbf{x}_n$ are nonzero, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is independent.

i. If one of $\mathbf{x}_1$, $\mathbf{x}_2$, $\dots$, $\mathbf{x}_n$ is zero, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is not independent.

j. If $a\mathbf{x}+b\mathbf{y}+c\mathbf{z}=\mathbf{0}$ where a, b, and c are in $\mathbb{R}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent.

k. If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, then $a\mathbf{x}+b\mathbf{y}+c\mathbf{z}=\mathbf{0}$ for some a, b, and c in $\mathbb{R}$.

l. If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is not independent, then $t_1\mathbf{x}_1+t_2\mathbf{x}_2+\cdots+t_n\mathbf{x}_n=\mathbf{0}$ for t_i in $\mathbb{R}$ not all zero.

m. If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is independent, then $t_1\mathbf{x}_1+t_2\mathbf{x}_2+\cdots+t_n\mathbf{x}_n=\mathbf{0}$ for some t_i in $\mathbb{R}$.

n. Every set of four non-zero vectors in $\mathbb{R}^4$ is a basis.

o. No basis of $\mathbb{R}^3$ can contain a vector with a component $\mathbf{0}$.

p. $\mathbb{R}^3$ has a basis of the form $\{\mathbf{x}, \mathbf{x}+\mathbf{y}, \mathbf{y}\}$ where $\mathbf{x}$ and $\mathbf{y}$ are vectors.

q. Every basis of $\mathbb{R}^5$ contains one column of I_5.

r. Every nonempty subset of a basis of $\mathbb{R}^3$ is again a basis of $\mathbb{R}^3$.

s. If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4\}$ are bases of $\mathbb{R}^4$, then $\{\mathbf{x}_1+\mathbf{y}_1, \mathbf{x}_2+\mathbf{y}_2, \mathbf{x}_3+\mathbf{y}_3, \mathbf{x}_4+\mathbf{y}_4\}$ is also a basis of $\mathbb{R}^4$.

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b. F

d. T

f. T

h. F

j. F

l. T

n. F

p. F

r. F

6. Vector Spaces

Contents

In this chapter we introduce vector spaces in full generality. The reader will notice some similarity with the discussion of the space $\mathbb{R}^n$ in Chapter 5. In fact much of the present material has been developed in that context, and there is some repetition. However, Chapter 6 deals with the notion of an *abstract* vector space, a concept that will be new to most readers. It turns out that there are many systems in which a natural addition and scalar multiplication are defined and satisfy the usual rules familiar from $\mathbb{R}^n$. The study of abstract vector spaces is a way to deal with all these examples simultaneously. The new aspect is that we are dealing with an abstract system in which *all we know* about the vectors is that they are objects that can be added and multiplied by a scalar and satisfy rules familiar from $\mathbb{R}^n$.

The novel thing is the *abstraction*. Getting used to this new conceptual level is facilitated by the work done in Chapter 5: First, the vector manipulations are familiar, giving the reader more time to become accustomed to the abstract setting; and, second, the mental images developed in the concrete setting of $\mathbb{R}^n$ serve as an aid to doing many of the exercises in Chapter 6.

The concept of a vector space was first introduced in 1844 by the German mathematician Hermann Grassmann (1809-1877), but his work did not receive the attention it deserved. It was not until 1888 that the Italian mathematician Guiseppe Peano (1858-1932) clarified Grassmann's work in his book *Calcolo Geometrico* and gave the vector space axioms in their present form. Vector spaces became established with the work of the Polish mathematician Stephan Banach (1892-1945), and the idea was finally accepted in 1918 when Hermann Weyl (1885-1955) used it in his widely read book *Raum-Zeit-Materie* ("Space-Time-Matter"), an introduction to the general theory of relativity.

6.1 Examples and Basic Properties

Many mathematical entities have the property that they can be added and multiplied by a number. Numbers themselves have this property, as do $m \times n$ matrices: The sum of two such matrices is again $m \times n$ as is any scalar multiple of such a matrix. Polynomials are another familiar example, as are the geometric vectors in Chapter 4. It turns out that there are many other types of mathematical objects that can be added and multiplied by a scalar, and the general study of such systems is introduced in this chapter. Remarkably, much of what we could say in Chapter 5 about the dimension of subspaces in $\mathbb{R}^n$ can be formulated in this generality.

Definition 6.1 Vector Spaces

*A **vector space** consists of a nonempty set V of objects (called **vectors**) that can be added, that can be multiplied by a real number (called a **scalar** in this context), and for which certain axioms hold.*[1] *If $\mathbf{v}$ and $\mathbf{w}$ are two vectors in V, their sum is expressed as $\mathbf{v}+\mathbf{w}$, and the scalar product of $\mathbf{v}$ by a real number a is denoted as $a\mathbf{v}$. These operations are called **vector addition** and **scalar multiplication**, respectively, and the following axioms are assumed to hold.*

Axioms for vector addition

A1. *If $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V.*

A2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ *for all $\mathbf{u}$ and $\mathbf{v}$ in V.*

A3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ *for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V.*

A4. *An element $\mathbf{0}$ in V exists such that $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ for every $\mathbf{v}$ in V.*

A5. *For each $\mathbf{v}$ in V, an element $-\mathbf{v}$ in V exists such that $-\mathbf{v}+\mathbf{v}=\mathbf{0}$ and $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.*

Axioms for scalar multiplication

S1. *If $\mathbf{v}$ is in V, then $a\mathbf{v}$ is in V for all a in $\mathbb{R}$.*

S2. $a(\mathbf{v}+\mathbf{w})=a\mathbf{v}+a\mathbf{w}$ *for all $\mathbf{v}$ and $\mathbf{w}$ in V and all a in $\mathbb{R}$.*

S3. $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$ *for all $\mathbf{v}$ in V and all a and b in $\mathbb{R}$.*

S4. $a(b\mathbf{v})=(ab)\mathbf{v}$ *for all $\mathbf{v}$ in V and all a and b in $\mathbb{R}$.*

S5. $1\mathbf{v}=\mathbf{v}$ *for all $\mathbf{v}$ in V.*

The content of axioms A1 and S1 is described by saying that V is **closed** under vector addition and scalar multiplication. The element $\mathbf{0}$ in axiom A4 is called the **zero vector**, and the vector $-\mathbf{v}$ in axiom A5 is called the **negative** of $\mathbf{v}$.

The rules of matrix arithmetic, when applied to $\mathbb{R}^n$, give

Example 6.1.1

$\mathbb{R}^n$ is a vector space using matrix addition and scalar multiplication.[2]

It is important to realize that, in a general vector space, the vectors need not be n-tuples as in $\mathbb{R}^n$. They can be any kind of objects at all as long as the addition and scalar multiplication are defined and the axioms are satisfied. The following examples illustrate the diversity of the concept.

[1] The scalars will usually be real numbers, but they could be complex numbers, or elements of an algebraic system called a field. Another example is the field $\mathbb{Q}$ of rational numbers. We will look briefly at finite fields in Section **??**.

[2] We will usually write the vectors in $\mathbb{R}^n$ as n-tuples. However, if it is convenient, we will sometimes denote them as rows or columns.

The space $\mathbb{R}^n$ consists of special types of matrices. More generally, let $\mathbf{M}_{mn}$ denote the set of all $m \times n$ matrices with real entries. Then Theorem 2.1.1 gives:

Example 6.1.2

The set $\mathbf{M}_{mn}$ of all $m \times n$ matrices is a vector space using matrix addition and scalar multiplication. The zero element in this vector space is the zero matrix of size $m \times n$, and the vector space negative of a matrix (required by axiom A5) is the usual matrix negative discussed in Section 2.1. Note that $\mathbf{M}_{mn}$ is just $\mathbb{R}^{mn}$ in different notation.

In Chapter 5 we identified many important subspaces of $\mathbb{R}^n$ such as $\mathbf{im}\, A$ and $\mathbf{null}\, A$ for a matrix A. These are all vector spaces.

Example 6.1.3

Show that every subspace of $\mathbb{R}^n$ is a vector space in its own right using the addition and scalar multiplication of $\mathbb{R}^n$.

Solution. Axioms A1 and S1 are two of the defining conditions for a subspace U of $\mathbb{R}^n$ (see Section 5.1). The other eight axioms for a vector space are inherited from $\mathbb{R}^n$. For example, if $\mathbf{x}$ and $\mathbf{y}$ are in U and a is a scalar, then $a(\mathbf{x}+\mathbf{y}) = a\mathbf{x}+a\mathbf{y}$ because $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$. This shows that axiom S2 holds for U; similarly, the other axioms also hold for U.

Example 6.1.4

Let V denote the set of all ordered pairs (x, y) and define addition in V as in $\mathbb{R}^2$. However, define a new scalar multiplication in V by

$$a(x, y) = (ay, ax)$$

Determine if V is a vector space with these operations.

Solution. Axioms A1 to A5 are valid for V because they hold for matrices. Also $a(x, y) = (ay, ax)$ is again in V, so axiom S1 holds. To verify axiom S2, let $\mathbf{v} = (x, y)$ and $\mathbf{w} = (x_1, y_1)$ be typical elements in V and compute

$$\begin{aligned} a(\mathbf{v}+\mathbf{w}) &= a(x+x_1, y+y_1) = (a(y+y_1), a(x+x_1)) \\ a\mathbf{v}+a\mathbf{w} &= (ay, ax)+(ay_1, ax_1) = (ay+ay_1, ax+ax_1) \end{aligned}$$

Because these are equal, axiom S2 holds. Similarly, the reader can verify that axiom S3 holds. However, axiom S4 fails because

$$a(b(x, y)) = a(by, bx) = (abx, aby)$$

need not equal $ab(x, y) = (aby, abx)$. Hence, V is *not* a vector space. (In fact, axiom S5 also fails.)

Sets of polynomials provide another important source of examples of vector spaces, so we review some basic facts. A **polynomial** in an indeterminate x is an expression

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $a_0,\ a_1,\ a_2,\ \ldots,\ a_n$ are real numbers called the **coefficients** of the polynomial. If all the coefficients are zero, the polynomial is called the **zero polynomial** and is denoted simply as 0. If $p(x) \neq 0$, the highest power of x with a nonzero coefficient is called the **degree** of $p(x)$ denoted as $\deg p(x)$. The coefficient itself is called the **leading coefficient** of $p(x)$. Hence $\deg(3+5x) = 1$, $\deg(1+x+x^2) = 2$, and $\deg(4) = 0$. (The degree of the zero polynomial is not defined.)

Let $\mathbf{P}$ denote the set of all polynomials and suppose that

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots$$
$$q(x) = b_0 + b_1x + b_2x^2 + \cdots$$

are two polynomials in $\mathbf{P}$ (possibly of different degrees). Then $p(x)$ and $q(x)$ are called **equal** [written $p(x) = q(x)$] if and only if all the corresponding coefficients are equal—that is, $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, and so on. In particular, $a_0 + a_1x + a_2x^2 + \cdots = 0$ means that $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $\ldots$, and this is the reason for calling x an **indeterminate**. The set $\mathbf{P}$ has an addition and scalar multiplication defined on it as follows: if $p(x)$ and $q(x)$ are as before and a is a real number,

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$
$$ap(x) = aa_0 + (aa_1)x + (aa_2)x^2 + \cdots$$

Evidently, these are again polynomials, so $\mathbf{P}$ is closed under these operations, called **pointwise** addition and scalar multiplication. The other vector space axioms are easily verified, and we have

Example 6.1.5

The set $\mathbf{P}$ of all polynomials is a vector space with the foregoing addition and scalar multiplication. The zero vector is the zero polynomial, and the negative of a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \ldots$ is the polynomial $-p(x) = -a_0 - a_1x - a_2x^2 - \ldots$ obtained by negating all the coefficients.

There is another vector space of polynomials that will be referred to later.

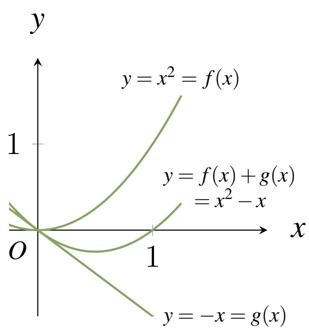
Example 6.1.6

Given $n \geq 1$, let $\mathbf{P}_n$ denote the set of all polynomials of degree at most n, together with the zero polynomial. That is

$$\mathbf{P}_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0,\ a_1,\ a_2,\ \ldots,\ a_n \text{ in } \mathbb{R}\}.$$

Then $\mathbf{P}_n$ is a vector space. Indeed, sums and scalar multiples of polynomials in $\mathbf{P}_n$ are again in $\mathbf{P}_n$, and the other vector space axioms are inherited from $\mathbf{P}$. In particular, the zero vector and the negative of a polynomial in $\mathbf{P}_n$ are the same as those in $\mathbf{P}$.

If a and b are real numbers and $a < b$, the **interval** $[a, b]$ is defined to be the set of all real numbers x such that $a \leq x \leq b$. A (real-valued) **function** f on $[a, b]$ is a rule that associates to every number x in $[a, b]$ a real number denoted $f(x)$. The rule is frequently specified by giving a formula for $f(x)$ in terms of x. For example, $f(x) = 2^x$, $f(x) = \sin x$, and $f(x) = x^2 + 1$ are familiar functions. In fact, every polynomial $p(x)$ can be regarded as the formula for a function p.



The set of all functions on $[a, b]$ is denoted $\mathbf{F}[a, b]$. Two functions f and g in $\mathbf{F}[a, b]$ are **equal** if $f(x) = g(x)$ for every x in $[a, b]$, and we describe this by saying that f and g have the **same action**. Note that two polynomials are equal in $\mathbf{P}$ (defined prior to Example 6.1.5) if and only if they are equal as functions.

If f and g are two functions in $\mathbf{F}[a, b]$, and if r is a real number, define the sum $f + g$ and the scalar product rf by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) && \text{for each } x \text{ in } [a, b] \\ (rf)(x) &= rf(x) && \text{for each } x \text{ in } [a, b] \end{aligned}$$

In other words, the action of $f + g$ upon x is to associate x with the number $f(x) + g(x)$, and rf associates x with $rf(x)$. The sum of $f(x) = x^2$ and $g(x) = -x$ is shown in the diagram. These operations on $\mathbf{F}[a, b]$ are called **pointwise addition and scalar multiplication** of functions and they are the usual operations familiar from elementary algebra and calculus.

Example 6.1.7

The set $\mathbf{F}[a, b]$ of all functions on the interval $[a, b]$ is a vector space using pointwise addition and scalar multiplication. The zero function (in axiom A4), denoted $\mathbf{0}$, is the constant function defined by

$$0(x) = 0 \quad \text{for each } x \text{ in } [a, b]$$

The negative of a function f is denoted $-f$ and has action defined by

$$(-f)(x) = -f(x) \quad \text{for each } x \text{ in } [a, b]$$

Axioms A1 and S1 are clearly satisfied because, if f and g are functions on $[a, b]$, then $f + g$ and rf are again such functions. The verification of the remaining axioms is left as Exercise 6.1.14.

Other examples of vector spaces will appear later, but these are sufficiently varied to indicate the scope of the concept and to illustrate the properties of vector spaces to be discussed. With such a variety of examples, it may come as a surprise that a well-developed *theory* of vector spaces exists. That is, many properties can be shown to hold for *all* vector spaces and hence hold in every example. Such properties are called *theorems* and can be deduced from the axioms. Here is an important example.

Theorem 6.1.1: Cancellation

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space V. If $\mathbf{v} + \mathbf{u} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{w}$.

Proof. We are given $\mathbf{v}+\mathbf{u} = \mathbf{v}+\mathbf{w}$. If these were numbers instead of vectors, we would simply subtract $\mathbf{v}$ from both sides of the equation to obtain $\mathbf{u} = \mathbf{w}$. This can be accomplished with vectors by adding $-\mathbf{v}$ to both sides of the equation. The steps (using only the axioms) are as follows:

$$\begin{aligned} \mathbf{v}+\mathbf{u} &= \mathbf{v}+\mathbf{w} \\ -\mathbf{v}+(\mathbf{v}+\mathbf{u}) &= -\mathbf{v}+(\mathbf{v}+\mathbf{w}) && \text{(axiom A5)} \\ (-\mathbf{v}+\mathbf{v})+\mathbf{u} &= (-\mathbf{v}+\mathbf{v})+\mathbf{w} && \text{(axiom A3)} \\ \mathbf{0}+\mathbf{u} &= \mathbf{0}+\mathbf{w} && \text{(axiom A5)} \\ \mathbf{u} &= \mathbf{w} && \text{(axiom A4)} \end{aligned}$$

This is the desired conclusion.[3] □

As with many good mathematical theorems, the technique of the proof of Theorem 6.1.1 is at least as important as the theorem itself. The idea was to mimic the well-known process of numerical subtraction in a vector space V as follows: To subtract a vector $\mathbf{v}$ from both sides of a vector equation, we added $-\mathbf{v}$ to both sides. With this in mind, we define **difference** $\mathbf{u}-\mathbf{v}$ of two vectors in V as

$$\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v})$$

We shall say that this vector is the result of having **subtracted** $\mathbf{v}$ from $\mathbf{u}$ and, as in arithmetic, this operation has the property given in Theorem 6.1.2.

Theorem 6.1.2

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a vector space V, the equation

$$\mathbf{x}+\mathbf{v} = \mathbf{u}$$

has one and only one solution $\mathbf{x}$ in V given by

$$\mathbf{x} = \mathbf{u}-\mathbf{v}$$

Proof. The difference $\mathbf{x} = \mathbf{u}-\mathbf{v}$ is indeed a solution to the equation because (using several axioms)

$$\mathbf{x}+\mathbf{v} = (\mathbf{u}-\mathbf{v})+\mathbf{v} = [\mathbf{u}+(-\mathbf{v})]+\mathbf{v} = \mathbf{u}+(-\mathbf{v}+\mathbf{v}) = \mathbf{u}+\mathbf{0} = \mathbf{u}$$

To see that this is the only solution, suppose $\mathbf{x}_1$ is another solution so that $\mathbf{x}_1+\mathbf{v} = \mathbf{u}$. Then $\mathbf{x}+\mathbf{v} = \mathbf{x}_1+\mathbf{v}$ (they both equal $\mathbf{u}$), so $\mathbf{x} = \mathbf{x}_1$ by cancellation. □

Similarly, cancellation shows that there is only one zero vector in any vector space and only one negative of each vector (Exercises 6.1.10 and 6.1.11). Hence we speak of *the* zero vector and *the* negative of a vector.

The next theorem derives some basic properties of scalar multiplication that hold in every vector space, and will be used extensively.

[3] Observe that none of the scalar multiplication axioms are needed here.

Theorem 6.1.3

Let $\mathbf{v}$ *denote a vector in a vector space* V *and let* a *denote a real number.*

1. $0\mathbf{v} = \mathbf{0}$.
2. $a\mathbf{0} = \mathbf{0}$.
3. *If* $a\mathbf{v} = \mathbf{0}$*, then either* $a = 0$ *or* $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.
5. $(-a)\mathbf{v} = -(a\mathbf{v}) = a(-\mathbf{v})$.

Proof.

1. Observe that $0\mathbf{v} + 0\mathbf{v} = (0+0)\mathbf{v} = 0\mathbf{v} = 0\mathbf{v} + \mathbf{0}$ where the first equality is by axiom S3. It follows that $0\mathbf{v} = \mathbf{0}$ by cancellation.

2. The proof is similar to that of (1), and is left as Exercise 6.1.12(a).

3. Assume that $a\mathbf{v} = \mathbf{0}$. If $a = 0$, there is nothing to prove; if $a \neq 0$, we must show that $\mathbf{v} = \mathbf{0}$. But $a \neq 0$ means we can scalar-multiply the equation $a\mathbf{v} = \mathbf{0}$ by the scalar $\frac{1}{a}$. The result (using (2) and Axioms S5 and S4) is

$$\mathbf{v} = 1\mathbf{v} = \left(\tfrac{1}{a}a\right)\mathbf{v} = \tfrac{1}{a}(a\mathbf{v}) = \tfrac{1}{a}\mathbf{0} = \mathbf{0}$$

4. We have $-\mathbf{v} + \mathbf{v} = \mathbf{0}$ by axiom A5. On the other hand,

$$(-1)\mathbf{v} + \mathbf{v} = (-1)\mathbf{v} + 1\mathbf{v} = (-1+1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$$

   using (1) and axioms S5 and S3. Hence $(-1)\mathbf{v} + \mathbf{v} = -\mathbf{v} + \mathbf{v}$ (because both are equal to $\mathbf{0}$), so $(-1)\mathbf{v} = -\mathbf{v}$ by cancellation.

5. The proof is left as Exercise 6.1.12.[4] □

The properties in Theorem 6.1.3 are familiar for matrices; the point here is that they hold in *every* vector space. It is hard to exaggerate the importance of this observation.

Axiom A3 ensures that the sum $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ is the same however it is formed, and we write it simply as $\mathbf{u} + \mathbf{v} + \mathbf{w}$. Similarly, there are different ways to form any sum $\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n$, and Axiom A3 guarantees that they are all equal. Moreover, Axiom A2 shows that the order in which the vectors are written does not matter (for example: $\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{z} = \mathbf{z} + \mathbf{u} + \mathbf{w} + \mathbf{v}$).

Similarly, Axioms S2 and S3 extend. For example

$$a(\mathbf{u} + \mathbf{v} + \mathbf{w}) = a[\mathbf{u} + (\mathbf{v} + \mathbf{w})] = a\mathbf{u} + a(\mathbf{v} + \mathbf{w}) = a\mathbf{u} + a\mathbf{v} + a\mathbf{w}$$

for all a, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Similarly $(a+b+c)\mathbf{v} = a\mathbf{v} + b\mathbf{v} + c\mathbf{v}$ hold for all values of a, b, c, and $\mathbf{v}$ (verify). More generally,

$$a(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + a\mathbf{v}_2 + \cdots + a\mathbf{v}_n$$
$$(a_1 + a_2 + \cdots + a_n)\mathbf{v} = a_1\mathbf{v} + a_2\mathbf{v} + \cdots + a_n\mathbf{v}$$

hold for all $n \geq 1$, all numbers a, a_1, $\dots$, a_n, and all vectors, $\mathbf{v}$, $\mathbf{v}_1$, $\dots$, $\mathbf{v}_n$. The verifications are by induction and are left to the reader (Exercise 6.1.13). These facts—together with the axioms, Theorem 6.1.3, and the definition of subtraction—enable us to simplify expressions involving sums of scalar multiples of vectors by collecting like terms, expanding, and taking out common factors. This has been discussed for the vector space of matrices in Section 2.1 (and for geometric vectors in Section 4.1); the manipulations in an arbitrary vector space are carried out in the same way. Here is an illustration.

Example 6.1.8

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space V, simplify the expression

$$2(\mathbf{u}+3\mathbf{w})-3(2\mathbf{w}-\mathbf{v})-3[2(2\mathbf{u}+\mathbf{v}-4\mathbf{w})-4(\mathbf{u}-2\mathbf{w})]$$

Solution. The reduction proceeds as though $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ were matrices or variables.

$$\begin{aligned} &2(\mathbf{u}+3\mathbf{w})-3(2\mathbf{w}-\mathbf{v})-3[2(2\mathbf{u}+\mathbf{v}-4\mathbf{w})-4(\mathbf{u}-2\mathbf{w})] \\ &= 2\mathbf{u}+6\mathbf{w}-6\mathbf{w}+3\mathbf{v}-3[4\mathbf{u}+2\mathbf{v}-8\mathbf{w}-4\mathbf{u}+8\mathbf{w}] \\ &= 2\mathbf{u}+3\mathbf{v}-3[2\mathbf{v}] \\ &= 2\mathbf{u}+3\mathbf{v}-6\mathbf{v} \\ &= 2\mathbf{u}-3\mathbf{v} \end{aligned}$$

Condition (2) in Theorem 6.1.3 points to another example of a vector space.

Example 6.1.9

A set $\{\mathbf{0}\}$ with one element becomes a vector space if we define

$$\mathbf{0}+\mathbf{0}=\mathbf{0} \quad \text{and} \quad a\mathbf{0}=\mathbf{0} \text{ for all scalars } a.$$

The resulting space is called the **zero vector space** and is denoted $\{\mathbf{0}\}$.

The vector space axioms are easily verified for $\{\mathbf{0}\}$. In any vector space V, Theorem 6.1.3 shows that the zero subspace (consisting of the zero vector of V alone) is a copy of the zero vector space.

Exercises for 6.1

Exercise 6.1.1 Let V denote the set of ordered triples (x, y, z) and define addition in V as in $\mathbb{R}^3$. For each of the following definitions of scalar multiplication, decide whether V is a vector space.

a. $a(x, y, z) = (ax, y, az)$

b. $a(x, y, z) = (ax, 0, az)$

c. $a(x, y, z) = (0, 0, 0)$

d. $a(x, y, z) = (2ax, 2ay, 2az)$

---

b. No; S5 fails.

d. No; S4 and S5 fail.

Exercise 6.1.2 Are the following sets vector spaces with the indicated operations? If not, why not?

a. The set V of nonnegative real numbers; ordinary addition and scalar multiplication.

b. The set V of all polynomials of degree ≥ 3, together with 0; operations of $\mathbf{P}$.

c. The set of all polynomials of degree ≤ 3; operations of $\mathbf{P}$.

d. The set $\{1, x, x^2, \dots\}$; operations of $\mathbf{P}$.

e. The set V of all 2×2 matrices of the form $\left[\begin{array}{cc} a & b \\ 0 & c \end{array} \right]$; operations of $\mathbf{M}_{22}$.

f. The set V of 2×2 matrices with equal column sums; operations of $\mathbf{M}_{22}$.

g. The set V of 2×2 matrices with zero determinant; usual matrix operations.

h. The set V of real numbers; usual operations.

i. The set V of complex numbers; usual addition and multiplication by a real number.

j. The set V of all ordered pairs (x, y) with the addition of $\mathbb{R}^2$, but using scalar multiplication $a(x, y) = (ax, -ay)$.

k. The set V of all ordered pairs (x, y) with the addition of $\mathbb{R}^2$, but using scalar multiplication $a(x, y) = (x, y)$ for all a in $\mathbb{R}$.

l. The set V of all functions $f : \mathbb{R} \to \mathbb{R}$ with pointwise addition, but scalar multiplication defined by $(af)(x) = f(ax)$.

m. The set V of all 2×2 matrices whose entries sum to 0; operations of $\mathbf{M}_{22}$.

n. The set V of all 2×2 matrices with the addition of $\mathbf{M}_{22}$ but scalar multiplication $*$ defined by $a * X = aX^T$.

---

b. No; only A1 fails.

d. No.

f. Yes.

h. Yes.

j. No.

l. No; only S3 fails.

n. No; only S4 and S5 fail.

Exercise 6.1.3 Let V be the set of positive real numbers with vector addition being ordinary multiplication, and scalar multiplication being $a \cdot v = v^a$. Show that V is a vector space.

Exercise 6.1.4 If V is the set of ordered pairs (x, y) of real numbers, show that it is a vector space with addition $(x, y) + (x_1, y_1) = (x + x_1, y + y_1 + 1)$ and scalar multiplication $a(x, y) = (ax, ay + a - 1)$. What is the zero vector in V?

The zero vector is $(0, -1)$; the negative of (x, y) is $(-x, -2-y)$.

Exercise 6.1.5 Find $\mathbf{x}$ and $\mathbf{y}$ (in terms of $\mathbf{u}$ and $\mathbf{v}$) such that:

a) $\begin{array}{l} 2\mathbf{x} + \mathbf{y} = \mathbf{u} \\ 5\mathbf{x} + 3\mathbf{y} = \mathbf{v} \end{array}$

b) $\begin{array}{l} 3\mathbf{x} - 2\mathbf{y} = \mathbf{u} \\ 4\mathbf{x} - 5\mathbf{y} = \mathbf{v} \end{array}$

---

b. $\mathbf{x} = \frac{1}{7}(5\mathbf{u} - 2\mathbf{v}),\ \mathbf{y} = \frac{1}{7}(4\mathbf{u} - 3\mathbf{v})$

Exercise 6.1.6 In each case show that the condition $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ in V implies that $a = b = c = 0$.

a. $V = \mathbb{R}^4$; $\mathbf{u} = (2, 1, 0, 2)$, $\mathbf{v} = (1, 1, -1, 0)$, $\mathbf{w} = (0, 1, 2, 1)$

b. $V = \mathbf{M}_{22}$; $\mathbf{u} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$, $\mathbf{v} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$, $\mathbf{w} = \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right]$

c. $V = \mathbf{P}$; $\mathbf{u} = x^3 + x$, $\mathbf{v} = x^2 + 1$, $\mathbf{w} = x^3 - x^2 + x + 1$

d. $V = \mathbf{F}[0, \pi]$; $\mathbf{u} = \sin x$, $\mathbf{v} = \cos x$, $\mathbf{w} = 1$—the constant function

---

b. Equating entries gives $a + c = 0$, $b + c = 0$, $b + c = 0$, $a - c = 0$. The solution is $a = b = c = 0$.

d. If $a \sin x + b \cos y + c = 0$ in $\mathbf{F}[0, \pi]$, then this must hold for *every* x in $[0, \pi]$. Taking $x = 0$, $\frac{\pi}{2}$, and π, respectively, gives $b + c = 0$, $a + c = 0$, $-b + c = 0$ whence, $a = b = c = 0$.

Exercise 6.1.7 Simplify each of the following.

a. $3[2(\mathbf{u} - 2\mathbf{v} - \mathbf{w}) + 3(\mathbf{w} - \mathbf{v})] - 7(\mathbf{u} - 3\mathbf{v} - \mathbf{w})$

b. $4(3\mathbf{u} - \mathbf{v} + \mathbf{w}) - 2[(3\mathbf{u} - 2\mathbf{v}) - 3(\mathbf{v} - \mathbf{w})] + 6(\mathbf{w} - \mathbf{u} - \mathbf{v})$

---

b. $4\mathbf{w}$

Exercise 6.1.8 Show that $\mathbf{x} = \mathbf{v}$ is the only solution to the equation $\mathbf{x} + \mathbf{x} = 2\mathbf{v}$ in a vector space V. Cite all axioms used.

Exercise 6.1.9 Show that $-\mathbf{0} = \mathbf{0}$ in any vector space. Cite all axioms used.

Exercise 6.1.10 Show that the zero vector $\mathbf{0}$ is uniquely determined by the property in axiom A4.

If $\mathbf{z} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$, then $\mathbf{z} + \mathbf{v} = \mathbf{0} + \mathbf{v}$, so $\mathbf{z} = \mathbf{0}$ by cancellation.

Exercise 6.1.11 Given a vector $\mathbf{v}$, show that its negative $-\mathbf{v}$ is uniquely determined by the property in axiom A5.

Exercise 6.1.12

a. Prove (2) of Theorem 6.1.3. [*Hint*: Axiom S2.]

b. Prove that $(-a)\mathbf{v} = -(a\mathbf{v})$ in Theorem 6.1.3 by first computing $(-a)\mathbf{v} + a\mathbf{v}$. Then do it using (4) of Theorem 6.1.3 and axiom S4.

c. Prove that $a(-\mathbf{v}) = -(a\mathbf{v})$ in Theorem 6.1.3 in two ways, as in part (b).

---

b. $(-a)\mathbf{v} + a\mathbf{v} = (-a + a)\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ by Theorem 6.1.3. Because also $-(a\mathbf{v}) + a\mathbf{v} = \mathbf{0}$ (by the definition of $-(a\mathbf{v})$ in axiom A5), this means that $(-a)\mathbf{v} = -(a\mathbf{v})$ by cancellation. Alternatively, use Theorem 6.1.3(4) to give $(-a)\mathbf{v} = [(-1)a]\mathbf{v} = (-1)(a\mathbf{v}) = -(a\mathbf{v})$.

Exercise 6.1.13 Let $\mathbf{v}$, $\mathbf{v}_1$, $\dots$, $\mathbf{v}_n$ denote vectors in a vector space V and let a, a_1, $\dots$, a_n denote numbers. Use induction on n to prove each of the following.

a. $a(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + a\mathbf{v}_2 + \cdots + a\mathbf{v}_n$

b. $(a_1 + a_2 + \cdots + a_n)\mathbf{v} = a_1\mathbf{v} + a_2\mathbf{v} + \cdots + a_n\mathbf{v}$

---

b. The case $n = 1$ is clear, and $n = 2$ is axiom S3. If $n > 2$, then $(a_1 + a_2 + \cdots + a_n)\mathbf{v} = [a_1 + (a_2 + \cdots + a_n)]\mathbf{v} = a_1\mathbf{v} + (a_2 + \cdots + a_n)\mathbf{v} = a_1\mathbf{v} + (a_2\mathbf{v} + \cdots + a_n\mathbf{v})$ using the induction hypothesis; so it holds for all n.

Exercise 6.1.14 Verify axioms A2—A5 and S2—S5 for the space $\mathbf{F}[a, b]$ of functions on $[a, b]$ (Example 6.1.7).

Exercise 6.1.15 Prove each of the following for vectors $\mathbf{u}$ and $\mathbf{v}$ and scalars a and b.

a. If $a\mathbf{v} = \mathbf{0}$, then $a = 0$ or $\mathbf{v} = \mathbf{0}$.

b. If $a\mathbf{v} = b\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, then $a = b$.

c. If $a\mathbf{v} = a\mathbf{w}$ and $a \neq 0$, then $\mathbf{v} = \mathbf{w}$.

---

c. If $a\mathbf{v} = a\mathbf{w}$, then $\mathbf{v} = 1\mathbf{v} = (a^{-1}a)\mathbf{v} = a^{-1}(a\mathbf{v}) = a^{-1}(a\mathbf{w}) = (a^{-1}a)\mathbf{w} = 1\mathbf{w} = \mathbf{w}$.

Exercise 6.1.16 By calculating $(1+1)(\mathbf{v}+\mathbf{w})$ in two ways (using axioms S2 and S3), show that axiom A2 follows from the other axioms.

Exercise 6.1.17 Let V be a vector space, and define V^n to be the set of all n-tuples $(\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n)$ of n vectors $\mathbf{v}_i$, each belonging to V. Define addition and scalar multiplication in V^n as follows:

$$\begin{aligned}(\mathbf{u}_1,\ \mathbf{u}_2,\ \dots,\ \mathbf{u}_n) &+ (\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n) \\ &= (\mathbf{u}_1+\mathbf{v}_1,\ \mathbf{u}_2+\mathbf{v}_2,\ \dots,\ \mathbf{u}_n+\mathbf{v}_n) \\ a(\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n) &= (a\mathbf{v}_1,\ a\mathbf{v}_2,\ \dots,\ a\mathbf{v}_n)\end{aligned}$$

Show that V^n is a vector space.

Exercise 6.1.18 Let V^n be the vector space of n-tuples from the preceding exercise, written as columns. If A is an $m \times n$ matrix, and X is in V^n, define AX in V^m by matrix multiplication. More precisely, if

$$A = [a_{ij}] \text{ and } X = \left[\begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array}\right], \quad \text{let } AX = \left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{array}\right]$$

where $\mathbf{u}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + \cdots + a_{in}\mathbf{v}_n$ for each i.

Prove that:

a. $B(AX) = (BA)X$

b. $(A + A_1)X = AX + A_1X$

c. $A(X + X_1) = AX + AX_1$

d. $(kA)X = k(AX) = A(kX)$ if k is any number

e. $IX = X$ if I is the $n \times n$ identity matrix

f. Let E be an elementary matrix obtained by performing a row operation on the rows of I_n (see Section 2.5). Show that EX is the column resulting from performing that same row operation on the vectors (call them rows) of X. [*Hint*: Lemma 2.5.1.]

6.2 Subspaces and Spanning Sets

Chapter 5 is essentially about the subspaces of $\mathbb{R}^n$. We now extend this notion.

Definition 6.2 Subspaces of a Vector Space

If V is a vector space, a nonempty subset $U \subseteq V$ is called a ***subspace*** *of V if U is itself a vector space using the addition and scalar multiplication of V.*

Subspaces of $\mathbb{R}^n$ (as defined in Section 5.1) are subspaces in the present sense by Example 6.1.3. Moreover, the defining properties for a subspace of $\mathbb{R}^n$ actually *characterize* subspaces in general.

Theorem 6.2.1: Subspace Test

A subset U of a vector space is a subspace of V if and only if it satisfies the following three conditions:

1. *$\mathbf{0}$ lies in U where $\mathbf{0}$ is the zero vector of V.*
2. *If $\mathbf{u}_1$ and $\mathbf{u}_2$ are in U, then $\mathbf{u}_1 + \mathbf{u}_2$ is also in U.*
3. *If $\mathbf{u}$ is in U, then $a\mathbf{u}$ is also in U for each scalar a.*

Proof. If U is a subspace of V, then (2) and (3) hold by axioms A1 and S1 respectively, applied to the vector space U. Since U is nonempty (it is a vector space), choose $\mathbf{u}$ in U. Then (1) holds because $\mathbf{0} = 0\mathbf{u}$ is in U by (3) and Theorem 6.1.3.

Conversely, if (1), (2), and (3) hold, then axioms A1 and S1 hold because of (2) and (3), and axioms A2, A3, S2, S3, S4, and S5 hold in U because they hold in V. Axiom A4 holds because the zero vector $\mathbf{0}$ of V is actually in U by (1), and so serves as the zero of U. Finally, given $\mathbf{u}$ in U, then its negative $-\mathbf{u}$ in V is again in U by (3) because $-\mathbf{u} = (-1)\mathbf{u}$ (again using Theorem 6.1.3). Hence $-\mathbf{u}$ serves as the negative of $\mathbf{u}$ in U. □

Note that the proof of Theorem 6.2.1 shows that if U is a subspace of V, then U and V share the same zero vector, and that the negative of a vector in the space U is the same as its negative in V.

Example 6.2.1

If V is any vector space, show that $\{\mathbf{0}\}$ and V are subspaces of V.

Solution. $U = V$ clearly satisfies the conditions of the subspace test. As to $U = \{\mathbf{0}\}$, it satisfies the conditions because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for all a in $\mathbb{R}$.

The vector space $\{\mathbf{0}\}$ is called the **zero subspace** of V.

Example 6.2.2

Let $\mathbf{v}$ be a vector in a vector space V. Show that the set

$$\mathbb{R}\mathbf{v} = \{a\mathbf{v} \mid a \text{ in } \mathbb{R}\}$$

of all scalar multiples of $\mathbf{v}$ is a subspace of V.

Solution. Because $\mathbf{0} = 0\mathbf{v}$, it is clear that $\mathbf{0}$ lies in $\mathbb{R}\mathbf{v}$. Given two vectors $a\mathbf{v}$ and $a_1\mathbf{v}$ in $\mathbb{R}\mathbf{v}$, their sum $a\mathbf{v} + a_1\mathbf{v} = (a + a_1)\mathbf{v}$ is also a scalar multiple of $\mathbf{v}$ and so lies in $\mathbb{R}\mathbf{v}$. Hence $\mathbb{R}\mathbf{v}$ is closed under addition. Finally, given $a\mathbf{v}$, $r(a\mathbf{v}) = (ra)\mathbf{v}$ lies in $\mathbb{R}\mathbf{v}$ for all $r \in \mathbb{R}$, so $\mathbb{R}\mathbf{v}$ is closed under scalar multiplication. Hence the subspace test applies.

In particular, given $\mathbf{d} \neq \mathbf{0}$ in $\mathbb{R}^3$, $\mathbb{R}\mathbf{d}$ is the line through the origin with direction vector $\mathbf{d}$.

The space $\mathbb{R}\mathbf{v}$ in Example 6.2.2 is described by giving the *form* of each vector in $\mathbb{R}\mathbf{v}$. The next example describes a subset U of the space $\mathbf{M}_{nn}$ by giving a *condition* that each matrix of U must satisfy.

Example 6.2.3

Let A be a fixed matrix in $\mathbf{M}_{nn}$. Show that $U = \{X \text{ in } \mathbf{M}_{nn} \mid AX = XA\}$ is a subspace of $\mathbf{M}_{nn}$.

Solution. If 0 is the $n \times n$ zero matrix, then $A0 = 0A$, so 0 satisfies the condition for membership in U. Next suppose that X and X_1 lie in U so that $AX = XA$ and $AX_1 = X_1A$. Then

$$A(X + X_1) = AX + AX_1 = XA + X_1A + (X + X_1)A$$
$$A(aX) = a(AX) = a(XA) = (aX)A$$

for all a in $\mathbb{R}$, so both $X + X_1$ and aX lie in U. Hence U is a subspace of $\mathbf{M}_{nn}$.

Suppose $p(x)$ is a polynomial and a is a number. Then the number $p(a)$ obtained by replacing x by a in the expression for $p(x)$ is called the **evaluation** of $p(x)$ at a. For example, if $p(x) = 5 - 6x + 2x^2$, then the evaluation of $p(x)$ at $a = 2$ is $p(2) = 5 - 12 + 8 = 1$. If $p(a) = 0$, the number a is called a **root** of $p(x)$.

Example 6.2.4

Consider the set U of all polynomials in $\mathbf{P}$ that have 3 as a root:

$$U = \{p(x) \in \mathbf{P} \mid p(3) = 0\}$$

Show that U is a subspace of $\mathbf{P}$.

Solution. Clearly, the zero polynomial lies in U. Now let $p(x)$ and $q(x)$ lie in U so $p(3) = 0$ and $q(3) = 0$. We have $(p+q)(x) = p(x) + q(x)$ for all x, so $(p+q)(3) = p(3) + q(3) = 0 + 0 = 0$, and U is closed under addition. The verification that U is closed under scalar multiplication is similar.

Recall that the space $\mathbf{P}_n$ consists of all polynomials of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, and so is closed under the addition and scalar multiplication in $\mathbf{P}$. Moreover, the zero polynomial is included in $\mathbf{P}_n$. Thus the subspace test gives Example 6.2.5.

Example 6.2.5

$\mathbf{P}_n$ is a subspace of $\mathbf{P}$ for each $n \geq 0$.

The next example involves the notion of the derivative f' of a function f. (If the reader is not familiar with calculus, this example may be omitted.) A function f defined on the interval $[a, b]$ is called **differentiable** if the derivative $f'(r)$ exists at every r in $[a, b]$.

Example 6.2.6

Show that the subset $\mathbf{D}[a, b]$ of all **differentiable functions** on $[a, b]$ is a subspace of the vector space $\mathbf{F}[a, b]$ of all functions on $[a, b]$.

Solution. The derivative of any constant function is the constant function 0; in particular, 0 itself is differentiable and so lies in $\mathbf{D}[a, b]$. If f and g both lie in $\mathbf{D}[a, b]$ (so that f' and g' exist), then it is a theorem of calculus that $f+g$ and rf are both differentiable for any $r \in \mathbb{R}$. In fact, $(f+g)' = f' + g'$ and $(rf)' = rf'$, so both lie in $\mathbf{D}[a, b]$. This shows that $\mathbf{D}[a, b]$ is a subspace of $\mathbf{F}[a, b]$.

Linear Combinations and Spanning Sets

Definition 6.3 Linear Combinations and Spanning

*Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors in a vector space V. As in $\mathbb{R}^n$, a vector $\mathbf{v}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if it can be expressed in the form*

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

*where $a_1, a_2, \dots, a_n$ are scalars, called the **coefficients** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. The set of all linear combinations of these vectors is called their **span**, and is denoted by*

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \mid a_i \text{ in } \mathbb{R}\}$$

If it happens that $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, these vectors are called a **spanning set** for V. For example, the span of two vectors $\mathbf{v}$ and $\mathbf{w}$ is the set

$$\text{span}\{\mathbf{v}, \mathbf{w}\} = \{s\mathbf{v} + t\mathbf{w} \mid s \text{ and } t \text{ in } \mathbb{R}\}$$

of all sums of scalar multiples of these vectors.

Example 6.2.7

Consider the vectors $p_1 = 1 + x + 4x^2$ and $p_2 = 1 + 5x + x^2$ in $\mathbf{P}_2$. Determine whether p_1 and p_2 lie in $\text{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Solution. For p_1, we want to determine if s and t exist such that

$$p_1 = s(1 + 2x - x^2) + t(3 + 5x + 2x^2)$$

Equating coefficients of powers of x (where $x^0 = 1$) gives

$$1 = s + 3t, \quad 1 = 2s + 5t, \quad \text{and} \quad 4 = -s + 2t$$

These equations have the solution $s = -2$ and $t = 1$, so p_1 is indeed in $\text{span}\{1 + 2x - x^2,\ 3 + 5x + 2x^2\}$.
Turning to $p_2 = 1 + 5x + x^2$, we are looking for s and t such that

$$p_2 = s(1 + 2x - x^2) + t(3 + 5x + 2x^2)$$

Again equating coefficients of powers of x gives equations $1 = s + 3t$, $5 = 2s + 5t$, and $1 = -s + 2t$. But in this case there is no solution, so p_2 is *not* in $\text{span}\{1 + 2x - x^2,\ 3 + 5x + 2x^2\}$.

We saw in Example 5.1.6 that $\mathbb{R}^m = \text{span}\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_m\}$ where the vectors $\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_m$ are the columns of the $m \times m$ identity matrix. Of course $\mathbb{R}^m = \mathbf{M}_{m1}$ is the set of all $m \times 1$ matrices, and there is an analogous spanning set for each space $\mathbf{M}_{mn}$. For example, each 2×2 matrix has the form

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = a\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + b\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] + c\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right] + d\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$$

so

$$\mathbf{M}_{22} = \text{span}\left\{\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]\right\}$$

Similarly, we obtain

Example 6.2.8

$\mathbf{M}_{mn}$ is the span of the set of all $m \times n$ matrices with exactly one entry equal to 1, and all other entries zero.

The fact that every polynomial in $\mathbf{P}_n$ has the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where each a_i is in $\mathbb{R}$ shows that

Example 6.2.9

$\mathbf{P}_n = \text{span}\{1,\ x,\ x^2,\ \dots,\ x^n\}$.

In Example 6.2.2 we saw that $\text{span}\{\mathbf{v}\} = \{a\mathbf{v} \mid a \text{ in } \mathbb{R}\} = \mathbb{R}\mathbf{v}$ is a subspace for any vector $\mathbf{v}$ in a vector space V. More generally, the span of *any* set of vectors is a subspace. In fact, the proof of Theorem 5.1.1 goes through to prove:

Theorem 6.2.2

Let $U = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ *in a vector space* V*. Then:*

1. U *is a subspace of* V *containing each of* $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$.

2. *U is the "smallest" subspace containing these vectors in the sense that any subspace that contains each of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ must contain U.*

Here is how condition 2 in Theorem 6.2.2 is used. Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in a vector space V and a subspace $U \subseteq V$, then:

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq U \Leftrightarrow \text{ each } \mathbf{v}_i \in U$$

The following examples illustrate this.

Example 6.2.10

Show that $\mathbf{P}_3 = \text{span}\{x^2 + x^3, x, 2x^2 + 1, 3\}$.

Solution. Write $U = \text{span}\{x^2 + x^3, x, 2x^2 + 1, 3\}$. Then $U \subseteq \mathbf{P}_3$, and we use the fact that $\mathbf{P}_3 = \text{span}\{1, x, x^2, x^3\}$ to show that $\mathbf{P}_3 \subseteq U$. In fact, x and $1 = \frac{1}{3} \cdot 3$ clearly lie in U. But then successively,

$$x^2 = \tfrac{1}{2}[(2x^2 + 1) - 1] \quad \text{and} \quad x^3 = (x^2 + x^3) - x^2$$

also lie in U. Hence $\mathbf{P}_3 \subseteq U$ by Theorem 6.2.2.

Example 6.2.11

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in a vector space V. Show that

$$\text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{u} + 2\mathbf{v}, \mathbf{u} - \mathbf{v}\}$$

Solution. We have $\text{span}\{\mathbf{u} + 2\mathbf{v}, \mathbf{u} - \mathbf{v}\} \subseteq \text{span}\{\mathbf{u}, \mathbf{v}\}$ by Theorem 6.2.2 because both $\mathbf{u} + 2\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ lie in $\text{span}\{\mathbf{u}, \mathbf{v}\}$. On the other hand,

$$\mathbf{u} = \tfrac{1}{3}(\mathbf{u} + 2\mathbf{v}) + \tfrac{2}{3}(\mathbf{u} - \mathbf{v}) \quad \text{and} \quad \mathbf{v} = \tfrac{1}{3}(\mathbf{u} + 2\mathbf{v}) - \tfrac{1}{3}(\mathbf{u} - \mathbf{v})$$

so $\text{span}\{\mathbf{u}, \mathbf{v}\} \subseteq \text{span}\{\mathbf{u} + 2\mathbf{v}, \mathbf{u} - \mathbf{v}\}$, again by Theorem 6.2.2.

Exercises for 6.2

Exercise 6.2.1 Which of the following are subspaces of $\mathbf{P}_3$? Support your answer.

a. $U = \{f(x) \mid f(x) \in \mathbf{P}_3, f(2) = 1\}$

b. $U = \{xg(x) \mid g(x) \in \mathbf{P}_2\}$

c. $U = \{xg(x) \mid g(x) \in \mathbf{P}_3\}$

d. $U = \{xg(x) + (1 - x)h(x) \mid g(x) \text{ and } h(x) \in \mathbf{P}_2\}$

e. $U =$ The set of all polynomials in $\mathbf{P}_3$ with constant term 0

f. $U = \{f(x) \mid f(x) \in \mathbf{P}_3,\ \deg f(x) = 3\}$

---

b. Yes

d. Yes

f. No; not closed under addition or scalar multiplication, and $\mathbf{0}$ is not in the set.

Exercise 6.2.2 Which of the following are subspaces of $\mathbf{M}_{22}$? Support your answer.

a. $U = \left\{ \left[\begin{array}{cc} a & b \\ 0 & c \end{array}\right] \middle| \ a,\ b,\ \text{and } c \text{ in } \mathbb{R} \right\}$

b. $U = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \middle| \ a+b = c+d;\ a,\ b,\ c,\ d \text{ in } \mathbb{R} \right\}$

c. $U = \{A \mid A \in \mathbf{M}_{22},\ A = A^T\}$

d. $U = \{A \mid A \in \mathbf{M}_{22},\ AB = 0\}$, B a fixed 2×2 matrix

e. $U = \{A \mid A \in \mathbf{M}_{22},\ A^2 = A\}$

f. $U = \{A \mid A \in \mathbf{M}_{22},\ A \text{ is not invertible}\}$

g. $U = \{A \mid A \in \mathbf{M}_{22},\ BAC = CAB\}$, B and C fixed 2×2 matrices

---

b. Yes.

d. Yes.

f. No; not closed under addition.

Exercise 6.2.3 Which of the following are subspaces of $\mathbf{F}[0,\ 1]$? Support your answer.

a. $U = \{f \mid f(0) = 0\}$

b. $U = \{f \mid f(0) = 1\}$

c. $U = \{f \mid f(0) = f(1)\}$

d. $U = \{f \mid f(x) \geq 0 \text{ for all } x \text{ in } [0,\ 1]\}$

e. $U = \{f \mid f(x) = f(y) \text{ for all } x \text{ and } y \text{ in } [0,\ 1]\}$

f. $U = \{f \mid f(x+y) = f(x) + f(y) \text{ for all } x \text{ and } y \text{ in } [0,\ 1]\}$

g. $U = \{f \mid f \text{ is integrable and } \int_0^1 f(x)dx = 0\}$

---

b. No; not closed under addition.

d. No; not closed under scalar multiplication.

f. Yes.

Exercise 6.2.4 Let A be an $m \times n$ matrix. For which columns $\mathbf{b}$ in $\mathbb{R}^m$ is $U = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n,\ A\mathbf{x} = \mathbf{b}\}$ a subspace of $\mathbb{R}^n$? Support your answer.

Exercise 6.2.5 Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ (written as a column), and define $U = \{A\mathbf{x} \mid A \in \mathbf{M}_{mn}\}$.

a. Show that U is a subspace of $\mathbb{R}^m$.

b. Show that $U = \mathbb{R}^m$ if $\mathbf{x} \neq \mathbf{0}$.

---

b. If entry k of $\mathbf{x}$ is $x_k \neq 0$, and if $\mathbf{y}$ is in $\mathbb{R}^n$, then $\mathbf{y} = A\mathbf{x}$ where the column of A is $x_k^{-1}\mathbf{y}$, and the other columns are zero.

Exercise 6.2.6 Write each of the following as a linear combination of $x+1$, x^2+x, and x^2+2.

a) $x^2 + 3x + 2$

b) $2x^2 - 3x + 1$

c) $x^2 + 1$

d) x

---

b. $-3(x+1) + 0(x^2+x) + 2(x^2+2)$

d. $\frac{2}{3}(x+1) + \frac{1}{3}(x^2+x) - \frac{1}{3}(x^2+2)$

Exercise 6.2.7 Determine whether $\mathbf{v}$ lies in span $\{\mathbf{u},\ \mathbf{w}\}$ in each case.

a. $\mathbf{v} = 3x^2 - 2x - 1$; $\mathbf{u} = x^2 + 1$, $\mathbf{w} = x + 2$

b. $\mathbf{v} = x$; $\mathbf{u} = x^2 + 1$, $\mathbf{w} = x + 2$

c. $\mathbf{v} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$; $\mathbf{u} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

d. $\mathbf{v} = \begin{bmatrix} 1 & -4 \\ 5 & 3 \end{bmatrix}$; $\mathbf{u} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

---

b. No.

d. Yes; $\mathbf{v} = 3\mathbf{u} - \mathbf{w}$.

Exercise 6.2.8 Which of the following functions lie in $\text{span}\{\cos^2 x,\ \sin^2 x\}$? (Work in $\mathbf{F}[0,\ \pi]$.)

a) $\cos 2x$ b) 1

c) x^2 d) $1 + x^2$

---

b. Yes; $1 = \cos^2 x + \sin^2 x$

d. No. If $1 + x^2 = a\cos^2 x + b\sin^2 x$, then taking $x = 0$ and $x = \pi$ gives $a = 1$ and $a = 1 + \pi^2$.

Exercise 6.2.9

a. Show that $\mathbb{R}^3$ is spanned by $\{(1,\ 0,\ 1),\ (1,\ 1,\ 0),\ (0,\ 1,\ 1)\}$.

b. Show that $\mathbf{P}_2$ is spanned by $\{1 + 2x^2,\ 3x,\ 1 + x\}$.

c. Show that $\mathbf{M}_{22}$ is spanned by $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$.

---

b. Because $\mathbf{P}_2 = \text{span}\{1,\ x,\ x^2\}$, it suffices to show that $\{1,\ x,\ x^2\} \subseteq \text{span}\{1 + 2x^2,\ 3x,\ 1 + x\}$. But $x = \frac{1}{3}(3x); 1 = (1 + x) - x$ and $x^2 = \frac{1}{2}[(1 + 2x^2) - 1]$.

Exercise 6.2.10 If X and Y are two sets of vectors in a vector space V, and if $X \subseteq Y$, show that $\text{span}\, X \subseteq \text{span}\, Y$.

Exercise 6.2.11 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ denote vectors in a vector space V. Show that:

a. $\text{span}\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\} = \text{span}\{\mathbf{u} + \mathbf{v},\ \mathbf{u} + \mathbf{w},\ \mathbf{v} + \mathbf{w}\}$

b. $\text{span}\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\} = \text{span}\{\mathbf{u} - \mathbf{v},\ \mathbf{u} + \mathbf{w},\ \mathbf{w}\}$

---

b. $\mathbf{u} = (\mathbf{u} + \mathbf{w}) - \mathbf{w}$, $\mathbf{v} = -(\mathbf{u} - \mathbf{v}) + (\mathbf{u} + \mathbf{w}) - \mathbf{w}$, and $\mathbf{w} = \mathbf{w}$

Exercise 6.2.12 Show that

$$\text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n,\ \mathbf{0}\} = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$$

holds for any set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$.

Exercise 6.2.13 If X and Y are nonempty subsets of a vector space V such that $\text{span}\, X = \text{span}\, Y = V$, must there be a vector common to both X and Y? Justify your answer.

Exercise 6.2.14 Is it possible that $\{(1,\ 2,\ 0),\ (1,\ 1,\ 1)\}$ can span the subspace $U = \{(a,\ b,\ 0) \mid a \text{ and } b \text{ in } \mathbb{R}\}$?

No.

Exercise 6.2.15 Describe $\text{span}\{\mathbf{0}\}$.

Exercise 6.2.16 Let $\mathbf{v}$ denote any vector in a vector space V. Show that $\text{span}\{\mathbf{v}\} = \text{span}\{a\mathbf{v}\}$ for any $a \neq 0$.

Exercise 6.2.17 Determine all subspaces of $\mathbb{R}\mathbf{v}$ where $\mathbf{v} \neq \mathbf{0}$ in some vector space V.

---

b. Yes.

Exercise 6.2.18 Suppose $V = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$. If $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where the a_i are in $\mathbb{R}$ and $a_1 \neq 0$, show that $V = \text{span}\{\mathbf{u},\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$.

$\mathbf{v}_1 = \frac{1}{a_1}\mathbf{u} - \frac{a_2}{a_1}\mathbf{v}_2 - \cdots - \frac{a_n}{a_1}\mathbf{v}_n$, so $V \subseteq \text{span}\{\mathbf{u},\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$

Exercise 6.2.19 If $\mathbf{M}_{nn} = \text{span}\{A_1,\ A_2,\ \dots,\ A_k\}$, show that $\mathbf{M}_{nn} = \text{span}\{A_1^T,\ A_2^T,\ \dots,\ A_k^T\}$.

Exercise 6.2.20 If $\mathbf{P}_n = \text{span}\{p_1(x),\ p_2(x),\ \dots,\ p_k(x)\}$ and a is in $\mathbb{R}$, show that $p_i(a) \neq 0$ for some i.

Exercise 6.2.21 Let U be a subspace of a vector space V.

a. If $a\mathbf{u}$ is in U where $a \neq 0$, show that $\mathbf{u}$ is in U.

b. If $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ are in U, show that $\mathbf{v}$ is in U.

---

b. $\mathbf{v} = (\mathbf{u}+\mathbf{v}) - \mathbf{u}$ is in U.

Exercise 6.2.22 Let U be a nonempty subset of a vector space V. Show that U is a subspace of V if and only if $\mathbf{u}_1 + a\mathbf{u}_2$ lies in U for all $\mathbf{u}_1$ and $\mathbf{u}_2$ in U and all a in $\mathbb{R}$.

Given the condition and $\mathbf{u} \in U$, $\mathbf{0} = \mathbf{u} + (-1)\mathbf{u} \in U$. The converse holds by the subspace test.

Exercise 6.2.23 Let $U = \{p(x) \text{ in } \mathbf{P} \mid p(3) = 0\}$ be the set in Example 6.2.4. Use the factor theorem (see Section **??**) to show that U consists of multiples of $x-3$; that is, show that $U = \{(x-3)q(x) \mid q(x) \in \mathbf{P}\}$. Use this to show that U is a subspace of $\mathbf{P}$.

Exercise 6.2.24 Let $A_1, A_2, \dots, A_m$ denote $n \times n$ matrices. If $\mathbf{0} \neq \mathbf{y} \in \mathbb{R}^n$ and $A_1\mathbf{y} = A_2\mathbf{y} = \cdots = A_m\mathbf{y} = \mathbf{0}$, show that $\{A_1, A_2, \dots, A_m\}$ cannot span $\mathbf{M}_{nn}$.

Exercise 6.2.25 Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be sets of vectors in a vector space, and let

$$X = \left[\begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array} \right] \quad Y = \left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{array} \right]$$

as in Exercise 6.1.18.

a. Show that $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ if and only if $AY = X$ for some $n \times n$ matrix A.

b. If $X = AY$ where A is invertible, show that $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$.

Exercise 6.2.26 If U and W are subspaces of a vector space V, let $U \cup W = \{\mathbf{v} \mid \mathbf{v} \text{ is in } U \text{ or } \mathbf{v} \text{ is in } W\}$. Show that $U \cup W$ is a subspace if and only if $U \subseteq W$ or $W \subseteq U$.

Exercise 6.2.27 Show that $\mathbf{P}$ cannot be spanned by a finite set of polynomials.

6.3 Linear Independence and Dimension

Definition 6.4 Linear Independence and Dependence

As in $\mathbb{R}^n$, a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ in a vector space V is called ***linearly independent*** *(or simply* ***independent****) if it satisfies the following condition:*

$$\text{If} \quad s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n = \mathbf{0}, \qquad \text{then} \quad s_1 = s_2 = \cdots = s_n = 0.$$

A set of vectors that is not linearly independent is said to be ***linearly dependent*** *(or simply* ***dependent****).*

The **trivial linear combination** of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$ is the one with every coefficient zero:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$$

This is obviously one way of expressing $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$, and they are linearly independent when it is the *only* way.

Example 6.3.1

Show that $\{1+x,\ 3x+x^2,\ 2+x-x^2\}$ is independent in $\mathbf{P}_2$.

Solution. Suppose a linear combination of these polynomials vanishes.

$$s_1(1+x) + s_2(3x+x^2) + s_3(2+x-x^2) = 0$$

Equating the coefficients of 1, x, and x^2 gives a set of linear equations.

$$\begin{aligned} s_1 + + 2s_3 &= 0 \\ s_1 + 3s_2 + s_3 &= 0 \\ s_2 - s_3 &= 0 \end{aligned}$$

The only solution is $s_1 = s_2 = s_3 = 0$.

Example 6.3.2

Show that $\{\sin x,\ \cos x\}$ is independent in the vector space $\mathbf{F}[0,\ 2\pi]$ of functions defined on the interval $[0,\ 2\pi]$.

Solution. Suppose that a linear combination of these functions vanishes.

$$s_1(\sin x) + s_2(\cos x) = 0$$

This must hold for *all* values of x in $[0,\ 2\pi]$ (by the definition of equality in $\mathbf{F}[0,\ 2\pi]$). Taking $x = 0$ yields $s_2 = 0$ (because $\sin 0 = 0$ and $\cos 0 = 1$). Similarly, $s_1 = 0$ follows from taking $x = \frac{\pi}{2}$ (because $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$).

Example 6.3.3

Suppose that $\{\mathbf{u},\ \mathbf{v}\}$ is an independent set in a vector space V. Show that $\{\mathbf{u}+2\mathbf{v},\ \mathbf{u}-3\mathbf{v}\}$ is also independent.

Solution. Suppose a linear combination of $\mathbf{u}+2\mathbf{v}$ and $\mathbf{u}-3\mathbf{v}$ vanishes:

$$s(\mathbf{u}+2\mathbf{v})+t(\mathbf{u}-3\mathbf{v})=\mathbf{0}$$

We must deduce that $s=t=0$. Collecting terms involving $\mathbf{u}$ and $\mathbf{v}$ gives

$$(s+t)\mathbf{u}+(2s-3t)\mathbf{v}=\mathbf{0}$$

Because $\{\mathbf{u},\ \mathbf{v}\}$ is independent, this yields linear equations $s+t=0$ and $2s-3t=0$. The only solution is $s=t=0$.

Example 6.3.4

Show that any set of polynomials of distinct degrees is independent.

Solution. Let $p_1,\ p_2,\ \dots,\ p_m$ be polynomials where $\deg(p_i)=d_i$. By relabelling if necessary, we may assume that $d_1>d_2>\cdots>d_m$. Suppose that a linear combination vanishes:

$$t_1p_1+t_2p_2+\cdots+t_mp_m=0$$

where each t_i is in $\mathbb{R}$. As $\deg(p_1)=d_1$, let ax^{d_1} be the term in p_1 of highest degree, where $a\neq 0$. Since $d_1>d_2>\cdots>d_m$, it follows that $t_1ax^{d_1}$ is the only term of degree d_1 in the linear combination $t_1p_1+t_2p_2+\cdots+t_mp_m=0$. This means that $t_1ax^{d_1}=0$, whence $t_1a=0$, hence $t_1=0$ (because $a\neq 0$). But then $t_2p_2+\cdots+t_mp_m=0$ so we can repeat the argument to show that $t_2=0$. Continuing, we obtain $t_i=0$ for each i, as desired.

Example 6.3.5

Suppose that A is an $n\times n$ matrix such that $A^k=0$ but $A^{k-1}\neq 0$. Show that $B=\{I,\ A,\ A^2,\ \dots,\ A^{k-1}\}$ is independent in $\mathbf{M}_{nn}$.

Solution. Suppose $r_0I+r_1A+r_2A^2+\cdots+r_{k-1}A^{k-1}=0$. Multiply by A^{k-1}:

$$r_0A^{k-1}+r_1A^k+r_2A^{k+1}+\cdots+r_{k-1}A^{2k-2}=0$$

Since $A^k=0$, all the higher powers are zero, so this becomes $r_0A^{k-1}=0$. But $A^{k-1}\neq 0$, so $r_0=0$, and we have $r_1A^1+r_2A^2+\cdots+r_{k-1}A^{k-1}=0$. Now multiply by A^{k-2} to conclude that $r_1=0$. Continuing, we obtain $r_i=0$ for each i, so B is independent.

The next example collects several useful properties of independence for reference.

Example 6.3.6

Let V denote a vector space.

1. If $\mathbf{v} \neq \mathbf{0}$ in V, then $\{\mathbf{v}\}$ is an independent set.
2. No independent set of vectors in V can contain the zero vector.

Solution.

1. Let $t\mathbf{v} = \mathbf{0}$, t in $\mathbb{R}$. If $t \neq 0$, then $\mathbf{v} = 1\mathbf{v} = \frac{1}{t}(t\mathbf{v}) = \frac{1}{t}\mathbf{0} = \mathbf{0}$, contrary to assumption. So $t = 0$.
2. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent and (say) $\mathbf{v}_2 = \mathbf{0}$, then $0\mathbf{v}_1 + 1\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$ is a nontrivial linear combination that vanishes, contrary to the independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

A set of vectors is independent if $\mathbf{0}$ is a linear combination in a unique way. The following theorem shows that *every* linear combination of these vectors has uniquely determined coefficients, and so extends Theorem 5.2.1.

Theorem 6.3.1

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a linearly independent set of vectors in a vector space V. If a vector $\mathbf{v}$ has two (ostensibly different) representations

$$\begin{aligned} \mathbf{v} &= s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n \\ \mathbf{v} &= t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_n\mathbf{v}_n \end{aligned}$$

as linear combinations of these vectors, then $s_1 = t_1$, $s_2 = t_2$, $\dots$, $s_n = t_n$. In other words, every vector in V can be written in a unique way as a linear combination of the $\mathbf{v}_i$.

Proof. Subtracting the equations given in the theorem gives

$$(s_1 - t_1)\mathbf{v}_1 + (s_2 - t_2)\mathbf{v}_2 + \cdots + (s_n - t_n)\mathbf{v}_n = \mathbf{0}$$

The independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ gives $s_i - t_i = 0$ for each i, as required. □

The following theorem extends (and proves) Theorem 5.2.4, and is one of the most useful results in linear algebra.

Theorem 6.3.2: Fundamental Theorem

can be spanned by n vectors. If any set of m vectors in V is linearly independent, then $m \leq n$.

Proof. Let $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, and suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is an independent set in V. Then $\mathbf{u}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where each a_i is in $\mathbb{R}$. As $\mathbf{u}_1 \neq \mathbf{0}$ (Example 6.3.6), not all of the

a_i are zero, say $a_1 \neq 0$ (after relabelling the $\mathbf{v}_i$). Then $V = \text{span}\{\mathbf{u}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \dots,\ \mathbf{v}_n\}$ as the reader can verify. Hence, write $\mathbf{u}_2 = b_1\mathbf{u}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n$. Then some $c_i \neq 0$ because $\{\mathbf{u}_1,\ \mathbf{u}_2\}$ is independent; so, as before, $V = \text{span}\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{v}_3,\ \dots,\ \mathbf{v}_n\}$, again after possible relabelling of the $\mathbf{v}_i$. If $m > n$, this procedure continues until all the vectors $\mathbf{v}_i$ are replaced by the vectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \dots,\ \mathbf{u}_n$. In particular, $V = \text{span}\{\mathbf{u}_1,\ \mathbf{u}_2,\ \dots,\ \mathbf{u}_n\}$. But then $\mathbf{u}_{n+1}$ is a linear combination of $\mathbf{u}_1,\ \mathbf{u}_2,\ \dots,\ \mathbf{u}_n$ contrary to the independence of the $\mathbf{u}_i$. Hence, the assumption $m > n$ cannot be valid, so $m \leq n$ and the theorem is proved. □

If $V = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$, and if $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \dots,\ \mathbf{u}_m\}$ is an independent set in V, the above proof shows not only that $m \leq n$ but also that m of the (spanning) vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$ can be replaced by the (independent) vectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \dots,\ \mathbf{u}_m$ and the resulting set will still span V. In this form the result is called the **Steinitz Exchange Lemma**.

Definition 6.5 Basis of a Vector Space

As in $\mathbb{R}^n$, a set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of vectors in a vector space V is called a ***basis*** *of V if it satisfies the following two conditions:*

1. $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ *is linearly independent*
2. $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$

Thus if a set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis, then *every* vector in V can be written as a linear combination of these vectors in a *unique* way (Theorem 6.3.1). But even more is true: Any two (finite) bases of V contain the same number of vectors.

Theorem 6.3.3: Invariance Theorem

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ be two bases of a vector space V. Then $n = m$.

Proof. Because $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is independent, it follows from Theorem 6.3.2 that $m \leq n$. Similarly $n \leq m$, so $n = m$, as asserted. □

Theorem 6.3.3 guarantees that no matter which basis of V is chosen it contains the same number of vectors as any other basis. Hence there is no ambiguity about the following definition.

Definition 6.6 Dimension of a Vector Space

If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis of the nonzero vector space V, the number n of vectors in the basis is called the ***dimension*** *of V, and we write*

$$\dim V = n$$

The zero vector space $\{\mathbf{0}\}$ is defined to have dimension 0:

$$\dim\{\mathbf{0}\} = 0$$

In our discussion to this point we have always assumed that a basis is nonempty and hence that the dimension of the space is at least 1. However, the zero space $\{\mathbf{0}\}$ has *no* basis (by Example 6.3.6) so our insistence that $\dim\{\mathbf{0}\} = 0$ amounts to saying that the *empty* set of vectors is a basis of $\{\mathbf{0}\}$. Thus the statement that "the dimension of a vector space is the number of vectors in any basis" holds even for the zero space.

We saw in Example 5.2.9 that $\dim(\mathbb{R}^n) = n$ and, if $\mathbf{e}_j$ denotes column j of I_n, that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis (called the standard basis). In Example 6.3.7 below, similar considerations apply to the space $\mathbf{M}_{mn}$ of all $m \times n$ matrices; the verifications are left to the reader.

Example 6.3.7

The space $\mathbf{M}_{mn}$ has dimension mn, and one basis consists of all $m \times n$ matrices with exactly one entry equal to 1 and all other entries equal to 0. We call this the **standard basis** of $\mathbf{M}_{mn}$.

Example 6.3.8

Show that $\dim \mathbf{P}_n = n+1$ and that $\{1,\ x,\ x^2,\ \dots,\ x^n\}$ is a basis, called the **standard basis** of $\mathbf{P}_n$.

Solution. Each polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ in $\mathbf{P}_n$ is clearly a linear combination of $1,\ x,\ \dots,\ x^n$, so $\mathbf{P}_n = \operatorname{span}\{1,\ x,\ \dots,\ x^n\}$. However, if a linear combination of these vectors vanishes, $a_0 1 + a_1 x + \cdots + a_n x^n = 0$, then $a_0 = a_1 = \cdots = a_n = 0$ because x is an indeterminate. So $\{1,\ x,\ \dots,\ x^n\}$ is linearly independent and hence is a basis containing $n+1$ vectors. Thus, $\dim(\mathbf{P}_n) = n+1$.

Example 6.3.9

If $\mathbf{v} \neq \mathbf{0}$ is any nonzero vector in a vector space V, show that $\operatorname{span}\{\mathbf{v}\} = \mathbb{R}\mathbf{v}$ has dimension 1.

Solution. $\{\mathbf{v}\}$ clearly spans $\mathbb{R}\mathbf{v}$, and it is linearly independent by Example 6.3.6. Hence $\{\mathbf{v}\}$ is a basis of $\mathbb{R}\mathbf{v}$, and so $\dim \mathbb{R}\mathbf{v} = 1$.

Example 6.3.10

Let $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$ and consider the subspace

$$U = \{X \text{ in } \mathbf{M}_{22} \mid AX = XA\}$$

of $\mathbf{M}_{22}$. Show that $\dim U = 2$ and find a basis of U.

Solution. It was shown in Example 6.2.3 that U is a subspace for any choice of the matrix A. In the present case, if $X = \left[\begin{array}{cc} x & y \\ z & w \end{array}\right]$ is in U, the condition $AX = XA$ gives $z = 0$ and $x = y + w$. Hence each matrix X in U can be written

$$X = \left[\begin{array}{cc} y+w & y \\ 0 & w \end{array}\right] = y\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right] + w\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

so $U = \operatorname{span} B$ where $B = \left\{\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right],\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right\}$. Moreover, the set B is linearly independent (verify this), so it is a basis of U and $\dim U = 2$.

Example 6.3.11

Show that the set V of all symmetric 2×2 matrices is a vector space, and find the dimension of V.

Solution. A matrix A is symmetric if $A^T = A$. If A and B lie in V, then

$$(A+B)^T = A^T + B^T = A+B \quad \text{and} \quad (kA)^T = kA^T = kA$$

using Theorem 2.1.2. Hence $A+B$ and kA are also symmetric. As the 2×2 zero matrix is also in V, this shows that V is a vector space (being a subspace of $\mathbf{M}_{22}$). Now a matrix A is symmetric when entries directly across the main diagonal are equal, so each 2×2 symmetric matrix has the form

$$\left[\begin{array}{cc} a & c \\ c & b \end{array}\right] = a\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + b\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] + c\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$$

Hence the set $B = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \right\}$ spans V, and the reader can verify that B is linearly independent. Thus B is a basis of V, so $\dim V = 3$.

It is frequently convenient to alter a basis by multiplying each basis vector by a nonzero scalar. The next example shows that this always produces another basis. The proof is left as Exercise 6.3.22.

Example 6.3.12

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be nonzero vectors in a vector space V. Given nonzero scalars $a_1, a_2, \dots, a_n$, write $D = \{a_1\mathbf{v}_1, a_2\mathbf{v}_2, \dots, a_n\mathbf{v}_n\}$. If B is independent or spans V, the same is true of D. In particular, if B is a basis of V, so also is D.

Exercises for 6.3

Exercise 6.3.1 Show that each of the following sets of vectors is independent.

a. $\{1+x, 1-x, x+x^2\}$ in $\mathbf{P}_2$

b. $\{x^2, x+1, 1-x-x^2\}$ in $\mathbf{P}_2$

c. $\left\{ \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 1 & -1 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right] \right\}$ in $\mathbf{M}_{22}$

d. $\left\{ \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right], \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right] \right\}$ in $\mathbf{M}_{22}$

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b. If $ax^2 + b(x+1) + c(1-x-x^2) = 0$, then $a+c = 0$, $b-c = 0$, $b+c = 0$, so $a = b = c = 0$.

d. If $a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $a+c+d=0$, $a+b+d=0$, $a+b+c=0$, and $b+c+d=0$, so $a=b=c=d=0$.

Exercise 6.3.2 Which of the following subsets of V are independent?

a. $V = \mathbf{P}_2$; $\{x^2+1,\ x+1,\ x\}$

b. $V = \mathbf{P}_2$; $\{x^2-x+3,\ 2x^2+x+5,\ x^2+5x+1\}$

c. $V = \mathbf{M}_{22}$; $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

d. $V = \mathbf{M}_{22}$; $\left\{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right\}$

e. $V = \mathbf{F}[1,\ 2]$; $\left\{\frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3}\right\}$

f. $V = \mathbf{F}[0,\ 1]$; $\left\{\frac{1}{x^2+x-6},\ \frac{1}{x^2-5x+6},\ \frac{1}{x^2-9}\right\}$

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b. $3(x^2-x+3)-2(2x^2+x+5)+(x^2+5x+1)=0$

d. $2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

f. $\frac{5}{x^2+x-6} + \frac{1}{x^2-5x+6} - \frac{6}{x^2-9} = 0$

Exercise 6.3.3 Which of the following are independent in $\mathbf{F}[0,\ 2\pi]$?

a. $\{\sin^2 x,\ \cos^2 x\}$

b. $\{1,\ \sin^2 x,\ \cos^2 x\}$

c. $\{x,\ \sin^2 x,\ \cos^2 x\}$

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b. Dependent: $1-\sin^2 x-\cos^2 x=0$

Exercise 6.3.4 Find all values of a such that the following are independent in $\mathbb{R}^3$.

a. $\{(1,\ -1,\ 0),\ (a,\ 1,\ 0),\ (0,\ 2,\ 3)\}$

b. $\{(2,\ a,\ 1),\ (1,\ 0,\ 1),\ (0,\ 1,\ 3)\}$

---

b. $x \neq -\frac{1}{3}$

Exercise 6.3.5 Show that the following are bases of the space V indicated.

a. $\{(1,\ 1,\ 0),\ (1,\ 0,\ 1),\ (0,\ 1,\ 1)\}$; $V = \mathbb{R}^3$

b. $\{(-1,\ 1,\ 1),\ (1,\ -1,\ 1),\ (1,\ 1,\ -1)\}$; $V = \mathbb{R}^3$

c. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$; $V = \mathbf{M}_{22}$

d. $\{1+x,\ x+x^2,\ x^2+x^3,\ x^3\}$; $V = \mathbf{P}_3$

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b. If $r(-1,\ 1,\ 1)+s(1,\ -1,\ 1)+t(1,\ 1,\ -1) = (0,\ 0,\ 0)$, then $-r+s+t=0$, $r-s+t=0$, and $r-s-t=0$, and this implies that $r=s=t=0$. This proves independence. To prove that they span $\mathbb{R}^3$, observe that $(0,\ 0,\ 1) = \frac{1}{2}[(-1,\ 1,\ 1)+(1,\ -1,\ 1)]$ so $(0,\ 0,\ 1)$ lies in span $\{(-1,\ 1,\ 1),\ (1,\ -1,\ 1),\ (1,\ 1,\ -1)\}$. The proof is similar for $(0,\ 1,\ 0)$ and $(1,\ 0,\ 0)$.

d. If $r(1+x)+s(x+x^2)+t(x^2+x^3)+ux^3=0$, then $r=0$, $r+s=0$, $s+t=0$, and $t+u=0$, so $r=s=t=u=0$. This proves independence. To show that they span $\mathbf{P}_3$, observe that $x^2=(x^2+x^3)-x^3$, $x=(x+x^2)-x^2$, and $1=(1+x)-x$, so $\{1,\ x,\ x^2,\ x^3\} \subseteq$ span $\{1+x,\ x+x^2,\ x^2+x^3,\ x^3\}$.

Exercise 6.3.6 Exhibit a basis and calculate the dimension of each of the following subspaces of $\mathbf{P}_2$.

a. $\{a(1+x)+b(x+x^2) \mid a \text{ and } b \text{ in } \mathbb{R}\}$

b. $\{a+b(x+x^2) \mid a \text{ and } b \text{ in } \mathbb{R}\}$

c. $\{p(x) \mid p(1)=0\}$

d. $\{p(x) \mid p(x)=p(-x)\}$

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b. $\{1,\ x+x^2\}$; dimension $= 2$

d. $\{1,\ x^2\}$; dimension $= 2$

Exercise 6.3.7 Exhibit a basis and calculate the dimension of each of the following subspaces of $\mathbf{M}_{22}$.

a. $\{A \mid A^T = -A\}$

b. $\left\{A \,\middle|\, A\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} A\right\}$

c. $\left\{A \,\middle|\, A\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

d. $\left\{A \,\middle|\, A\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} A\right\}$

---

b. $\left\{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$; dimension $= 2$

d. $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$; dimension $= 2$

Exercise 6.3.8 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and define $U = \{X \mid X \in \mathbf{M}_{22} \text{ and } AX = X\}$.

a. Find a basis of U containing A.

b. Find a basis of U not containing A.

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b. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

Exercise 6.3.9 Show that the set $\mathbb{C}$ of all complex numbers is a vector space with the usual operations, and find its dimension.

Exercise 6.3.10

a. Let V denote the set of all 2×2 matrices with equal column sums. Show that V is a subspace of $\mathbf{M}_{22}$, and compute $\dim V$.

b. Repeat part (a) for 3×3 matrices.

c. Repeat part (a) for $n \times n$ matrices.

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b. $\dim V = 7$

Exercise 6.3.11

a. Let $V = \{(x^2+x+1)p(x) \mid p(x) \text{ in } \mathbf{P}_2\}$. Show that V is a subspace of $\mathbf{P}_4$ and find $\dim V$. [*Hint*: If $f(x)g(x) = 0$ in $\mathbf{P}$, then $f(x) = 0$ or $g(x) = 0$.]

b. Repeat with $V = \{(x^2-x)p(x) \mid p(x) \text{ in } \mathbf{P}_3\}$, a subset of $\mathbf{P}_5$.

c. Generalize.

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b. $\{x^2 - x,\ \ x(x^2-x),\ \ x^2(x^2-x),\ \ x^3(x^2-x)\}$; $\dim V = 4$

Exercise 6.3.12 In each case, either prove the assertion or give an example showing that it is false.

a. Every set of four nonzero polynomials in $\mathbf{P}_3$ is a basis.

b. $\mathbf{P}_2$ has a basis of polynomials $f(x)$ such that $f(0) = 0$.

c. $\mathbf{P}_2$ has a basis of polynomials $f(x)$ such that $f(0) = 1$.

d. Every basis of $\mathbf{M}_{22}$ contains a noninvertible matrix.

e. No independent subset of $\mathbf{M}_{22}$ contains a matrix A with $A^2 = 0$.

f. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent then, $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$ for some a, b, c.

g. $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent if $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$ for some a, b, c.

h. If $\{\mathbf{u},\ \mathbf{v}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{u}+\mathbf{v}\}$.

i. If $\{\mathbf{u},\ \mathbf{v}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{v},\ \mathbf{u}+\mathbf{v}\}$.

j. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, so is $\{\mathbf{u},\ \mathbf{v}\}$.

k. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, so is $\{\mathbf{u}+\mathbf{w},\ \mathbf{v}+\mathbf{w}\}$.

l. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, so is $\{\mathbf{u}+\mathbf{v}+\mathbf{w}\}$.

m. If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ then $\{\mathbf{u},\ \mathbf{v}\}$ is dependent if and only if one is a scalar multiple of the other.

n. If $\dim V = n$, then no set of more than n vectors can be independent.

o. If $\dim V = n$, then no set of fewer than n vectors can span V.

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b. No. Any linear combination f of such polynomials has $f(0) = 0$.

d. No. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$; consists of invertible matrices.

f. Yes. $0\mathbf{u}+0\mathbf{v}+0\mathbf{w} = \mathbf{0}$ for every set $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$.

h. Yes. $s\mathbf{u}+t(\mathbf{u}+\mathbf{v}) = \mathbf{0}$ gives $(s+t)\mathbf{u}+t\mathbf{v} = \mathbf{0}$, whence $s+t = 0 = t$.

j. Yes. If $r\mathbf{u}+s\mathbf{v} = \mathbf{0}$, then $r\mathbf{u}+s\mathbf{v}+0\mathbf{w} = \mathbf{0}$, so $r = 0 = s$.

l. Yes. $\mathbf{u}+\mathbf{v}+\mathbf{w} \neq \mathbf{0}$ because $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent.

n. Yes. If I is independent, then $|I| \leq n$ by the fundamental theorem because any basis spans V.

Exercise 6.3.13 Let $A \neq 0$ and $B \neq 0$ be $n \times n$ matrices, and assume that A is symmetric and B is skew-symmetric (that is, $B^T = -B$). Show that $\{A,\ B\}$ is independent.

Exercise 6.3.14 Show that every set of vectors containing a dependent set is again dependent.

Exercise 6.3.15 Show that every nonempty subset of an independent set of vectors is again independent.

If a linear combination of the subset vanishes, it is a linear combination of the vectors in the larger set (coefficients outside the subset are zero) so it is trivial.

Exercise 6.3.16 Let f and g be functions on $[a,\ b]$, and assume that $f(a) = 1 = g(b)$ and $f(b) = 0 = g(a)$. Show that $\{f,\ g\}$ is independent in $\mathbf{F}[a,\ b]$.

Exercise 6.3.17 Let $\{A_1,\ A_2,\ \dots,\ A_k\}$ be independent in $\mathbf{M}_{mn}$, and suppose that U and V are invertible matrices of size $m \times m$ and $n \times n$, respectively. Show that $\{UA_1V,\ UA_2V,\ \dots,\ UA_kV\}$ is independent.

Exercise 6.3.18 Show that $\{\mathbf{v},\ \mathbf{w}\}$ is independent if and only if neither $\mathbf{v}$ nor $\mathbf{w}$ is a scalar multiple of the other.

Exercise 6.3.19 Assume that $\{\mathbf{u},\ \mathbf{v}\}$ is independent in a vector space V. Write $\mathbf{u}' = a\mathbf{u}+b\mathbf{v}$ and $\mathbf{v}' = c\mathbf{u}+d\mathbf{v}$, where a, b, c, and d are numbers. Show that $\{\mathbf{u}',\ \mathbf{v}'\}$ is independent if and only if the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is invertible. [*Hint*: Theorem 2.4.5.]

Because $\{\mathbf{u},\ \mathbf{v}\}$ is linearly independent, $s\mathbf{u}'+t\mathbf{v}' = \mathbf{0}$ is equivalent to $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Now apply Theorem 2.4.5.

Exercise 6.3.20 If $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ is independent and $\mathbf{w}$ is not in $\operatorname{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$, show that:

a. $\{\mathbf{w},\ \mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ is independent.

b. $\{\mathbf{v}_1+\mathbf{w},\ \mathbf{v}_2+\mathbf{w},\ \dots,\ \mathbf{v}_k+\mathbf{w}\}$ is independent.

Exercise 6.3.21 If $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ is independent, show that $\{\mathbf{v}_1,\ \mathbf{v}_1+\mathbf{v}_2,\ \dots,\ \mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_k\}$ is also independent.

Exercise 6.3.22 Prove Example 6.3.12.

Exercise 6.3.23 Let $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w},\ \mathbf{z}\}$ be independent. Which of the following are dependent?

a. $\{\mathbf{u}-\mathbf{v},\ \mathbf{v}-\mathbf{w},\ \mathbf{w}-\mathbf{u}\}$

b. $\{\mathbf{u}+\mathbf{v},\ \mathbf{v}+\mathbf{w},\ \mathbf{w}+\mathbf{u}\}$

c. $\{\mathbf{u}-\mathbf{v},\ \mathbf{v}-\mathbf{w},\ \mathbf{w}-\mathbf{z},\ \mathbf{z}-\mathbf{u}\}$

d. $\{\mathbf{u}+\mathbf{v},\ \mathbf{v}+\mathbf{w},\ \mathbf{w}+\mathbf{z},\ \mathbf{z}+\mathbf{u}\}$

---

b. Independent.

d. Dependent. For example, $(\mathbf{u}+\mathbf{v})-(\mathbf{v}+\mathbf{w})+(\mathbf{w}+\mathbf{z})-(\mathbf{z}+\mathbf{u})=\mathbf{0}$.

Exercise 6.3.24 Let U and W be subspaces of V with bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ respectively. If U and W have only the zero vector in common, show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{w}_2\}$ is independent.

Exercise 6.3.25 Let $\{p, q\}$ be independent polynomials. Show that $\{p, q, pq\}$ is independent if and only if $\deg p \geq 1$ and $\deg q \geq 1$.

Exercise 6.3.26 If z is a complex number, show that $\{z, z^2\}$ is independent if and only if z is not real.

If z is not real and $az+bz^2=0$, then $a+bz=0 (z\neq 0)$. Hence if $b\neq 0$, then $z=-ab^{-1}$ is real. So $b=0$, and so $a=0$. Conversely, if z is real, say $z=a$, then $(-a)z+1z^2=0$, contrary to the independence of $\{z, z^2\}$.

Exercise 6.3.27 Let $B=\{A_1, A_2, \dots, A_n\}\subseteq \mathbf{M}_{mn}$, and write $B'=\{A_1^T, A_2^T, \dots, A_n^T\}\subseteq \mathbf{M}_{nm}$. Show that:

a. B is independent if and only if B' is independent.

b. B spans $\mathbf{M}_{mn}$ if and only if B' spans $\mathbf{M}_{nm}$.

Exercise 6.3.28 If $V=\mathbf{F}[a, b]$ as in Example 6.1.7, show that the set of constant functions is a subspace of dimension 1 (f is **constant** if there is a number c such that $f(x)=c$ for all x).

Exercise 6.3.29

a. If U is an invertible $n\times n$ matrix and $\{A_1, A_2, \dots, A_{mn}\}$ is a basis of $\mathbf{M}_{mn}$, show that $\{A_1U, A_2U, \dots, A_{mn}U\}$ is also a basis.

b. Show that part (a) fails if U is not invertible. [*Hint*: Theorem 2.4.5.]

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b. If $U\mathbf{x}=\mathbf{0}$, $\mathbf{x}\neq\mathbf{0}$ in $\mathbb{R}^n$, then $R\mathbf{x}=\mathbf{0}$ where $R\neq 0$ is row 1 of U. If $B\in\mathbf{M}_{mn}$ has each row equal to R, then $B\mathbf{x}\neq\mathbf{0}$. But if $B=\sum r_iA_iU$, then $B\mathbf{x}=\sum r_iA_iU\mathbf{x}=\mathbf{0}$. So $\{A_iU\}$ cannot span $\mathbf{M}_{mn}$.

Exercise 6.3.30 Show that $\{(a, b), (a_1, b_1)\}$ is a basis of $\mathbb{R}^2$ if and only if $\{a+bx, a_1+b_1x\}$ is a basis of $\mathbf{P}_1$.

Exercise 6.3.31 Find the dimension of the subspace $\text{span}\{1, \sin^2\theta, \cos 2\theta\}$ of $\mathbf{F}[0, 2\pi]$.

Exercise 6.3.32 Show that $\mathbf{F}[0, 1]$ is not finite dimensional.

Exercise 6.3.33 If U and W are subspaces of V, define their intersection $U\cap W$ as follows: $U\cap W=\{\mathbf{v}\mid \mathbf{v}\text{ is in both } U \text{ and } W\}$

a. Show that $U\cap W$ is a subspace contained in U and W.

b. Show that $U\cap W=\{\mathbf{0}\}$ if and only if $\{\mathbf{u}, \mathbf{w}\}$ is independent for any nonzero vectors $\mathbf{u}$ in U and $\mathbf{w}$ in W.

c. If B and D are bases of U and W, and if $U\cap W=\{\mathbf{0}\}$, show that $B\cup D=\{\mathbf{v}\mid \mathbf{v}\text{ is in } B \text{ or } D\}$ is independent.

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b. If $U\cap W=0$ and $r\mathbf{u}+s\mathbf{w}=\mathbf{0}$, then $r\mathbf{u}=-s\mathbf{w}$ is in $U\cap W$, so $r\mathbf{u}=\mathbf{0}=s\mathbf{w}$. Hence $r=0=s$ because $\mathbf{u}\neq\mathbf{0}\neq\mathbf{w}$. Conversely, if $\mathbf{v}\neq\mathbf{0}$ lies in $U\cap W$, then $1\mathbf{v}+(-1)\mathbf{v}=\mathbf{0}$, contrary to hypothesis.

Exercise 6.3.34 If U and W are vector spaces, let $V=\{(\mathbf{u}, \mathbf{w})\mid \mathbf{u}\text{ in } U \text{ and } \mathbf{w} \text{ in } W\}$.

a. Show that V is a vector space if $(\mathbf{u}, \mathbf{w})+(\mathbf{u}_1, \mathbf{w}_1)=(\mathbf{u}+\mathbf{u}_1, \mathbf{w}+\mathbf{w}_1)$ and $a(\mathbf{u}, \mathbf{w})=(a\mathbf{u}, a\mathbf{w})$.

b. If $\dim U=m$ and $\dim W=n$, show that $\dim V=m+n$.

c. If $V_1, \dots, V_m$ are vector spaces, let

$$\begin{aligned} V &= V_1\times\cdots\times V_m \\ &= \{(\mathbf{v}_1, \dots, \mathbf{v}_m)\mid \mathbf{v}_i\in V_i \text{ for each } i\} \end{aligned}$$

denote the space of n-tuples from the V_i with componentwise operations (see Exercise 6.1.17). If $\dim V_i=n_i$ for each i, show that $\dim V=n_1+\cdots+n_m$.

Exercise 6.3.35 Let $\mathbf{D}_n$ denote the set of all functions f from the set $\{1, 2, \dots, n\}$ to $\mathbb{R}$.

a. Show that $\mathbf{D}_n$ is a vector space with pointwise addition and scalar multiplication.

b. Show that $\{S_1, S_2, \dots, S_n\}$ is a basis of $\mathbf{D}_n$ where, for each $k = 1, 2, \dots, n$, the function S_k is defined by $S_k(k) = 1$, whereas $S_k(j) = 0$ if $j \neq k$.

Exercise 6.3.36 A polynomial $p(x)$ is called **even** if $p(-x) = p(x)$ and **odd** if $p(-x) = -p(x)$. Let E_n and O_n denote the sets of even and odd polynomials in $\mathbf{P}_n$.

a. Show that E_n is a subspace of $\mathbf{P}_n$ and find $\dim E_n$.

b. Show that O_n is a subspace of $\mathbf{P}_n$ and find $\dim O_n$.

---

b. $\dim O_n = \frac{n}{2}$ if n is even and $\dim O_n = \frac{n+1}{2}$ if n is odd.

Exercise 6.3.37 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be independent in a vector space V, and let A be an $n \times n$ matrix. Define $\mathbf{u}_1, \dots, \mathbf{u}_n$ by

$$\begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = A \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$$

(See Exercise 6.1.18.) Show that $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is independent if and only if A is invertible.

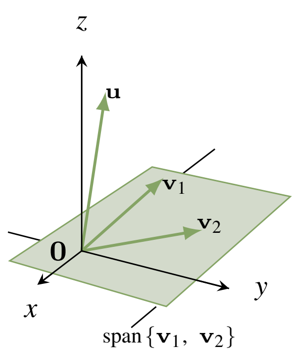
6.4 Finite Dimensional Spaces

Up to this point, we have had no guarantee that an arbitrary vector space *has* a basis—and hence no guarantee that one can speak *at all* of the dimension of V. However, Theorem 6.4.1 will show that any space that is spanned by a finite set of vectors has a (finite) basis: The proof requires the following basic lemma, of interest in itself, that gives a way to enlarge a given independent set of vectors.

Lemma 6.4.1: Independent Lemma

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ *be an independent set of vectors in a vector space* V. *If* $\mathbf{u} \in V$ *but*[5] $\mathbf{u} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, *then* $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ *is also independent.*

Proof. Let $t\mathbf{u} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$; we must show that all the coefficients are zero. First, $t = 0$ because, otherwise, $\mathbf{u} = -\frac{t_1}{t}\mathbf{v}_1 - \frac{t_2}{t}\mathbf{v}_2 - \cdots - \frac{t_k}{t}\mathbf{v}_k$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, contrary to our assumption. Hence $t = 0$. But then $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ so the rest of the t_i are zero by the independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. This is what we wanted. □



Note that the converse of Lemma 6.4.1 is also true: if $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent, then $\mathbf{u}$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

As an illustration, suppose that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is independent in $\mathbb{R}^3$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, so $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane through the origin (shaded in the diagram). By Lemma 6.4.1, $\mathbf{u}$ is not in this plane if and only if $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2\}$ is independent.

Definition 6.7 Finite Dimensional and Infinite Dimensional Vector Spaces

A vector space V *is called* ***finite dimensional*** *if it is spanned by a finite set of vectors. Otherwise,* V *is called* ***infinite dimensional***.

Thus the zero vector space $\{\mathbf{0}\}$ is finite dimensional because $\{\mathbf{0}\}$ is a spanning set.

Lemma 6.4.2

Let V *be a finite dimensional vector space. If* U *is any subspace of* V, *then any independent subset of* U *can be enlarged to a finite basis of* U.

Proof. $\operatorname{span} I = U$ then I is already a basis of U. If $\operatorname{span} I \neq U$, choose $\mathbf{u}_1 \in U$ such that $\mathbf{u}_1 \notin \operatorname{span} I$. Hence the set $I \cup \{\mathbf{u}_1\}$ is independent by Lemma 6.4.1. If $\operatorname{span}(I \cup \{\mathbf{u}_1\}) = U$ we are done; otherwise choose $\mathbf{u}_2 \in U$ such that $\mathbf{u}_2 \notin \operatorname{span}(I \cup \{\mathbf{u}_1\})$. Hence $I \cup \{\mathbf{u}_1, \mathbf{u}_2\}$ is independent, and the process

[5] If X is a set, we write $a \in X$ to indicate that a is an element of the set X. If a is not an element of X, we write $a \notin X$.

continues. We claim that a basis of U will be reached eventually. Indeed, if no basis of U is ever reached, the process creates arbitrarily large independent sets in V. But this is impossible by the fundamental theorem because V is finite dimensional and so is spanned by a finite set of vectors. □

Theorem 6.4.1

Let V be a finite dimensional vector space spanned by m vectors.

1. *V has a finite basis, and $\dim V \leq m$.*
2. *Every independent set of vectors in V can be enlarged to a basis of V by adding vectors from any fixed basis of V.*
3. *If U is a subspace of V, then*
   a. *U is finite dimensional and $\dim U \leq \dim V$.*
   b. *If $\dim U = \dim V$ then $U = V$.*

Proof.

1. If $V = \{\mathbf{0}\}$, then V has an empty basis and $\dim V = 0 \leq m$. Otherwise, let $\mathbf{v} \neq \mathbf{0}$ be a vector in V. Then $\{\mathbf{v}\}$ is independent, so (1) follows from Lemma 6.4.2 with $U = V$.
2. We refine the proof of Lemma 6.4.2. Fix a basis B of V and let I be an independent subset of V. If $\text{span}\, I = V$ then I is already a basis of V. If $\text{span}\, I \neq V$, then B is not contained in I (because B spans V). Hence choose $\mathbf{b}_1 \in B$ such that $\mathbf{b}_1 \notin \text{span}\, I$. Hence the set $I \cup \{\mathbf{b}_1\}$ is independent by Lemma 6.4.1. If $\text{span}\,(I \cup \{\mathbf{b}_1\}) = V$ we are done; otherwise a similar argument shows that $(I \cup \{\mathbf{b}_1,\ \mathbf{b}_2\})$ is independent for some $\mathbf{b}_2 \in B$. Continue this process. As in the proof of Lemma 6.4.2, a basis of V will be reached eventually.
3. a. This is clear if $U = \{\mathbf{0}\}$. Otherwise, let $\mathbf{u} \neq \mathbf{0}$ in U. Then $\{\mathbf{u}\}$ can be enlarged to a finite basis B of U by Lemma 6.4.2, proving that U is finite dimensional. But B is independent in V, so $\dim U \leq \dim V$ by the fundamental theorem.
   b. This is clear if $U = \{\mathbf{0}\}$ because V *has* a basis; otherwise, it follows from (2). □

Theorem 6.4.1 shows that a vector space V is finite dimensional if and only if it has a finite basis (possibly empty), and that every subspace of a finite dimensional space is again finite dimensional.

Example 6.4.1

Enlarge the independent set $D = \left\{ \left[\begin{array}{rr} 1 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{rr} 0 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{rr} 1 & 0 \\ 1 & 1 \end{array} \right] \right\}$ to a basis of $\mathbf{M}_{22}$.

Solution. The standard basis of $\mathbf{M}_{22}$ is $\left\{ \left[\begin{array}{rr} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{rr} 0 & 1 \\ 0 & 0 \end{array} \right], \left[\begin{array}{rr} 0 & 0 \\ 1 & 0 \end{array} \right], \left[\begin{array}{rr} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}$, so including one of these in D will produce a basis by Theorem 6.4.1. In fact including *any* of these matrices in D produces an independent set (verify), and hence a basis by

Theorem 6.4.4. Of course these vectors are not the only possibilities, for example, including $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ works as well.

Example 6.4.2

Find a basis of $\mathbf{P}_3$ containing the independent set $\{1+x,\ 1+x^2\}$.

Solution. The standard basis of $\mathbf{P}_3$ is $\{1,\ x,\ x^2,\ x^3\}$, so including two of these vectors will do. If we use 1 and x^3, the result is $\{1,\ 1+x,\ 1+x^2,\ x^3\}$. This is independent because the polynomials have distinct degrees (Example 6.3.4), and so is a basis by Theorem 6.4.1. Of course, including $\{1,\ x\}$ or $\{1,\ x^2\}$ would *not* work!

Example 6.4.3

Show that the space $\mathbf{P}$ of all polynomials is infinite dimensional.

Solution. For each $n \geq 1$, $\mathbf{P}$ has a subspace $\mathbf{P}_n$ of dimension $n+1$. Suppose $\mathbf{P}$ is finite dimensional, say $\dim \mathbf{P} = m$. Then $\dim \mathbf{P}_n \leq \dim \mathbf{P}$ by Theorem 6.4.1, that is $n+1 \leq m$. This is impossible since n is arbitrary, so $\mathbf{P}$ must be infinite dimensional.

The next example illustrates how (2) of Theorem 6.4.1 can be used.

Example 6.4.4

If $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_k$ are independent columns in $\mathbb{R}^n$, show that they are the first k columns in some invertible $n \times n$ matrix.

Solution. By Theorem 6.4.1, expand $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_k\}$ to a basis $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_k,\ \mathbf{c}_{k+1},\ \dots,\ \mathbf{c}_n\}$ of $\mathbb{R}^n$. Then the matrix $A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_k & \mathbf{c}_{k+1} & \dots & \mathbf{c}_n \end{bmatrix}$ with this basis as its columns is an $n \times n$ matrix and it is invertible by Theorem 5.2.3.

Theorem 6.4.2

Let U and W be subspaces of the finite dimensional space V.

1. *If $U \subseteq W$, then $\dim U \leq \dim W$.*
2. *If $U \subseteq W$ and $\dim U = \dim W$, then $U = W$.*

Proof. Since W is finite dimensional, (1) follows by taking $V = W$ in part (3) of Theorem 6.4.1. Now assume $\dim U = \dim W = n$, and let B be a basis of U. Then B is an independent set in W. If $U \neq W$, then $\operatorname{span} B \neq W$, so B can be extended to an independent set of $n+1$ vectors in W by

Lemma 6.4.1. This contradicts the fundamental theorem (Theorem 6.3.2) because W is spanned by $\dim W = n$ vectors. Hence $U = W$, proving (2). □

Theorem 6.4.2 is very useful. This was illustrated in Example 5.2.13 for $\mathbb{R}^2$ and $\mathbb{R}^3$; here is another example.

Example 6.4.5

If a is a number, let W denote the subspace of all polynomials in $\mathbf{P}_n$ that have a as a root:

$$W = \{p(x) \mid p(x) \in \mathbf{P}_n \text{ and } p(a) = 0\}$$

Show that $\{(x-a),\ (x-a)^2,\ \dots,\ (x-a)^n\}$ is a basis of W.

Solution. Observe first that $(x-a),\ (x-a)^2,\ \dots,\ (x-a)^n$ are members of W, and that they are independent because they have distinct degrees (Example 6.3.4). Write

$$U = \operatorname{span}\{(x-a),\ (x-a)^2,\ \dots,\ (x-a)^n\}$$

Then we have $U \subseteq W \subseteq \mathbf{P}_n$, $\dim U = n$, and $\dim \mathbf{P}_n = n+1$. Hence $n \leq \dim W \leq n+1$ by Theorem 6.4.2. Since $\dim W$ is an integer, we must have $\dim W = n$ or $\dim W = n+1$. But then $W = U$ or $W = \mathbf{P}_n$, again by Theorem 6.4.2. Because $W \neq \mathbf{P}_n$, it follows that $W = U$, as required.

A set of vectors is called **dependent** if it is *not* independent, that is if some nontrivial linear combination vanishes. The next result is a convenient test for dependence.

Lemma 6.4.3: Dependent Lemma

A set $D = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ of vectors in a vector space V is dependent if and only if some vector in D is a linear combination of the others.

Proof. Let $\mathbf{v}_2$ (say) be a linear combination of the rest: $\mathbf{v}_2 = s_1\mathbf{v}_1 + s_3\mathbf{v}_3 + \cdots + s_k\mathbf{v}_k$. Then

$$s_1\mathbf{v}_1 + (-1)\mathbf{v}_2 + s_3\mathbf{v}_3 + \cdots + s_k\mathbf{v}_k = \mathbf{0}$$

is a nontrivial linear combination that vanishes, so D is dependent. Conversely, if D is dependent, let $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ where some coefficient is nonzero. If (say) $t_2 \neq 0$, then $\mathbf{v}_2 = -\frac{t_1}{t_2}\mathbf{v}_1 - \frac{t_3}{t_2}\mathbf{v}_3 - \cdots - \frac{t_k}{t_2}\mathbf{v}_k$ is a linear combination of the others. □

Lemma 6.4.1 gives a way to enlarge independent sets to a basis; by contrast, Lemma 6.4.3 shows that spanning sets can be cut down to a basis.

Theorem 6.4.3

Let V be a finite dimensional vector space. Any spanning set for V can be cut down (by deleting vectors) to a basis of V.

Proof. Since V is finite dimensional, it has a finite spanning set S. Among all spanning sets contained in S, choose S_0 containing the smallest number of vectors. It suffices to show that S_0 is independent (then S_0 is a basis, proving the theorem). Suppose, on the contrary, that S_0 is not independent. Then, by Lemma 6.4.3, some vector $\mathbf{u} \in S_0$ is a linear combination of the set $S_1 = S_0 \setminus \{\mathbf{u}\}$ of vectors in S_0 other than $\mathbf{u}$. It follows that $\text{span } S_0 = \text{span } S_1$, that is, $V = \text{span } S_1$. But S_1 has fewer elements than S_0 so this contradicts the choice of S_0. Hence S_0 is independent after all. □

Note that, with Theorem 6.4.1, Theorem 6.4.3 completes the promised proof of Theorem 5.2.6 for the case $V = \mathbb{R}^n$.

Example 6.4.6

Find a basis of $\mathbf{P}_3$ in the spanning set $S = \{1,\ x+x^2,\ 2x-3x^2,\ 1+3x-2x^2,\ x^3\}$.

Solution. Since $\dim \mathbf{P}_3 = 4$, we must eliminate one polynomial from S. It cannot be x^3 because the span of the rest of S is contained in $\mathbf{P}_2$. But eliminating $1+3x-2x^2$ does leave a basis (verify). Note that $1+3x-2x^2$ is the sum of the first three polynomials in S.

Theorems 6.4.1 and 6.4.3 have other useful consequences.

Theorem 6.4.4

Let V be a vector space with $\dim V = n$, and suppose S is a set of exactly n vectors in V. Then S is independent if and only if S spans V.

Proof. Assume first that S is independent. By Theorem 6.4.1, S is contained in a basis B of V. Hence $|S| = n = |B|$ so, since $S \subseteq B$, it follows that $S = B$. In particular S spans V.

Conversely, assume that S spans V, so S contains a basis B by Theorem 6.4.3. Again $|S| = n = |B|$ so, since $S \supseteq B$, it follows that $S = B$. Hence S is independent. □

One of independence or spanning is often easier to establish than the other when showing that a set of vectors is a basis. For example if $V = \mathbb{R}^n$ it is easy to check whether a subset S of $\mathbb{R}^n$ is orthogonal (hence independent) but checking spanning can be tedious. Here are three more examples.

Example 6.4.7

Consider the set $S = \{p_0(x),\ p_1(x),\ \dots,\ p_n(x)\}$ of polynomials in $\mathbf{P}_n$. If $\deg p_k(x) = k$ for each k, show that S is a basis of $\mathbf{P}_n$.

Solution. The set S is independent—the degrees are distinct—see Example 6.3.4. Hence S is a basis of $\mathbf{P}_n$ by Theorem 6.4.4 because $\dim \mathbf{P}_n = n+1$.

Example 6.4.8

Let V denote the space of all symmetric 2×2 matrices. Find a basis of V consisting of invertible matrices.

Solution. We know that $\dim V = 3$ (Example 6.3.11), so what is needed is a set of three invertible, symmetric matrices that (using Theorem 6.4.4) is either independent or spans V. The set $\left\{ \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right], \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}$ is independent (verify) and so is a basis of the required type.

Example 6.4.9

Let A be any $n \times n$ matrix. Show that there exist n^2+1 scalars $a_0, a_1, a_2, \dots, a_{n^2}$ not all zero, such that

$$a_0I + a_1A + a_2A^2 + \cdots + a_{n^2}A^{n^2} = 0$$

where I denotes the $n \times n$ identity matrix.

Solution. The space $\mathbf{M}_{nn}$ of all $n \times n$ matrices has dimension n^2 by Example 6.3.7. Hence the n^2+1 matrices $I, A, A^2, \dots, A^{n^2}$ cannot be independent by Theorem 6.4.4, so a nontrivial linear combination vanishes. This is the desired conclusion.

The result in Example 6.4.9 can be written as $f(A) = 0$ where $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n^2}x^{n^2}$. In other words, A satisfies a nonzero polynomial $f(x)$ of degree at most n^2. In fact we know that A satisfies a nonzero polynomial of degree n (this is the Cayley-Hamilton theorem—see Theorem **??**), but the brevity of the solution in Example 6.4.6 is an indication of the power of these methods.

If U and W are subspaces of a vector space V, there are two related subspaces that are of interest, their **sum** $U+W$ and their **intersection** $U \cap W$, defined by

$$U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$$
$$U \cap W = \{\mathbf{v} \in V \mid \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$$

It is routine to verify that these are indeed subspaces of V, that $U \cap W$ is contained in both U and W, and that $U+W$ contains both U and W. We conclude this section with a useful fact about the dimensions of these spaces. The proof is a good illustration of how the theorems in this section are used.

Theorem 6.4.5

Suppose that U and W are finite dimensional subspaces of a vector space V. Then $U+W$ is finite dimensional and

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Proof. Since $U \cap W \subseteq U$, it has a finite basis, say $\{\mathbf{x}_1, \dots, \mathbf{x}_d\}$. Extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{u}_1, \dots, \mathbf{u}_m\}$ of U by Theorem 6.4.1. Similarly extend $\{\mathbf{x}_1, \dots, \mathbf{x}_d\}$ to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{w}_1, \dots, \mathbf{w}_p\}$ of W.

Then

$$U+W = \text{span}\{\mathbf{x}_1,\ \dots,\ \mathbf{x}_d,\ \mathbf{u}_1,\ \dots,\ \mathbf{u}_m,\ \mathbf{w}_1,\ \dots,\ \mathbf{w}_p\}$$

as the reader can verify, so $U+W$ is finite dimensional. For the rest, it suffices to show that $\{\mathbf{x}_1,\ \dots,\ \mathbf{x}_d,\ \mathbf{u}_1,\ \dots,\ \mathbf{u}_m,\ \mathbf{w}_1,\ \dots,\ \mathbf{w}_p\}$ is independent (verify). Suppose that

$$r_1\mathbf{x}_1+\cdots+r_d\mathbf{x}_d+s_1\mathbf{u}_1+\cdots+s_m\mathbf{u}_m+t_1\mathbf{w}_1+\cdots+t_p\mathbf{w}_p=\mathbf{0} \tag{6.1}$$

where the r_i, s_j, and t_k are scalars. Then

$$r_1\mathbf{x}_1+\cdots+r_d\mathbf{x}_d+s_1\mathbf{u}_1+\cdots+s_m\mathbf{u}_m=-(t_1\mathbf{w}_1+\cdots+t_p\mathbf{w}_p)$$

is in U (left side) and also in W (right side), and so is in $U \cap W$. Hence $(t_1\mathbf{w}_1+\cdots+t_p\mathbf{w}_p)$ is a linear combination of $\{\mathbf{x}_1,\ \dots,\ \mathbf{x}_d\}$, so $t_1=\cdots=t_p=0$, because $\{\mathbf{x}_1,\ \dots,\ \mathbf{x}_d,\ \mathbf{w}_1,\ \dots,\ \mathbf{w}_p\}$ is independent. Similarly, $s_1=\cdots=s_m=0$, so (6.1) becomes $r_1\mathbf{x}_1+\cdots+r_d\mathbf{x}_d=\mathbf{0}$. It follows that $r_1=\cdots=r_d=0$, as required. □

Theorem 6.4.5 is particularly interesting if $U \cap W=\{\mathbf{0}\}$. Then there are *no* vectors $\mathbf{x}_i$ in the above proof, and the argument shows that if $\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_m\}$ and $\{\mathbf{w}_1,\ \dots,\ \mathbf{w}_p\}$ are bases of U and W respectively, then $\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_m,\ \mathbf{w}_1,\ \dots,\ \mathbf{w}_p\}$ is a basis of $U+W$. In this case $U+W$ is said to be a **direct sum** (written $U \oplus W$); we return to this in Chapter **??**.

Exercises for 6.4

Exercise 6.4.1 In each case, find a basis for V that includes the vector $\mathbf{v}$.

a. $V=\mathbb{R}^3$, $\mathbf{v}=(1,\ -1,\ 1)$

b. $V=\mathbb{R}^3$, $\mathbf{v}=(0,\ 1,\ 1)$

c. $V=\mathbf{M}_{22}$, $\mathbf{v}=\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$

d. $V=\mathbf{P}_2$, $\mathbf{v}=x^2-x+1$

---

b. $\{(0,\ 1,\ 1),\ (1,\ 0,\ 0),\ (0,\ 1,\ 0)\}$

d. $\{x^2-x+1,\ 1,\ x\}$

Exercise 6.4.2 In each case, find a basis for V among the given vectors.

a. $V=\mathbb{R}^3$,
$\{(1,\ 1,\ -1),\ (2,\ 0,\ 1),\ (-1,\ 1,\ -2),\ (1,\ 2,\ 1)\}$

b. $V=\mathbf{P}_2$, $\{x^2+3,\ x+2,\ x^2-2x-1,\ x^2+x\}$

---

b. Any three except $\{x^2+3,\ x+2,\ x^2-2x-1\}$

Exercise 6.4.3 In each case, find a basis of V containing $\mathbf{v}$ and $\mathbf{w}$.

a. $V=\mathbb{R}^4$, $\mathbf{v}=(1,\ -1,\ 1,\ -1)$, $\mathbf{w}=(0,\ 1,\ 0,\ 1)$

b. $V=\mathbb{R}^4$, $\mathbf{v}=(0,\ 0,\ 1,\ 1)$, $\mathbf{w}=(1,\ 1,\ 1,\ 1)$

c. $V=\mathbf{M}_{22}$, $\mathbf{v}=\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$, $\mathbf{w}=\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$

d. $V=\mathbf{P}_3$, $\mathbf{v}=x^2+1$, $\mathbf{w}=x^2+x$

---

b. Add $(0,\ 1,\ 0,\ 0)$ and $(0,\ 0,\ 1,\ 0)$.

d. Add 1 and x^3.

Exercise 6.4.4

a. If z is not a real number, show that $\{z, z^2\}$ is a basis of the real vector space $\mathbb{C}$ of all complex numbers.

b. If z is neither real nor pure imaginary, show that $\{z, \bar{z}\}$ is a basis of $\mathbb{C}$.

---

b. If $z = a + bi$, then $a \neq 0$ and $b \neq 0$. If $rz + s\bar{z} = 0$, then $(r+s)a = 0$ and $(r-s)b = 0$. This means that $r+s = 0 = r-s$, so $r = s = 0$. Thus $\{z, \bar{z}\}$ is independent; it is a basis because $\dim \mathbb{C} = 2$.

Exercise 6.4.5 In each case use Theorem 6.4.4 to decide if S is a basis of V.

a. $V = \mathbf{M}_{22}$;
$$S = \left\{ \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}$$

b. $V = \mathbf{P}_3$; $S = \{2x^2,\ 1+x,\ 3,\ 1+x+x^2+x^3\}$

---

b. The polynomials in S have distinct degrees.

Exercise 6.4.6

a. Find a basis of $\mathbf{M}_{22}$ consisting of matrices with the property that $A^2 = A$.

b. Find a basis of $\mathbf{P}_3$ consisting of polynomials whose coefficients sum to 4. What if they sum to 0?

---

b. $\{4, 4x, 4x^2, 4x^3\}$ is one such basis of $\mathbf{P}_3$. However, there is *no* basis of $\mathbf{P}_3$ consisting of polynomials that have the property that their coefficients sum to zero. For if such a basis exists, then every polynomial in $\mathbf{P}_3$ would have this property (because sums and scalar multiples of such polynomials have the same property).

Exercise 6.4.7 If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of V, determine which of the following are bases.

a. $\{\mathbf{u}+\mathbf{v},\ \mathbf{u}+\mathbf{w},\ \mathbf{v}+\mathbf{w}\}$

b. $\{2\mathbf{u}+\mathbf{v}+3\mathbf{w},\ 3\mathbf{u}+\mathbf{v}-\mathbf{w},\ \mathbf{u}-4\mathbf{w}\}$

c. $\{\mathbf{u},\ \mathbf{u}+\mathbf{v}+\mathbf{w}\}$

d. $\{\mathbf{u},\ \mathbf{u}+\mathbf{w},\ \mathbf{u}-\mathbf{w},\ \mathbf{v}+\mathbf{w}\}$

---

b. Not a basis.

d. Not a basis.

Exercise 6.4.8

a. Can two vectors span $\mathbb{R}^3$? Can they be linearly independent? Explain.

b. Can four vectors span $\mathbb{R}^3$? Can they be linearly independent? Explain.

---

b. Yes; no.

Exercise 6.4.9 Show that any nonzero vector in a finite dimensional vector space is part of a basis.

Exercise 6.4.10 If A is a square matrix, show that $\det A = 0$ if and only if some row is a linear combination of the others.

$\det A = 0$ if and only if A is not invertible; if and only if the rows of A are dependent (Theorem 5.2.3); if and only if some row is a linear combination of the others (Lemma 6.4.2).

Exercise 6.4.11 Let D, I, and X denote finite, nonempty sets of vectors in a vector space V. Assume that D is dependent and I is independent. In each case answer yes or no, and defend your answer.

a. If $X \supseteq D$, must X be dependent?

b. If $X \subseteq D$, must X be dependent?

c. If $X \supseteq I$, must X be independent?

d. If $X \subseteq I$, must X be independent?

b. No. $\{(0,\ 1),\ (1,\ 0)\} \subseteq \{(0,\ 1),\ (1,\ 0),\ (1,\ 1)\}$.

d. Yes. See Exercise 6.3.15.

Exercise 6.4.12 If U and W are subspaces of V and $\dim U = 2$, show that either $U \subseteq W$ or $\dim(U \cap W) \leq 1$.

Exercise 6.4.13 Let A be a nonzero 2×2 matrix and write $U = \{X \text{ in } \mathbf{M}_{22} \mid XA = AX\}$. Show that $\dim U \geq 2$. [*Hint*: I and A are in U.]

Exercise 6.4.14 If $U \subseteq \mathbb{R}^2$ is a subspace, show that $U = \{\mathbf{0}\}$, $U = \mathbb{R}^2$, or U is a line through the origin.

Exercise 6.4.15 Given $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \dots,\ \mathbf{v}_k$, and $\mathbf{v}$, let $U = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ and $W = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k,\ \mathbf{v}\}$. Show that either $\dim W = \dim U$ or $\dim W = 1 + \dim U$.

If $\mathbf{v} \in U$ then $W = U$; if $\mathbf{v} \notin U$ then $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k,\ \mathbf{v}\}$ is a basis of W by the independent lemma.

Exercise 6.4.16 Suppose U is a subspace of $\mathbf{P}_1$, $U \neq \{0\}$, and $U \neq \mathbf{P}_1$. Show that either $U = \mathbb{R}$ or $U = \mathbb{R}(a + x)$ for some a in $\mathbb{R}$.

Exercise 6.4.17 Let U be a subspace of V and assume $\dim V = 4$ and $\dim U = 2$. Does every basis of V result from adding (two) vectors to some basis of U? Defend your answer.

Exercise 6.4.18 Let U and W be subspaces of a vector space V.

a. If $\dim V = 3$, $\dim U = \dim W = 2$, and $U \neq W$, show that $\dim(U \cap W) = 1$.

b. Interpret (a.) geometrically if $V = \mathbb{R}^3$.

b. Two distinct planes through the origin (U and W) meet in a line through the origin ($U \cap W$).

Exercise 6.4.19 Let $U \subseteq W$ be subspaces of V with $\dim U = k$ and $\dim W = m$, where $k < m$. If $k < l < m$, show that a subspace X exists where $U \subseteq X \subseteq W$ and $\dim X = l$.

Exercise 6.4.20 Let $B = \{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ be a *maximal* independent set in a vector space V. That is, no set of more than n vectors S is independent. Show that B is a basis of V.

Exercise 6.4.21 Let $B = \{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ be a *minimal* spanning set for a vector space V. That is, V cannot be spanned by fewer than n vectors. Show that B is a basis of V.

Exercise 6.4.22

a. Let $p(x)$ and $q(x)$ lie in $\mathbf{P}_1$ and suppose that $p(1) \neq 0$, $q(2) \neq 0$, and $p(2) = 0 = q(1)$. Show that $\{p(x),\ q(x)\}$ is a basis of $\mathbf{P}_1$. [*Hint*: If $rp(x) + sq(x) = 0$, evaluate at $x = 1$, $x = 2$.]

b. Let $B = \{p_0(x),\ p_1(x),\ \dots,\ p_n(x)\}$ be a set of polynomials in $\mathbf{P}_n$. Assume that there exist numbers $a_0,\ a_1,\ \dots,\ a_n$ such that $p_i(a_i) \neq 0$ for each i but $p_i(a_j) = 0$ if i is different from j. Show that B is a basis of $\mathbf{P}_n$.

Exercise 6.4.23 Let V be the set of all infinite sequences $(a_0,\ a_1,\ a_2,\ \dots)$ of real numbers. Define addition and scalar multiplication by

$$(a_0,\ a_1,\ \dots) + (b_0,\ b_1,\ \dots) = (a_0 + b_0,\ a_1 + b_1,\ \dots)$$

and

$$r(a_0,\ a_1,\ \dots) = (ra_0,\ ra_1,\ \dots)$$

a. Show that V is a vector space.

b. Show that V is not finite dimensional.

c. [For those with some calculus.] Show that the set of convergent sequences (that is, $\lim\limits_{n\to\infty} a_n$ exists) is a subspace, also of infinite dimension.

b. The set $\{(1,\ 0,\ 0,\ 0,\ \dots),\ (0,\ 1,\ 0,\ 0,\ 0,\ \dots),\ (0,\ 0,\ 1,\ 0,\ 0,\ \dots),\ \dots\}$ contains independent subsets of arbitrary size.

Exercise 6.4.24 Let A be an $n \times n$ matrix of rank r. If $U = \{X \text{ in } \mathbf{M}_{nn} \mid AX = 0\}$, show that $\dim U = n(n - r)$. [*Hint*: Exercise 6.3.34.]

Exercise 6.4.25 Let U and W be subspaces of V.

a. Show that $U+W$ is a subspace of V containing both U and W.

b. Show that $\text{span}\{\mathbf{u}, \mathbf{w}\} = \mathbb{R}\mathbf{u}+\mathbb{R}\mathbf{w}$ for any vectors $\mathbf{u}$ and $\mathbf{w}$.

c. Show that

$$\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_n\} = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} + \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$$

for any vectors $\mathbf{u}_i$ in U and $\mathbf{w}_j$ in W.

---

b. $\mathbb{R}\mathbf{u}+\mathbb{R}\mathbf{w} = \{r\mathbf{u}+s\mathbf{w} \mid r, s \text{ in } \mathbb{R}\} = \text{span}\{\mathbf{u}, \mathbf{w}\}$

Exercise 6.4.26 If A and B are $m \times n$ matrices, show that $\text{rank}(A+B) \leq \text{rank} A + \text{rank} B$. [*Hint*: If U and V are the column spaces of A and B, respectively, show that the column space of $A+B$ is contained in $U+V$ and that $\dim(U+V) \leq \dim U + \dim V$. (See Theorem 6.4.5.)]

Supplementary Exercises for Chapter 6

Exercise 6.1 (Requires calculus) Let V denote the space of all functions $f:\mathbb{R}\to\mathbb{R}$ for which the derivatives f' and f'' exist. Show that f_1, f_2, and f_3 in V are linearly independent provided that their **wronskian** $w(x)$ is nonzero for some x, where

$$w(x) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix}$$

Exercise 6.2 Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$ (written as columns), and let A be an $n\times n$ matrix.

a. If A is invertible, show that $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$.

b. If $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$, show that A is invertible.

---

b. If $YA = 0$, Y a row, we show that $Y = 0$; thus A^T (and hence A) is invertible. Given a column $\mathbf{c}$ in $\mathbb{R}^n$ write $\mathbf{c} = \sum_i r_i(A\mathbf{v}_i)$ where each r_i is in $\mathbb{R}$. Then $Y\mathbf{c} = \sum_i r_i YA\mathbf{v}_i$, so $Y = YI_n = Y\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} Y\mathbf{e}_1 & Y\mathbf{e}_2 & \cdots & Y\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} = 0$, as required.

Exercise 6.3 If A is an $m\times n$ matrix, show that A has rank m if and only if col A contains every column of I_m.

Exercise 6.4 Show that $\text{null } A = \text{null }(A^TA)$ for any real matrix A.

We have $\text{null } A \subseteq \text{null }(A^TA)$ because $A\mathbf{x} = \mathbf{0}$ implies $(A^TA)\mathbf{x} = \mathbf{0}$. Conversely, if $(A^TA)\mathbf{x} = \mathbf{0}$, then $\|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = 0$. Thus $A\mathbf{x} = \mathbf{0}$.

Exercise 6.5 Let A be an $m\times n$ matrix of rank r. Show that $\dim(\text{null } A) = n - r$ (Theorem 5.4.3) as follows. Choose a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ of $\text{null } A$ and extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{z}_1, \dots, \mathbf{z}_m\}$ of $\mathbb{R}^n$. Show that $\{A\mathbf{z}_1, \dots, A\mathbf{z}_m\}$ is a basis of $\text{col } A$.

7. Linear Transformations

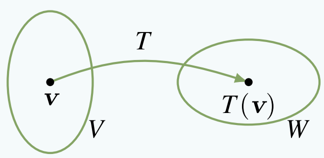
Contents

If V and W are vector spaces, a function $T : V \to W$ is a rule that assigns to each vector $\mathbf{v}$ in V a uniquely determined vector $T(\mathbf{v})$ in W. As mentioned in Section 2.2, two functions $S : V \to W$ and $T : V \to W$ are equal if $S(\mathbf{v}) = T(\mathbf{v})$ for every $\mathbf{v}$ in V. A function $T : V \to W$ is called a *linear transformation* if $T(\mathbf{v}+\mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1)$ for all $\mathbf{v}$, $\mathbf{v}_1$ in V and $T(r\mathbf{v}) = rT(\mathbf{v})$ for all $\mathbf{v}$ in V and all scalars r. $T(\mathbf{v})$ is called the *image* of $\mathbf{v}$ under T. We have already studied linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and shown (in Section 2.6) that they are all given by multiplication by a uniquely determined $m \times n$ matrix A; that is $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In the case of linear operators $\mathbb{R}^2 \to \mathbb{R}^2$, this yields an important way to describe geometric functions such as rotations about the origin and reflections in a line through the origin.

In the present chapter we will describe linear transformations in general, introduce the *kernel* and *image* of a linear transformation, and prove a useful result (called the *dimension theorem*) that relates the dimensions of the kernel and image, and unifies and extends several earlier results. Finally we study the notion of *isomorphic* vector spaces, that is, spaces that are identical except for notation, and relate this to composition of transformations that was introduced in Section 2.3.

7.1 Examples and Elementary Properties

Definition 7.1 Linear Transformations of Vector Spaces



If V and W are two vector spaces, a function $T : V \to W$ is called a ***linear transformation*** *if it satisfies the following axioms.*

T1.	$T(\mathbf{v}+\mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1)$	*for all $\mathbf{v}$ and $\mathbf{v}_1$ in V.*
T2.	$T(r\mathbf{v}) = rT(\mathbf{v})$	*for all $\mathbf{v}$ in V and r in $\mathbb{R}$.*

A linear transformation $T : V \to V$ is called a ***linear operator*** *on V. The situation can be visualized as in the diagram.*

Axiom T1 is just the requirement that T *preserves* vector addition. It asserts that the result $T(\mathbf{v}+\mathbf{v}_1)$ of adding $\mathbf{v}$ and $\mathbf{v}_1$ first and then applying T is the same as applying T first to get $T(\mathbf{v})$ and $T(\mathbf{v}_1)$ and then adding. Similarly, axiom T2 means that T *preserves* scalar multiplication. Note that, even though the additions in axiom T1 are both denoted by the same symbol $+$, the addition on the left forming $\mathbf{v}+\mathbf{v}_1$ is carried out in V, whereas the addition $T(\mathbf{v})+T(\mathbf{v}_1)$ is done in W. Similarly, the scalar multiplications $r\mathbf{v}$ and $rT(\mathbf{v})$ in axiom T2 refer to the spaces V and W, respectively.

We have already seen many examples of linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$. In fact, writing vectors in $\mathbb{R}^n$ as columns, Theorem 2.6.2 shows that, for each such T, there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$. Moreover, the matrix A is given by $A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array} \right]$ where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$. We denote this transformation by $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Example 7.1.1 lists three important linear transformations that will be referred to later. The verification of axioms T1 and T2 is left to the reader.

Example 7.1.1

If V and W are vector spaces, the following are linear transformations:

Identity operator $V \to V$	$1_V : V \to V$	where $1_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V
Zero transformation $V \to W$	$0 : V \to W$	where $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in V
Scalar operator $V \to V$	$a : V \to V$	where $a(\mathbf{v}) = a\mathbf{v}$ for all $\mathbf{v}$ in V
		(Here a is any real number.)

The symbol 0 will be used to denote the zero transformation from V to W for *any* spaces V and W. It was also used earlier to denote the zero function $[a, b] \to \mathbb{R}$.

The next example gives two important transformations of matrices. Recall that the trace $\operatorname{tr} A$ of an $n \times n$ matrix A is the sum of the entries on the main diagonal.

Example 7.1.2

Show that the transposition and trace are linear transformations. More precisely,

$$\begin{array}{ll} R : \mathbf{M}_{mn} \to \mathbf{M}_{nm} & \text{where } R(A) = A^T \text{ for all } A \text{ in } \mathbf{M}_{mn} \\ S : \mathbf{M}_{mn} \to \mathbb{R} & \text{where } S(A) = \text{tr}\, A \text{ for all } A \text{ in } \mathbf{M}_{nn} \end{array}$$

are both linear transformations.

Solution. Axioms T1 and T2 for transposition are $(A+B)^T = A^T + B^T$ and $(rA)^T = r(A^T)$, respectively (using Theorem 2.1.2). The verifications for the trace are left to the reader.

Example 7.1.3

If a is a scalar, define $E_a : \mathbf{P}_n \to \mathbb{R}$ by $E_a(p) = p(a)$ for each polynomial p in $\mathbf{P}_n$. Show that E_a is a linear transformation (called **evaluation** at a).

Solution. If p and q are polynomials and r is in $\mathbb{R}$, we use the fact that the sum $p+q$ and scalar product rp are defined as for functions:

$$(p+q)(x) = p(x) + q(x) \quad \text{and} \quad (rp)(x) = rp(x)$$

for all x. Hence, for all p and q in $\mathbf{P}_n$ and all r in $\mathbb{R}$:

$$\begin{aligned} E_a(p+q) &= (p+q)(a) = p(a) + q(a) = E_a(p) + E_a(q), \quad \text{and} \\ E_a(rp) &= (rp)(a) = rp(a) = rE_a(p). \end{aligned}$$

Hence E_a is a linear transformation.

The next example involves some calculus.

Example 7.1.4

Show that the differentiation and integration operations on $\mathbf{P}_n$ are linear transformations. More precisely,

$$\begin{array}{ll} D : \mathbf{P}_n \to \mathbf{P}_{n-1} & \text{where } D[p(x)] = p'(x) \text{ for all } p(x) \text{ in } \mathbf{P}_n \\ I : \mathbf{P}_n \to \mathbf{P}_{n+1} & \text{where } I[p(x)] = \displaystyle\int_0^x p(t)dt \text{ for all } p(x) \text{ in } \mathbf{P}_n \end{array}$$

are linear transformations.

Solution. These restate the following fundamental properties of differentiation and integration.

$$[p(x) + q(x)]' = p'(x) + q'(x) \quad \text{and} \quad [rp(x)]' = (rp)'(x)$$

$$\textstyle\int_0^x [p(t) + q(t)]\, dt = \int_0^x p(t)dt + \int_0^x q(t)dt \quad \text{and} \quad \int_0^x rp(t)dt = r\int_0^x p(t)dt$$

The next theorem collects three useful properties of *all* linear transformations. They can be described by saying that, in addition to preserving addition and scalar multiplication (these are the axioms), linear transformations preserve the zero vector, negatives, and linear combinations.

Theorem 7.1.1

Let $T : V \to W$ be a linear transformation.

1. $T(\mathbf{0}) = \mathbf{0}$.
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v}$ *in* V.
3. $T(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + r_2T(\mathbf{v}_2) + \cdots + r_kT(\mathbf{v}_k)$ *for all* $\mathbf{v}_i$ *in* V *and all* r_i *in* $\mathbb{R}$.

Proof.

1. $T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}$ for any $\mathbf{v}$ in V.
2. $T(-\mathbf{v}) = T[(-1)\mathbf{v}] = (-1)T(\mathbf{v}) = -T(\mathbf{v})$ for any $\mathbf{v}$ in V.
3. The proof of Theorem 2.6.1 goes through. □

The ability to use the last part of Theorem 7.1.1 effectively is vital to obtaining the benefits of linear transformations. Example 7.1.5 and Theorem 7.1.2 provide illustrations.

Example 7.1.5

Let $T : V \to W$ be a linear transformation. If $T(\mathbf{v} - 3\mathbf{v}_1) = \mathbf{w}$ and $T(2\mathbf{v} - \mathbf{v}_1) = \mathbf{w}_1$, find $T(\mathbf{v})$ and $T(\mathbf{v}_1)$ in terms of $\mathbf{w}$ and $\mathbf{w}_1$.

Solution. The given relations imply that

$$\begin{aligned} T(\mathbf{v}) - 3T(\mathbf{v}_1) &= \mathbf{w} \\ 2T(\mathbf{v}) - T(\mathbf{v}_1) &= \mathbf{w}_1 \end{aligned}$$

by Theorem 7.1.1. Subtracting twice the first from the second gives $T(\mathbf{v}_1) = \frac{1}{5}(\mathbf{w}_1 - 2\mathbf{w})$. Then substitution gives $T(\mathbf{v}) = \frac{1}{5}(3\mathbf{w}_1 - \mathbf{w})$.

The full effect of property (3) in Theorem 7.1.1 is this: If $T : V \to W$ is a linear transformation and $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)$ are known, then $T(\mathbf{v})$ can be computed for *every* vector $\mathbf{v}$ in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. In particular, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans V, then $T(\mathbf{v})$ is determined for all $\mathbf{v}$ in V by the choice of $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)$. The next theorem states this somewhat differently. As for functions in general, two linear transformations $T : V \to W$ and $S : V \to W$ are called **equal** (written $T = S$) if they have the same **action**; that is, if $T(\mathbf{v}) = S(\mathbf{v})$ for all $\mathbf{v}$ in V.

Theorem 7.1.2

Let $T : V \to W$ and $S : V \to W$ be two linear transformations. Suppose that $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. If $T(\mathbf{v}_i) = S(\mathbf{v}_i)$ for each i, then $T = S$.

Proof. If $\mathbf{v}$ is any vector in $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, write $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where each a_i is in $\mathbb{R}$. Since $T(\mathbf{v}_i) = S(\mathbf{v}_i)$ for each i, Theorem 7.1.1 gives

$$\begin{aligned} T(\mathbf{v}) &= T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \cdots + a_nT(\mathbf{v}_n) \\ &= a_1S(\mathbf{v}_1) + a_2S(\mathbf{v}_2) + \cdots + a_nS(\mathbf{v}_n) \\ &= S(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= S(\mathbf{v}) \end{aligned}$$

Since $\mathbf{v}$ was arbitrary in V, this shows that $T = S$. □

Example 7.1.6

Let $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Let $T : V \to W$ be a linear transformation. If $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_n) = \mathbf{0}$, show that $T = 0$, the zero transformation from V to W.

Solution. The zero transformation $0 : V \to W$ is defined by $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in V (Example 7.1.1), so $T(\mathbf{v}_i) = 0(\mathbf{v}_i)$ holds for each i. Hence $T = 0$ by Theorem 7.1.2.

Theorem 7.1.2 can be expressed as follows: If we know what a linear transformation $T : V \to W$ does to each vector in a spanning set for V, then we know what T does to *every* vector in V. If the spanning set is a basis, we can say much more.

Theorem 7.1.3

Let V and W be vector spaces and let $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a basis of V. Given any vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ in W (they need not be distinct), there exists a unique linear transformation $T : V \to W$ satisfying $T(\mathbf{b}_i) = \mathbf{w}_i$ for each $i = 1, 2, \dots, n$. In fact, the action of T is as follows:
Given $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n$ in V, v_i in $\mathbb{R}$, then

$$T(\mathbf{v}) = T(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{w}_1 + v_2\mathbf{w}_2 + \cdots + v_n\mathbf{w}_n.$$

Proof. If a transformation T *does* exist with $T(\mathbf{b}_i) = \mathbf{w}_i$ for each i, and if S is any other such transformation, then $T(\mathbf{b}_i) = \mathbf{w}_i = S(\mathbf{b}_i)$ holds for each i, so $S = T$ by Theorem 7.1.2. Hence T is unique if it exists, and it remains to show that there really is such a linear transformation. Given $\mathbf{v}$ in V, we must specify $T(\mathbf{v})$ in W. Because $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of V, we have $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$, where $v_1, \dots, v_n$ are *uniquely* determined by $\mathbf{v}$ (this is Theorem 6.3.1). Hence we may define $T : V \to W$ by

$$T(\mathbf{v}) = T(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{w}_1 + v_2\mathbf{w}_2 + \cdots + v_n\mathbf{w}_n$$

for all $\mathbf{v} = v_1\mathbf{b}_1 + \cdots + v_n\mathbf{b}_n$ in V. This satisfies $T(\mathbf{b}_i) = \mathbf{w}_i$ for each i; the verification that T is linear is left to the reader. □

This theorem shows that linear transformations can be defined almost at will: Simply specify where the basis vectors go, and the rest of the action is dictated by the linearity. Moreover, Theorem 7.1.2 shows that deciding whether two linear transformations are equal comes down to determining whether they have the same effect on the basis vectors. So, given a basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space V, there is a different linear transformation $V \to W$ for every ordered selection $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ of vectors in W (not necessarily distinct).

Example 7.1.7

Find a linear transformation $T : \mathbf{P}_2 \to \mathbf{M}_{22}$ such that

$$T(1+x) = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \quad T(x+x^2) = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \quad \text{and} \quad T(1+x^2) = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right].$$

Solution. The set $\{1+x,\ x+x^2,\ 1+x^2\}$ is a basis of $\mathbf{P}_2$, so every vector $p = a+bx+cx^2$ in $\mathbf{P}_2$ is a linear combination of these vectors. In fact

$$p(x) = \tfrac{1}{2}(a+b-c)(1+x) + \tfrac{1}{2}(-a+b+c)(x+x^2) + \tfrac{1}{2}(a-b+c)(1+x^2)$$

Hence Theorem 7.1.3 gives

$$\begin{aligned} T[p(x)] &= \tfrac{1}{2}(a+b-c)\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + \tfrac{1}{2}(-a+b+c)\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] + \tfrac{1}{2}(a-b+c)\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] \\ &= \tfrac{1}{2}\left[\begin{array}{cc} a+b-c & -a+b+c \\ -a+b+c & a-b+c \end{array}\right] \end{aligned}$$

Exercises for 7.1

Exercise 7.1.1 Show that each of the following functions is a linear transformation.

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x,\ y) = (x,\ -y)$ (reflection in the x axis)

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x,\ y,\ z) = (x,\ y,\ -z)$ (reflection in the x-y plane)

c. $T : \mathbb{C} \to \mathbb{C}$; $T(z) = \overline{z}$ (conjugation)

d. $T : \mathbf{M}_{mn} \to \mathbf{M}_{kl}$; $T(A) = PAQ$, P a $k \times m$ matrix, Q an $n \times l$ matrix, both fixed

e. $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$; $T(A) = A^T + A$

f. $T : \mathbf{P}_n \to \mathbb{R}$; $T[p(x)] = p(0)$

g. $T : \mathbf{P}_n \to \mathbb{R}$; $T(r_0 + r_1x + \cdots + r_nx^n) = r_n$

h. $T : \mathbb{R}^n \to \mathbb{R}$; $T(\mathbf{x}) = \mathbf{x} \cdot \mathbf{z}$, $\mathbf{z}$ a fixed vector in $\mathbb{R}^n$

i. $T : \mathbf{P}_n \to \mathbf{P}_n$; $T[p(x)] = p(x+1)$

j. $T : \mathbb{R}^n \to V$; $T(r_1,\ \cdots,\ r_n) = r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n$ where $\{\mathbf{e}_1,\ \dots,\ \mathbf{e}_n\}$ is a fixed basis of V

k. $T : V \to \mathbb{R}$; $T(r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n) = r_1$, where $\{\mathbf{e}_1,\ \dots,\ \mathbf{e}_n\}$ is a fixed basis of V

---

b. $T(\mathbf{v}) = \mathbf{v}A$ where $A = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$

d. $T(A+B) = P(A+B)Q = PAQ + PBQ = T(A) + T(B)$; $T(rA) = P(rA)Q = rPAQ = rT(A)$

f. $T[(p+q)(x)] = (p+q)(0) = p(0) + q(0) = T[p(x)] + T[q(x)]$;
$T[(rp)(x)] = (rp)(0) = r(p(0)) = rT[p(x)]$

h. $T(X+Y) = (X+Y) \cdot Z = X \cdot Z + Y \cdot Z = T(X) + T(Y)$, and $T(rX) = (rX) \cdot Z = r(X \cdot Z) = rT(X)$

j. If $\mathbf{v} = (v_1, \dots, v_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$, then $T(\mathbf{v} + \mathbf{w}) = (v_1 + w_1)\mathbf{e}_1 + \cdots + (v_n + w_n)\mathbf{e}_n = (v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) + (w_1\mathbf{e}_1 + \cdots + w_n\mathbf{e}_n) = T(\mathbf{v}) + T(\mathbf{w})$
$T(a\mathbf{v}) = (av_1)\mathbf{e} + \cdots + (av_n)\mathbf{e}_n = a(v\mathbf{e} + \cdots + v_n\mathbf{e}_n) = aT(\mathbf{v})$

Exercise 7.1.2 In each case, show that T is *not* a linear transformation.

a. $T : \mathbf{M}_{nn} \to \mathbb{R}$; $T(A) = \det A$

b. $T : \mathbf{M}_{nm} \to \mathbb{R}$; $T(A) = \operatorname{rank} A$

c. $T : \mathbb{R} \to \mathbb{R}$; $T(x) = x^2$

d. $T : V \to V$; $T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$ where $\mathbf{u} \neq \mathbf{0}$ is a fixed vector in V (T is called the **translation** by $\mathbf{u}$)

---

b. $\operatorname{rank}(A+B) \neq \operatorname{rank} A + \operatorname{rank} B$ in general. For example, $A = \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right]$ and $B = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$.

d. $T(\mathbf{0}) = \mathbf{0} + \mathbf{u} = \mathbf{u} \neq \mathbf{0}$, so T is not linear by Theorem 7.1.1.

Exercise 7.1.3 In each case, assume that T is a linear transformation.

a. If $T : V \to \mathbb{R}$ and $T(\mathbf{v}_1) = 1$, $T(\mathbf{v}_2) = -1$, find $T(3\mathbf{v}_1 - 5\mathbf{v}_2)$.

b. If $T : V \to \mathbb{R}$ and $T(\mathbf{v}_1) = 2$, $T(\mathbf{v}_2) = -3$, find $T(3\mathbf{v}_1 + 2\mathbf{v}_2)$.

c. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T\left[\begin{array}{r} 1 \\ 3 \end{array}\right] = \left[\begin{array}{r} 1 \\ 1 \end{array}\right]$, $T\left[\begin{array}{r} 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 0 \\ 1 \end{array}\right]$, find $T\left[\begin{array}{r} -1 \\ 3 \end{array}\right]$.

d. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T\left[\begin{array}{r} 1 \\ -1 \end{array}\right] = \left[\begin{array}{r} 0 \\ 1 \end{array}\right]$, $T\left[\begin{array}{r} 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 1 \\ 0 \end{array}\right]$, find $T\left[\begin{array}{r} 1 \\ -7 \end{array}\right]$.

e. If $T : \mathbf{P}_2 \to \mathbf{P}_2$ and $T(x+1) = x$, $T(x-1) = 1$, $T(x^2) = 0$, find $T(2 + 3x - x^2)$.

f. If $T : \mathbf{P}_2 \to \mathbb{R}$ and $T(x+2) = 1$, $T(1) = 5$, $T(x^2 + x) = 0$, find $T(2 - x + 3x^2)$.

---

b. $T(3\mathbf{v}_1 + 2\mathbf{v}_2) = 0$

d. $T\left[\begin{array}{r} 1 \\ -7 \end{array}\right] = \left[\begin{array}{r} -3 \\ 4 \end{array}\right]$

f. $T(2 - x + 3x^2) = 46$

Exercise 7.1.4 In each case, find a linear transformation with the given properties and compute $T(\mathbf{v})$.

a. $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T(1, 2) = (1, 0, 1)$, $T(-1, 0) = (0, 1, 1)$; $\mathbf{v} = (2, 1)$

b. $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T(2, -1) = (1, -1, 1)$, $T(1, 1) = (0, 1, 0)$; $\mathbf{v} = (-1, 2)$

c. $T : \mathbf{P}_2 \to \mathbf{P}_3$; $T(x^2) = x^3$, $T(x+1) = 0$, $T(x-1) = x$; $\mathbf{v} = x^2 + x + 1$

d. $T : \mathbf{M}_{22} \to \mathbb{R}$; $T\left[\begin{array}{rr} 1 & 0 \\ 0 & 0 \end{array}\right] = 3$, $T\left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right] = -1$, $T\left[\begin{array}{rr} 1 & 0 \\ 1 & 0 \end{array}\right] = 0 = T\left[\begin{array}{rr} 0 & 0 \\ 0 & 1 \end{array}\right]$; $\mathbf{v} = \left[\begin{array}{rr} a & b \\ c & d \end{array}\right]$

---

b. $T(x, y) = \frac{1}{3}(x - y, 3y, x - y)$; $T(-1, 2) = (-1, 2, -1)$

d. $T\left[\begin{array}{rr} a & b \\ c & d \end{array}\right] = 3a - 3c + 2b$

Exercise 7.1.5 If $T : V \to V$ is a linear transformation, find $T(\mathbf{v})$ and $T(\mathbf{w})$ if:

a. $T(\mathbf{v}+\mathbf{w}) = \mathbf{v}-2\mathbf{w}$ and $T(2\mathbf{v}-\mathbf{w}) = 2\mathbf{v}$

b. $T(\mathbf{v}+2\mathbf{w}) = 3\mathbf{v}-\mathbf{w}$ and $T(\mathbf{v}-\mathbf{w}) = 2\mathbf{v}-4\mathbf{w}$

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b. $T(\mathbf{v}) = \frac{1}{3}(7\mathbf{v}-9\mathbf{w})$, $T(\mathbf{w}) = \frac{1}{3}(\mathbf{v}+3\mathbf{w})$

Exercise 7.1.6 If $T : V \to W$ is a linear transformation, show that $T(\mathbf{v}-\mathbf{v}_1) = T(\mathbf{v}) - T(\mathbf{v}_1)$ for all $\mathbf{v}$ and $\mathbf{v}_1$ in V.

Exercise 7.1.7 Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of $\mathbb{R}^2$. Is it possible to have a linear transformation T such that $T(\mathbf{e}_1)$ lies in $\mathbb{R}$ while $T(\mathbf{e}_2)$ lies in $\mathbb{R}^2$? Explain your answer.

Exercise 7.1.8 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V and let $T : V \to V$ be a linear transformation.

a. If $T(\mathbf{v}_i) = \mathbf{v}_i$ for each i, show that $T = 1_V$.

b. If $T(\mathbf{v}_i) = -\mathbf{v}_i$ for each i, show that $T = -1$ is the scalar operator (see Example 7.1.1).

---

b. $T(\mathbf{v}) = (-1)\mathbf{v}$ for all $\mathbf{v}$ in V, so T is the scalar operator -1.

Exercise 7.1.9 If A is an $m \times n$ matrix, let $C_k(A)$ denote column k of A. Show that $C_k : \mathbf{M}_{mn} \to \mathbb{R}^m$ is a linear transformation for each $k = 1, \dots, n$.

Exercise 7.1.10 Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis of $\mathbb{R}^n$. Given k, $1 \le k \le n$, define $P_k : \mathbb{R}^n \to \mathbb{R}^n$ by $P_k(r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n) = r_k\mathbf{e}_k$. Show that P_k a linear transformation for each k.

Exercise 7.1.11 Let $S : V \to W$ and $T : V \to W$ be linear transformations. Given a in $\mathbb{R}$, define functions $(S+T) : V \to W$ and $(aT) : V \to W$ by $(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$ and $(aT)(\mathbf{v}) = aT(\mathbf{v})$ for all $\mathbf{v}$ in V. Show that $S+T$ and aT are linear transformations.

Exercise 7.1.12 Describe all linear transformations $T : \mathbb{R} \to V$.

If $T(1) = \mathbf{v}$, then $T(r) = T(r \cdot 1) = rT(1) = r\mathbf{v}$ for all r in $\mathbb{R}$.

Exercise 7.1.13 Let V and W be vector spaces, let V be finite dimensional, and let $\mathbf{v} \neq \mathbf{0}$ in V. Given any $\mathbf{w}$ in W, show that there exists a linear transformation $T : V \to W$ with $T(\mathbf{v}) = \mathbf{w}$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

Exercise 7.1.14 Given $\mathbf{y}$ in $\mathbb{R}^n$, define $S_\mathbf{y} : \mathbb{R}^n \to \mathbb{R}$ by $S_\mathbf{y}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ (where $\cdot$ is the dot product introduced in Section 5.3).

a. Show that $S_\mathbf{y} : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation for any $\mathbf{y}$ in $\mathbb{R}^n$.

b. Show that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ arises in this way; that is, $T = S_\mathbf{y}$ for some $\mathbf{y}$ in $\mathbb{R}^n$. [*Hint*: If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, write $S_\mathbf{y}(\mathbf{e}_i) = y_i$ for each i. Use Theorem 7.1.1.]

Exercise 7.1.15 Let $T : V \to W$ be a linear transformation.

a. If U is a subspace of V, show that $T(U) = \{T(\mathbf{u}) \mid \mathbf{u} \text{ in } U\}$ is a subspace of W (called the **image** of U under T).

b. If P is a subspace of W, show that $\{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) \text{ in } P\}$ is a subspace of V (called the **preimage** of P under T).

---

b. $\mathbf{0}$ is in $U = \{\mathbf{v} \in V \mid T(\mathbf{v}) \in P\}$ because $T(\mathbf{0}) = \mathbf{0}$ is in P. If $\mathbf{v}$ and $\mathbf{w}$ are in U, then $T(\mathbf{v})$ and $T(\mathbf{w})$ are in P. Hence $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ is in P and $T(r\mathbf{v}) = rT(\mathbf{v})$ is in P, so $\mathbf{v}+\mathbf{w}$ and $r\mathbf{v}$ are in U.

Exercise 7.1.16 Show that differentiation is the only linear transformation $\mathbf{P}_n \to \mathbf{P}_n$ that satisfies $T(x^k) = kx^{k-1}$ for each $k = 0, 1, 2, \dots, n$.

Exercise 7.1.17 Let $T : V \to W$ be a linear transformation and let $\mathbf{v}_1, \dots, \mathbf{v}_n$ denote vectors in V.

a. If $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is linearly independent, show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is also independent.

b. Find $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which the converse of part (a) is false.

Exercise 7.1.18 Suppose $T : V \to V$ is a linear operator with the property that $T[T(\mathbf{v})] = \mathbf{v}$ for all $\mathbf{v}$ in V. (For example, transposition in $\mathbf{M}_{nn}$ or conjugation in $\mathbb{C}$.) If $\mathbf{v} \neq \mathbf{0}$ in V, show that $\{\mathbf{v},\ T(\mathbf{v})\}$ is linearly independent if and only if $T(\mathbf{v}) \neq \mathbf{v}$ and $T(\mathbf{v}) \neq -\mathbf{v}$.

Suppose $r\mathbf{v} + sT(\mathbf{v}) = \mathbf{0}$. If $s = 0$, then $r = 0$ (because $\mathbf{v} \neq \mathbf{0}$). If $s \neq 0$, then $T(\mathbf{v}) = a\mathbf{v}$ where $a = -s^{-1}r$. Thus $\mathbf{v} = T^2(\mathbf{v}) = T(a\mathbf{v}) = a^2\mathbf{v}$, so $a^2 = 1$, again because $\mathbf{v} \neq \mathbf{0}$. Hence $a = \pm 1$. Conversely, if $T(\mathbf{v}) = \pm\mathbf{v}$, then $\{\mathbf{v},\ T(\mathbf{v})\}$ is certainly not independent.

Exercise 7.1.19 If a and b are real numbers, define $T_{a,\ b} : \mathbb{C} \to \mathbb{C}$ by $T_{a,\ b}(r + si) = ra + sbi$ for all $r + si$ in $\mathbb{C}$.

a. Show that $T_{a,\ b}$ is linear and $T_{a,\ b}(\overline{z}) = \overline{T_{a,\ b}(z)}$ for all z in $\mathbb{C}$. (Here $\overline{z}$ denotes the conjugate of z.)

b. If $T : \mathbb{C} \to \mathbb{C}$ is linear and $T(\overline{z}) = \overline{T(z)}$ for all z in $\mathbb{C}$, show that $T = T_{a,\ b}$ for some real a and b.

Exercise 7.1.20 Show that the following conditions are equivalent for a linear transformation $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$.

1. $\text{tr}\,[T(A)] = \text{tr}\,A$ for all A in $\mathbf{M}_{22}$.

2. $T\left[\begin{array}{cc} r_{11} & r_{12} \\ r_{21} & r_{22} \end{array}\right] = r_{11}B_{11} + r_{12}B_{12} + r_{21}B_{21} + r_{22}B_{22}$ for matrices B_{ij} such that $\text{tr}\,B_{11} = 1 = \text{tr}\,B_{22}$ and $\text{tr}\,B_{12} = 0 = \text{tr}\,B_{21}$.

Exercise 7.1.21 Given a in $\mathbb{R}$, consider the **evaluation** map $E_a : \mathbf{P}_n \to \mathbb{R}$ defined in Example 7.1.3.

a. Show that E_a is a linear transformation satisfying the additional condition that $E_a(x^k) = [E_a(x)]^k$ holds for all $k = 0,\ 1,\ 2,\ \dots$. [*Note*: $x^0 = 1$.]

b. If $T : \mathbf{P}_n \to \mathbb{R}$ is a linear transformation satisfying $T(x^k) = [T(x)]^k$ for all $k = 0,\ 1,\ 2,\ \dots$, show that $T = E_a$ for some a in R.

---

b. Given such a T, write $T(x) = a$. If $p = p(x) = \sum_{i=0}^{n} a_i x^i$, then $T(p) = \sum a_i T(x^i) = \sum a_i [T(x)]^i = \sum a_i a^i = p(a) = E_a(p)$. Hence $T = E_a$.

Exercise 7.1.22 If $T : \mathbf{M}_{nn} \to \mathbb{R}$ is any linear transformation satisfying $T(AB) = T(BA)$ for all A and B in $\mathbf{M}_{nn}$, show that there exists a number k such that $T(A) = k\,\text{tr}\,A$ for all A. (See Lemma 5.5.1.) [*Hint*: Let E_{ij} denote the $n \times n$ matrix with 1 in the $(i,\ j)$ position and zeros elsewhere. Show that $E_{ik}E_{lj} = \left\{\begin{array}{ll} 0 & \text{if } k \neq l \\ E_{ij} & \text{if } k = l \end{array}\right.$. Use this to show that $T(E_{ij}) = 0$ if $i \neq j$ and $T(E_{11}) = T(E_{22}) = \cdots = T(E_{nn})$. Put $k = T(E_{11})$ and use the fact that $\{E_{ij} \mid 1 \leq i,\ j \leq n\}$ is a basis of $\mathbf{M}_{nn}$.]

Exercise 7.1.23 Let $T : \mathbb{C} \to \mathbb{C}$ be a linear transformation of the real vector space $\mathbb{C}$ and assume that $T(a) = a$ for every real number a. Show that the following are equivalent:

a. $T(zw) = T(z)T(w)$ for all z and w in $\mathbb{C}$.

b. Either $T = 1_{\mathbb{C}}$ or $T(z) = \overline{z}$ for each z in $\mathbb{C}$ (where $\overline{z}$ denotes the conjugate).

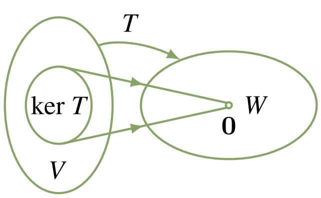
7.2 Kernel and Image of a Linear Transformation

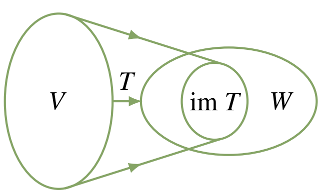
This section is devoted to two important subspaces associated with a linear transformation $T : V \to W$.

Definition 7.2 Kernel and Image of a Linear Transformation

The ***kernel*** *of* T *(denoted* $\ker T$*) and the* ***image*** *of* T *(denoted* $\text{im } T$ *or* $T(V)$*) are defined by*

$$\ker T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$$
$$\text{im } T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$$





The kernel of T is often called the **nullspace** of T because it consists of all vectors $\mathbf{v}$ in V satisfying the *condition* that $T(\mathbf{v}) = \mathbf{0}$. The image of T is often called the **range** of T and consists of all vectors $\mathbf{w}$ in W of the *form* $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. These subspaces are depicted in the diagrams.

Example 7.2.1

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by the $m \times n$ matrix A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. Then

$$\ker T_A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} = \text{null } A \quad \text{and}$$
$$\text{im } T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \text{im } A$$

Hence the following theorem extends Example 5.1.2.

Theorem 7.2.1

Let $T : V \to W$ *be a linear transformation.*

1. $\ker T$ *is a subspace of* V.
2. $\text{im } T$ *is a subspace of* W.

Proof. The fact that $T(\mathbf{0}) = \mathbf{0}$ shows that $\ker T$ and $\text{im } T$ contain the zero vector of V and W respectively.

1. If $\mathbf{v}$ and $\mathbf{v}_1$ lie in $\ker T$, then $T(\mathbf{v}) = \mathbf{0} = T(\mathbf{v}_1)$, so

$$T(\mathbf{v} + \mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
$$T(r\mathbf{v}) = rT(\mathbf{v}) = r\mathbf{0} = \mathbf{0} \quad \text{for all } r \text{ in } \mathbb{R}$$

Hence $\mathbf{v}+\mathbf{v}_1$ and $r\mathbf{v}$ lie in $\ker T$ (they satisfy the required condition), so $\ker T$ is a subspace of V by the subspace test (Theorem 6.2.1).

2. If $\mathbf{w}$ and $\mathbf{w}_1$ lie in $\operatorname{im} T$, write $\mathbf{w}=T(\mathbf{v})$ and $\mathbf{w}_1=T(\mathbf{v}_1)$ where $\mathbf{v},\ \mathbf{v}_1 \in V$. Then

$$\begin{aligned} \mathbf{w}+\mathbf{w}_1 &= T(\mathbf{v})+T(\mathbf{v}_1)=T(\mathbf{v}+\mathbf{v}_1) \\ r\mathbf{w} &= rT(\mathbf{v})=T(r\mathbf{v}) \quad \text{for all } r \text{ in } \mathbb{R} \end{aligned}$$

Hence $\mathbf{w}+\mathbf{w}_1$ and $r\mathbf{w}$ both lie in $\operatorname{im} T$ (they have the required form), so $\operatorname{im} T$ is a subspace of W.

□

Given a linear transformation $T: V \to W$:

$\dim(\ker T)$ is called the **nullity** of T and denoted as $\operatorname{nullity}(T)$
$\dim(\operatorname{im} T)$ is called the **rank** of T and denoted as $\operatorname{rank}(T)$

The rank of a matrix A was defined earlier to be the dimension of $\operatorname{col} A$, the column space of A. The two usages of the word *rank* are consistent in the following sense. Recall the definition of T_A in Example 7.2.1.

Example 7.2.2

Given an $m \times n$ matrix A, show that $\operatorname{im} T_A = \operatorname{col} A$, so $\operatorname{rank} T_A = \operatorname{rank} A$.

Solution. Write $A = \left[\begin{array}{ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{array}\right]$ in terms of its columns. Then

$$\operatorname{im} T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \{x_1\mathbf{c}_1+\cdots+x_n\mathbf{c}_n \mid x_i \text{ in } \mathbb{R}\}$$

using Definition 2.5. Hence $\operatorname{im} T_A$ is the column space of A; the rest follows.

Often, a useful way to study a subspace of a vector space is to exhibit it as the kernel or image of a linear transformation. Here is an example.

Example 7.2.3

Define a transformation $P: \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ by $P(A) = A - A^T$ for all A in $\mathbf{M}_{nn}$. Show that P is linear and that:

a. $\ker P$ consists of all symmetric matrices.

b. $\operatorname{im} P$ consists of all skew-symmetric matrices.

Solution. The verification that P is linear is left to the reader. To prove part (a), note that a matrix A lies in $\ker P$ just when $0 = P(A) = A - A^T$, and this occurs if and only if $A = A^T$—that is, A is symmetric. Turning to part (b), the space $\operatorname{im} P$ consists of all matrices $P(A)$, A in $\mathbf{M}_{nn}$. Every such matrix is skew-symmetric because

$$P(A)^T = (A-A^T)^T = A^T - A = -P(A)$$

On the other hand, if S is skew-symmetric (that is, $S^T = -S$), then S lies in $\text{im } P$. In fact,

$$P\left[\tfrac{1}{2}S\right] = \tfrac{1}{2}S - \left[\tfrac{1}{2}S\right]^T = \tfrac{1}{2}(S - S^T) = \tfrac{1}{2}(S + S) = S$$

One-to-One and Onto Transformations

Definition 7.3 One-to-one and Onto Linear Transformations

Let $T : V \to W$ be a linear transformation.

1. *T is said to be* **onto** *if* $\text{im } T = W$.
2. *T is said to be* **one-to-one** *if $T(\mathbf{v}) = T(\mathbf{v}_1)$ implies $\mathbf{v} = \mathbf{v}_1$.*

A vector $\mathbf{w}$ in W is said to be **hit** by T if $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. Then T is onto if every vector in W is hit at least once, and T is one-to-one if no element of W gets hit twice. Clearly the onto transformations T are those for which $\text{im } T = W$ is as large a subspace of W as possible. By contrast, Theorem 7.2.2 shows that the one-to-one transformations T are the ones with $\ker T$ as *small* a subspace of V as possible.

Theorem 7.2.2

If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Proof. If T is one-to-one, let $\mathbf{v}$ be any vector in $\ker T$. Then $T(\mathbf{v}) = \mathbf{0}$, so $T(\mathbf{v}) = T(\mathbf{0})$. Hence $\mathbf{v} = \mathbf{0}$ because T is one-to-one. Hence $\ker T = \{\mathbf{0}\}$.

Conversely, assume that $\ker T = \{\mathbf{0}\}$ and let $T(\mathbf{v}) = T(\mathbf{v}_1)$ with $\mathbf{v}$ and $\mathbf{v}_1$ in V. Then $T(\mathbf{v} - \mathbf{v}_1) = T(\mathbf{v}) - T(\mathbf{v}_1) = \mathbf{0}$, so $\mathbf{v} - \mathbf{v}_1$ lies in $\ker T = \{\mathbf{0}\}$. This means that $\mathbf{v} - \mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v} = \mathbf{v}_1$, proving that T is one-to-one. □

Example 7.2.4

The identity transformation $1_V : V \to V$ is both one-to-one and onto for any vector space V.

Example 7.2.5

Consider the linear transformations

$$\begin{aligned} S : \mathbb{R}^3 \to \mathbb{R}^2 &\quad \text{given by } S(x,\ y,\ z) = (x + y,\ x - y) \\ T : \mathbb{R}^2 \to \mathbb{R}^3 &\quad \text{given by } T(x,\ y) = (x + y,\ x - y,\ x) \end{aligned}$$

Show that T is one-to-one but not onto, whereas S is onto but not one-to-one.

Solution. The verification that they are linear is omitted. T is one-to-one because

$$\ker T = \{(x,\ y) \mid x+y = x-y = x = 0\} = \{(0,\ 0)\}$$

However, it is not onto. For example $(0,\ 0,\ 1)$ does not lie in $\text{im}\, T$ because if $(0,\ 0,\ 1) = (x+y,\ x-y,\ x)$ for some x and y, then $x+y = 0 = x-y$ and $x = 1$, an impossibility. Turning to S, it is not one-to-one by Theorem 7.2.2 because $(0,\ 0,\ 1)$ lies in $\ker S$. But every element $(s,\ t)$ in $\mathbb{R}^2$ lies in $\text{im}\, S$ because $(s,\ t) = (x+y,\ x-y) = S(x,\ y,\ z)$ for some x, y, and z (in fact, $x = \frac{1}{2}(s+t)$, $y = \frac{1}{2}(s-t)$, and $z = 0$). Hence S is onto.

Example 7.2.6

Let U be an invertible $m \times m$ matrix and define

$$T : \mathbf{M}_{mn} \to \mathbf{M}_{mn} \quad \text{by} \quad T(X) = UX \text{ for all } X \text{ in } \mathbf{M}_{mn}$$

Show that T is a linear transformation that is both one-to-one and onto.

Solution. The verification that T is linear is left to the reader. To see that T is one-to-one, let $T(X) = 0$. Then $UX = 0$, so left-multiplication by U^{-1} gives $X = 0$. Hence $\ker T = \{\mathbf{0}\}$, so T is one-to-one. Finally, if Y is any member of $\mathbf{M}_{mn}$, then $U^{-1}Y$ lies in $\mathbf{M}_{mn}$ too, and $T(U^{-1}Y) = U(U^{-1}Y) = Y$. This shows that T is onto.

The linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ all have the form T_A for some $m \times n$ matrix A (Theorem 2.6.2). The next theorem gives conditions under which they are onto or one-to-one. Note the connection with Theorem 5.4.3 and Theorem 5.4.4.

Theorem 7.2.3

Let A be an $m \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$.

1. *T_A is onto if and only if $\text{rank}\, A = m$.*
2. *T_A is one-to-one if and only if $\text{rank}\, A = n$.*

Proof.

1. We have that $\text{im}\, T_A$ is the column space of A (see Example 7.2.2), so T_A is onto if and only if the column space of A is $\mathbb{R}^m$. Because the rank of A is the dimension of the column space, this holds if and only if $\text{rank}\, A = m$.
2. $\ker T_A = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, so (using Theorem 7.2.2) T_A is one-to-one if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. This is equivalent to $\text{rank}\, A = n$ by Theorem 5.4.3. □

The Dimension Theorem

Let A denote an $m \times n$ matrix of rank r and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ denote the corresponding matrix transformation given by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. It follows from Example 7.2.1 and Example 7.2.2 that $\text{im } T_A = \text{col } A$, so $\dim(\text{im } T_A) = \dim(\text{col } A) = r$. On the other hand Theorem 5.4.2 shows that $\dim(\ker T_A) = \dim(\text{null } A) = n - r$. Combining these we see that

$$\dim(\text{im } T_A) + \dim(\ker T_A) = n \quad \text{for every } m \times n \text{ matrix } A$$

The main result of this section is a deep generalization of this observation.

Theorem 7.2.4: Dimension Theorem

Let $T : V \to W$ be any linear transformation and assume that $\ker T$ and $\text{im } T$ are both finite dimensional. Then V is also finite dimensional and

$$\dim V = \dim(\ker T) + \dim(\text{im } T)$$

In other words, $\dim V = \text{nullity}(T) + \text{rank}(T)$.

Proof. Every vector in $\text{im } T = T(V)$ has the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V. Hence let $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_r)\}$ be a basis of $\text{im } T$, where the $\mathbf{e}_i$ lie in V. Let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k\}$ be any basis of $\ker T$. Then $\dim(\text{im } T) = r$ and $\dim(\ker T) = k$, so it suffices to show that $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{f}_1, \dots, \mathbf{f}_k\}$ is a basis of V.

1. *B spans V.* If $\mathbf{v}$ lies in V, then $T(\mathbf{v})$ lies in $\text{im } V$, so

$$T(\mathbf{v}) = t_1 T(\mathbf{e}_1) + t_2 T(\mathbf{e}_2) + \cdots + t_r T(\mathbf{e}_r) \quad t_i \text{ in } \mathbb{R}$$

This implies that $\mathbf{v} - t_1\mathbf{e}_1 - t_2\mathbf{e}_2 - \cdots - t_r\mathbf{e}_r$ lies in $\ker T$ and so is a linear combination of $\mathbf{f}_1, \dots, \mathbf{f}_k$. Hence $\mathbf{v}$ is a linear combination of the vectors in B.

2. B is *linearly independent.* Suppose that t_i and s_j in $\mathbb{R}$ satisfy

$$t_1\mathbf{e}_1 + \cdots + t_r\mathbf{e}_r + s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0} \tag{7.1}$$

Applying T gives $t_1 T(\mathbf{e}_1) + \cdots + t_r T(\mathbf{e}_r) = \mathbf{0}$ (because $T(\mathbf{f}_i) = \mathbf{0}$ for each i). Hence the independence of $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ yields $t_1 = \cdots = t_r = 0$. But then (7.1) becomes

$$s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0}$$

so $s_1 = \cdots = s_k = 0$ by the independence of $\{\mathbf{f}_1, \dots, \mathbf{f}_k\}$. This proves that B is linearly independent.

□

Note that the vector space V is not assumed to be finite dimensional in Theorem 7.2.4. In fact, verifying that $\ker T$ and $\text{im } T$ are both finite dimensional is often an important way to *prove* that V is finite dimensional.

Note further that $r+k=n$ in the proof so, after relabelling, we end up with a basis

$$B = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_r,\ \mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$$

of V with the property that $\{\mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$ is a basis of $\ker T$ and $\{T(\mathbf{e}_1),\ \dots,\ T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T$. In fact, if V is known in advance to be finite dimensional, then *any* basis $\{\mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$ of $\ker T$ can be extended to a basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_r,\ \mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$ of V by Theorem 6.4.1. Moreover, it turns out that, no matter how this is done, the vectors $\{T(\mathbf{e}_1),\ \dots,\ T(\mathbf{e}_r)\}$ will be a basis of $\operatorname{im} T$. This result is useful, and we record it for reference. The proof is much like that of Theorem 7.2.4 and is left as Exercise 7.2.26.

Theorem 7.2.5

Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{e}_1,\ \dots,\ \mathbf{e}_r,\ \mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$ is a basis of $\ker T$. Then $\{T(\mathbf{e}_1),\ \dots,\ T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T$, and hence $r = \operatorname{rank} T$.

The dimension theorem is one of the most useful results in all of linear algebra. It shows that if either $\dim(\ker T)$ or $\dim(\operatorname{im} T)$ can be found, then the other is automatically known. In many cases it is easier to compute one than the other, so the theorem is a real asset. The rest of this section is devoted to illustrations of this fact. The next example uses the dimension theorem to give a different proof of the first part of Theorem 5.4.2.

Example 7.2.7

Let A be an $m \times n$ matrix of $\operatorname{rank} r$. Show that the space $\operatorname{null} A$ of all solutions of the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n variables has dimension $n-r$.

Solution. The space in question is just $\ker T_A$, where $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. But $\dim(\operatorname{im} T_A) = \operatorname{rank} T_A = \operatorname{rank} A = r$ by Example 7.2.2, so $\dim(\ker T_A) = n-r$ by the dimension theorem.

Example 7.2.8

If $T : V \to W$ is a linear transformation where V is finite dimensional, then

$$\dim(\ker T) \leq \dim V \quad \text{and} \quad \dim(\operatorname{im} T) \leq \dim V$$

Indeed, $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$ by Theorem 7.2.4. Of course, the first inequality also follows because $\ker T$ is a subspace of V.

Example 7.2.9

Let $D : \mathbf{P}_n \to \mathbf{P}_{n-1}$ be the differentiation map defined by $D[p(x)] = p'(x)$. Compute $\ker D$ and hence conclude that D is onto.

Solution. Because $p'(x) = 0$ means $p(x)$ is constant, we have $\dim(\ker D) = 1$. Since $\dim \mathbf{P}_n = n+1$, the dimension theorem gives

$$\dim(\operatorname{im} D) = (n+1) - \dim(\ker D) = n = \dim(\mathbf{P}_{n-1})$$

This implies that $\operatorname{im} D = \mathbf{P}_{n-1}$, so D is onto.

Of course it is not difficult to verify directly that each polynomial $q(x)$ in $\mathbf{P}_{n-1}$ is the derivative of some polynomial in $\mathbf{P}_n$ (simply integrate $q(x)$!), so the dimension theorem is not needed in this case. However, in some situations it is difficult to see directly that a linear transformation is onto, and the method used in Example 7.2.9 may be by far the easiest way to prove it. Here is another illustration.

Example 7.2.10

Given a in $\mathbb{R}$, the evaluation map $E_a : \mathbf{P}_n \to \mathbb{R}$ is given by $E_a[p(x)] = p(a)$. Show that E_a is linear and onto, and hence conclude that $\{(x-a), (x-a)^2, \dots, (x-a)^n\}$ is a basis of $\ker E_a$, the subspace of all polynomials $p(x)$ for which $p(a) = 0$.

Solution. E_a is linear by Example 7.1.3; the verification that it is onto is left to the reader. Hence $\dim(\operatorname{im} E_a) = \dim(\mathbb{R}) = 1$, so $\dim(\ker E_a) = (n+1) - 1 = n$ by the dimension theorem. Now each of the n polynomials $(x-a), (x-a)^2, \dots, (x-a)^n$ clearly lies in $\ker E_a$, and they are linearly independent (they have distinct degrees). Hence they are a basis because $\dim(\ker E_a) = n$.

We conclude by applying the dimension theorem to the rank of a matrix.

Example 7.2.11

If A is any $m \times n$ matrix, show that $\operatorname{rank} A = \operatorname{rank} A^T A = \operatorname{rank} AA^T$.

Solution. It suffices to show that $\operatorname{rank} A = \operatorname{rank} A^T A$ (the rest follows by replacing A with A^T). Write $B = A^T A$, and consider the associated matrix transformations

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{and} \quad T_B : \mathbb{R}^n \to \mathbb{R}^n$$

The dimension theorem and Example 7.2.2 give

$$\begin{aligned} \operatorname{rank} A &= \operatorname{rank} T_A = \dim(\operatorname{im} T_A) = n - \dim(\ker T_A) \\ \operatorname{rank} B &= \operatorname{rank} T_B = \dim(\operatorname{im} T_B) = n - \dim(\ker T_B) \end{aligned}$$

so it suffices to show that $\ker T_A = \ker T_B$. Now $A\mathbf{x} = \mathbf{0}$ implies that $B\mathbf{x} = A^T A\mathbf{x} = \mathbf{0}$, so $\ker T_A$ is contained in $\ker T_B$. On the other hand, if $B\mathbf{x} = \mathbf{0}$, then $A^T A\mathbf{x} = \mathbf{0}$, so

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^T (A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T \mathbf{0} = 0$$

This implies that $A\mathbf{x} = \mathbf{0}$, so $\ker T_B$ is contained in $\ker T_A$.

Exercises for 7.2

Exercise 7.2.1 For each matrix A, find a basis for the kernel and image of T_A, and find the rank and nullity of T_A.

a) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & -4 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \\ 0 & 3 & -6 \end{bmatrix}$

---

b. $\left\{ \begin{bmatrix} -3 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$; 2, 2

d. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}; \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\}$; 2, 1

Exercise 7.2.2 In each case, (i) find a basis of $\ker T$, and (ii) find a basis of $\operatorname{im} T$. You may assume that T is linear.

a. $T : \mathbf{P}_2 \to \mathbb{R}^2$; $T(a+bx+cx^2) = (a, b)$

b. $T : \mathbf{P}_2 \to \mathbb{R}^2$; $T(p(x)) = (p(0), p(1))$

c. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x+y, x+y, 0)$

d. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x, x, y, y)$

e. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix}$

f. $T : \mathbf{M}_{22} \to \mathbb{R}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+d$

g. $T : \mathbf{P}_n \to \mathbb{R}$; $T(r_0 + r_1 x + \cdots + r_n x^n) = r_n$

h. $T : \mathbb{R}^n \to \mathbb{R}$; $T(r_1, r_2, \dots, r_n) = r_1 + r_2 + \cdots + r_n$

i. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = XA - AX$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

j. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = XA$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

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b. $\{x^2 - x\}$; $\{(1, 0), (0, 1)\}$

d. $\{(0, 0, 1)\}$; $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$

f. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$; $\{1\}$

h. $\{(1, 0, 0, \dots, 0, -1), (0, 1, 0, \dots, 0, -1), \dots, (0, 0, 0, \dots, 1, -1)\}$; $\{1\}$

j. $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$; $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

Exercise 7.2.3 Let $P : V \to \mathbb{R}$ and $Q : V \to \mathbb{R}$ be linear transformations, where V is a vector space. Define $T : V \to \mathbb{R}^2$ by $T(\mathbf{v}) = (P(\mathbf{v}), Q(\mathbf{v}))$.

a. Show that T is a linear transformation.

b. Show that $\ker T = \ker P \cap \ker Q$, the set of vectors in both $\ker P$ and $\ker Q$.

---

b. $T(\mathbf{v}) = \mathbf{0} = (0, 0)$ if and only if $P(\mathbf{v}) = 0$ and $Q(\mathbf{v}) = 0$; that is, if and only if $\mathbf{v}$ is in $\ker P \cap \ker Q$.

Exercise 7.2.4 In each case, find a basis $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$, and verify Theorem 7.2.5.

a. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x - y + 2z, x + y - z, 2x + z, 2y - 3z)$

b. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x + y + z, 2x - y + 3z, z - 3y, 3x + 4z)$

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b. $\ker T = \text{span}\{(-4,\ 1,\ 3)\}$; $B = \{(1,\ 0,\ 0),\ (0,\ 1,\ 0),\ (-4,\ 1,\ 3)\}$, $\text{im}\, T = \text{span}\{(1,\ 2,\ 0,\ 3),\ (1,\ -1,\ -3,\ 0)\}$

Exercise 7.2.5 Show that every matrix X in $\mathbf{M}_{nn}$ has the form $X = A^T - 2A$ for some matrix A in $\mathbf{M}_{nn}$. [*Hint*: The dimension theorem.]

Exercise 7.2.6 In each case either prove the statement or give an example in which it is false. Throughout, let $T : V \to W$ be a linear transformation where V and W are finite dimensional.

a. If $V = W$, then $\ker T \subseteq \text{im}\, T$.

b. If $\dim V = 5$, $\dim W = 3$, and $\dim(\ker T) = 2$, then T is onto.

c. If $\dim V = 5$ and $\dim W = 4$, then $\ker T \neq \{\mathbf{0}\}$.

d. If $\ker T = V$, then $W = \{\mathbf{0}\}$.

e. If $W = \{\mathbf{0}\}$, then $\ker T = V$.

f. If $W = V$, and $\text{im}\, T \subseteq \ker T$, then $T = 0$.

g. If $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ is a basis of V and $T(\mathbf{e}_1) = \mathbf{0} = T(\mathbf{e}_2)$, then $\dim(\text{im}\, T) \leq 1$.

h. If $\dim(\ker T) \leq \dim W$, then $\dim W \geq \frac{1}{2}\dim V$.

i. If T is one-to-one, then $\dim V \leq \dim W$.

j. If $\dim V \leq \dim W$, then T is one-to-one.

k. If T is onto, then $\dim V \geq \dim W$.

l. If $\dim V \geq \dim W$, then T is onto.

m. If $\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_k)\}$ is independent, then $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_k\}$ is independent.

n. If $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_k\}$ spans V, then $\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_k)\}$ spans W.

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b. Yes. $\dim(\text{im}\, T) = 5 - \dim(\ker T) = 3$, so $\text{im}\, T = W$ as $\dim W = 3$.

d. No. $T = 0 : \mathbb{R}^2 \to \mathbb{R}^2$

f. No. $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,\ y) = (y,\ 0)$. Then $\ker T = \text{im}\, T$

h. Yes. $\dim V = \dim(\ker T) + \dim(\text{im}\, T) \leq \dim W + \dim W = 2\dim W$

j. No. Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x,\ y) = (y,\ 0)$.

l. No. Same example as **(j)**.

n. No. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,\ y) = (x,\ 0)$. If $\mathbf{v}_1 = (1,\ 0)$ and $\mathbf{v}_2 = (0,\ 1)$, then $\mathbb{R}^2 = \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2\}$ but $\mathbb{R}^2 \neq \text{span}\{T(\mathbf{v}_1),\ T(\mathbf{v}_2)\}$.

Exercise 7.2.7 Show that linear independence is preserved by one-to-one transformations and that spanning sets are preserved by onto transformations. More precisely, if $T : V \to W$ is a linear transformation, show that:

a. If T is one-to-one and $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ is independent in V, then $\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_n)\}$ is independent in W.

b. If T is onto and $V = \text{span}\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$, then $W = \text{span}\{T(\mathbf{v}_1),\ \dots,\ T(\mathbf{v}_n)\}$.

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b. Given $\mathbf{w}$ in W, let $\mathbf{w} = T(\mathbf{v})$, $\mathbf{v}$ in V, and write $\mathbf{v} = r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n$. Then $\mathbf{w} = T(\mathbf{v}) = r_1T(\mathbf{v}_1) + \cdots + r_nT(\mathbf{v}_n)$.

Exercise 7.2.8 Given $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ in a vector space V, define $T : \mathbb{R}^n \to V$ by $T(r_1,\ \dots,\ r_n) = r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n$. Show that T is linear, and that:

a. T is one-to-one if and only if $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$ is independent.

b. T is onto if and only if $V = \text{span}\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\}$.

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b. $\text{im}\, T = \{\sum_i r_i\mathbf{v}_i \mid r_i \text{ in } \mathbb{R}\} = \text{span}\{\mathbf{v}_i\}$.

Exercise 7.2.9 Let $T : V \to V$ be a linear transformation where V is finite dimensional. Show that exactly one of (i) and (ii) holds: (i) $T(\mathbf{v}) = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$ in V; (ii) $T(\mathbf{x}) = \mathbf{v}$ has a solution $\mathbf{x}$ in V for every $\mathbf{v}$ in V.

Exercise 7.2.10 Let $T : \mathbf{M}_{nn} \to \mathbb{R}$ denote the trace map: $T(A) = \text{tr}\, A$ for all A in $\mathbf{M}_{nn}$. Show that $\dim(\ker T) = n^2 - 1$.

T is linear and onto. Hence $1 = \dim \mathbb{R} = \dim(\text{im}\, T) = \dim(\mathbf{M}_{nn}) - \dim(\ker T) = n^2 - \dim(\ker T)$.

Exercise 7.2.11 Show that the following are equivalent for a linear transformation $T : V \to W$.

1. $\ker T = V$
2. $\text{im}\, T = \{\mathbf{0}\}$
3. $T = 0$

Exercise 7.2.12 Let A and B be $m \times n$ and $k \times n$ matrices, respectively. Assume that $A\mathbf{x} = \mathbf{0}$ implies $B\mathbf{x} = \mathbf{0}$ for every n-column $\mathbf{x}$. Show that $\text{rank}\, A \geq \text{rank}\, B$.
[*Hint*: Theorem 7.2.4.]

The condition means $\ker(T_A) \subseteq \ker(T_B)$, so $\dim[\ker(T_A)] \leq \dim[\ker(T_B)]$. Then Theorem 7.2.4 gives $\dim[\text{im}(T_A)] \geq \dim[\text{im}(T_B)]$; that is, $\text{rank}\, A \geq \text{rank}\, B$.

Exercise 7.2.13 Let A be an $m \times n$ matrix of rank r. Thinking of $\mathbb{R}^n$ as rows, define $V = \{\mathbf{x} \text{ in } \mathbb{R}^m \mid \mathbf{x}A = \mathbf{0}\}$. Show that $\dim V = m - r$.

Exercise 7.2.14 Consider

$$V = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \middle| a + c = b + d \right\}$$

a. Consider $S : \mathbf{M}_{22} \to \mathbb{R}$ with $S \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = a + c - b - d$. Show that S is linear and onto and that V is a subspace of $\mathbf{M}_{22}$. Compute $\dim V$.

b. Consider $T : V \to \mathbb{R}$ with $T \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = a + c$. Show that T is linear and onto, and use this information to compute $\dim(\ker T)$.

Exercise 7.2.15 Define $T : \mathbf{P}_n \to \mathbb{R}$ by $T[p(x)] =$ the sum of all the coefficients of $p(x)$.

a. Use the dimension theorem to show that $\dim(\ker T) = n$.

b. Conclude that $\{x - 1,\ x^2 - 1,\ \dots,\ x^n - 1\}$ is a basis of $\ker T$.

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b. $B = \{x - 1,\ \dots,\ x^n - 1\}$ is independent (distinct degrees) and contained in $\ker T$. Hence B is a basis of $\ker T$ by **(a)**.

Exercise 7.2.16 Use the dimension theorem to prove Theorem 1.3.1: If A is an $m \times n$ matrix with $m < n$, the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n variables always has a nontrivial solution.

Exercise 7.2.17 Let B be an $n \times n$ matrix, and consider the subspaces $U = \{A \mid A \text{ in } \mathbf{M}_{mn},\ AB = 0\}$ and $V = \{AB \mid A \text{ in } \mathbf{M}_{mn}\}$. Show that $\dim U + \dim V = mn$.

Exercise 7.2.18 Let U and V denote, respectively, the spaces of even and odd polynomials in $\mathbf{P}_n$. Show that $\dim U + \dim V = n + 1$. [*Hint*: Consider $T : \mathbf{P}_n \to \mathbf{P}_n$ where $T[p(x)] = p(x) - p(-x)$.]

Exercise 7.2.19 Show that every polynomial $f(x)$ in $\mathbf{P}_{n-1}$ can be written as $f(x) = p(x+1) - p(x)$ for some polynomial $p(x)$ in $\mathbf{P}_n$. [*Hint*: Define $T : \mathbf{P}_n \to \mathbf{P}_{n-1}$ by $T[p(x)] = p(x+1) - p(x)$.]

Exercise 7.2.20 Let U and V denote the spaces of symmetric and skew-symmetric $n \times n$ matrices. Show that $\dim U + \dim V = n^2$.

Define $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ by $T(A) = A - A^T$ for all A in $\mathbf{M}_{nn}$. Then $\ker T = U$ and $\text{im}\, T = V$ by Example 7.2.3, so the dimension theorem gives $n^2 = \dim \mathbf{M}_{nn} = \dim(U) + \dim(V)$.

Exercise 7.2.21 Assume that B in $\mathbf{M}_{nn}$ satisfies $B^k = 0$ for some $k \geq 1$. Show that every matrix in $\mathbf{M}_{nn}$ has the form $BA - A$ for some A in $\mathbf{M}_{nn}$. [*Hint*: Show that $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ is linear and one-to-one where
$T(A) = BA - A$ for each A.]

Exercise 7.2.22 Fix a column $\mathbf{y} \neq \mathbf{0}$ in $\mathbb{R}^n$ and let $U = \{A \text{ in } \mathbf{M}_{nn} \mid A\mathbf{y} = \mathbf{0}\}$. Show that $\dim U = n(n-1)$.

Define $T : \mathbf{M}_{nn} \to \mathbb{R}^n$ by $T(A) = A\mathbf{y}$ for all A in $\mathbf{M}_{nn}$. Then T is linear with $\ker T = U$, so it is enough to show that T is onto (then $\dim U = n^2 - \dim(\text{im}\, T) = n^2 - n$). We have $T(0) = \mathbf{0}$. Let $\mathbf{y} = \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \right]^T \neq \mathbf{0}$ in $\mathbb{R}^n$. If $y_k \neq \mathbf{0}$

let $\mathbf{c}_k = y_k^{-1}\mathbf{y}$, and let $\mathbf{c}_j = \mathbf{0}$ if $j \neq k$. If $A = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array} \right]$, then $T(A) = A\mathbf{y} = y_1\mathbf{c}_1 + \cdots + y_k\mathbf{c}_k + \cdots + y_n\mathbf{c}_n = \mathbf{y}$. This shows that T is onto, as required.

Exercise 7.2.23 If B in $\mathbf{M}_{mn}$ has rank r, let $U = \{A$ in $\mathbf{M}_{nn} \mid BA = 0\}$ and $W = \{BA \mid A$ in $\mathbf{M}_{nn}\}$. Show that $\dim U = n(n-r)$ and $\dim W = nr$. [*Hint*: Show that U consists of all matrices A whose columns are in the null space of B. Use Example 7.2.7.]

Exercise 7.2.24 Let $T : V \to V$ be a linear transformation where $\dim V = n$. If $\ker T \cap \operatorname{im} T = \{\mathbf{0}\}$, show that every vector $\mathbf{v}$ in V can be written $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u}$ in $\ker T$ and $\mathbf{w}$ in $\operatorname{im} T$. [*Hint*: Choose bases $B \subseteq \ker T$ and $D \subseteq \operatorname{im} T$, and use Exercise 6.3.33.]

Exercise 7.2.25 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator of rank 1, where $\mathbb{R}^n$ is written as rows. Show that there exist numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ such that $T(X) = XA$ for all rows X in $\mathbb{R}^n$, where

$$A = \left[\begin{array}{cccc} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{array} \right]$$

[*Hint*: $\operatorname{im} T = \mathbb{R}\mathbf{w}$ for $\mathbf{w} = (b_1, \dots, b_n)$ in $\mathbb{R}^n$.]

Exercise 7.2.26 Prove Theorem 7.2.5.

Exercise 7.2.27 Let $T : V \to \mathbb{R}$ be a nonzero linear transformation, where $\dim V = n$. Show that there is a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of V so that $T(r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + \cdots + r_n\mathbf{e}_n) = r_1$.

Exercise 7.2.28 Let $f \neq 0$ be a fixed polynomial of degree $m \geq 1$. If p is any polynomial, recall that $(p \circ f)(x) = p[f(x)]$. Define $T_f : P_n \to P_{n+m}$ by $T_f(p) = p \circ f$.

a. Show that T_f is linear.

b. Show that T_f is one-to-one.

Exercise 7.2.29 Let U be a subspace of a finite dimensional vector space V.

a. Show that $U = \ker T$ for some linear operator $T : V \to V$.

b. Show that $U = \operatorname{im} S$ for some linear operator $S : V \to V$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

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b. By Lemma 6.4.2, let $\{\mathbf{u}_1, \dots, \mathbf{u}_m, \dots, \mathbf{u}_n\}$ be a basis of V where $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a basis of U. By Theorem 7.1.3 there is a linear transformation $S : V \to V$ such that $S(\mathbf{u}_i) = \mathbf{u}_i$ for $1 \leq i \leq m$, and $S(\mathbf{u}_i) = \mathbf{0}$ if $i > m$. Because each $\mathbf{u}_i$ is in $\operatorname{im} S$, $U \subseteq \operatorname{im} S$. But if $S(\mathbf{v})$ is in $\operatorname{im} S$, write $\mathbf{v} = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m + \cdots + r_n\mathbf{u}_n$. Then $S(\mathbf{v}) = r_1S(\mathbf{u}_1) + \cdots + r_mS(\mathbf{u}_m) = r_1\mathbf{u}_1 + \cdots + r_m\mathbf{u}_m$ is in U. So $\operatorname{im} S \subseteq U$.

Exercise 7.2.30 Let V and W be finite dimensional vector spaces.

a. Show that $\dim W \leq \dim V$ if and only if there exists an onto linear transformation $T : V \to W$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

b. Show that $\dim W \geq \dim V$ if and only if there exists a one-to-one linear transformation $T : V \to W$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

Exercise 7.2.31 Let A and B be $n \times n$ matrices, and assume that $AXB = 0$, $X \in \mathbf{M}_{nn}$, implies $X = 0$. Show that A and B are both invertible. [Hint: Dimension Theorem.]

7.3 Isomorphisms and Composition

Often two vector spaces can consist of quite different types of vectors but, on closer examination, turn out to be the same underlying space displayed in different symbols. For example, consider the spaces

$$\mathbb{R}^2 = \{(a,\ b) \mid a,\ b \in \mathbb{R}\} \quad \text{and} \quad \mathbf{P}_1 = \{a + bx \mid a,\ b \in \mathbb{R}\}$$

Compare the addition and scalar multiplication in these spaces:

$$(a,\ b) + (a_1,\ b_1) = (a + a_1,\ b + b_1) \qquad (a + bx) + (a_1 + b_1x) = (a + a_1) + (b + b_1)x$$
$$r(a,\ b) = (ra,\ rb) \qquad r(a + bx) = (ra) + (rb)x$$

Clearly these are the *same* vector space expressed in different notation: if we change each $(a,\ b)$ in $\mathbb{R}^2$ to $a + bx$, then $\mathbb{R}^2$ *becomes* $\mathbf{P}_1$, complete with addition and scalar multiplication. This can be expressed by noting that the map $(a,\ b) \mapsto a + bx$ is a linear transformation $\mathbb{R}^2 \to \mathbf{P}_1$ that is both one-to-one and onto. In this form, we can describe the general situation.

Definition 7.4 Isomorphic Vector Spaces

*A linear transformation $T : V \to W$ is called an **isomorphism** if it is both onto and one-to-one. The vector spaces V and W are said to be **isomorphic** if there exists an isomorphism $T : V \to W$, and we write $V \cong W$ when this is the case.*

Example 7.3.1

The identity transformation $1_V : V \to V$ is an isomorphism for any vector space V.

Example 7.3.2

If $T : \mathbf{M}_{mn} \to \mathbf{M}_{nm}$ is defined by $T(A) = A^T$ for all A in $\mathbf{M}_{mn}$, then T is an isomorphism (verify). Hence $\mathbf{M}_{mn} \cong \mathbf{M}_{nm}$.

Example 7.3.3

Isomorphic spaces can "look" quite different. For example, $\mathbf{M}_{22} \cong \mathbf{P}_3$ because the map $T : \mathbf{M}_{22} \to \mathbf{P}_3$ given by $T\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = a + bx + cx^2 + dx^3$ is an isomorphism (verify).

The word *isomorphism* comes from two Greek roots: *iso*, meaning "same," and *morphos*, meaning "form." An isomorphism $T : V \to W$ induces a pairing

$$\mathbf{v} \leftrightarrow T(\mathbf{v})$$

between vectors $\mathbf{v}$ in V and vectors $T(\mathbf{v})$ in W that preserves vector addition and scalar multiplication. Hence, *as far as their vector space properties are concerned*, the spaces V and W are identical except for notation. Because addition and scalar multiplication in either space are completely determined by the same operations in the other space, all *vector space* properties of either space are completely determined by those of the other.

One of the most important examples of isomorphic spaces was considered in Chapter 4. Let A denote the set of all "arrows" with tail at the origin in space, and make A into a vector space using the parallelogram law and the scalar multiple law (see Section 4.1). Then define a transformation $T : \mathbb{R}^3 \to A$ by taking

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{the arrow } \mathbf{v} \text{ from the origin to the point } P(x,\ y,\ z).$$

In Section 4.1 matrix addition and scalar multiplication were shown to correspond to the parallelogram law and the scalar multiplication law for these arrows, so the map T is a linear transformation. Moreover T is an isomorphism: it is one-to-one by Theorem 4.1.2, and it is onto because, given an arrow $\mathbf{v}$ in A with tip $P(x,\ y,\ z)$, we have $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{v}$. This justifies the identification $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in Chapter 4 of the geometric arrows with the algebraic matrices. This identification is very useful. The arrows give a "picture" of the matrices and so bring geometric intuition into $\mathbb{R}^3$; the matrices are useful for detailed calculations and so bring analytic precision into geometry. This is one of the best examples of the power of an isomorphism to shed light on both spaces being considered.

The following theorem gives a very useful characterization of isomorphisms: They are the linear transformations that preserve bases.

Theorem 7.3.1

If V and W are finite dimensional spaces, the following conditions are equivalent for a linear transformation $T : V \to W$.

1. *T is an isomorphism.*
2. *If $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ is any basis of V, then $\{T(\mathbf{e}_1),\ T(\mathbf{e}_2),\ \dots,\ T(\mathbf{e}_n)\}$ is a basis of W.*
3. *There exists a basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ of V such that $\{T(\mathbf{e}_1),\ T(\mathbf{e}_2),\ \dots,\ T(\mathbf{e}_n)\}$ is a basis of W.*

Proof. (1) ⇒ (2). Let $\{\mathbf{e}_1,\ \dots,\ \mathbf{e}_n\}$ be a basis of V. If $t_1T(\mathbf{e}_1) + \cdots + t_nT(\mathbf{e}_n) = \mathbf{0}$ with t_i in $\mathbb{R}$, then $T(t_1\mathbf{e}_1 + \cdots + t_n\mathbf{e}_n) = \mathbf{0}$, so $t_1\mathbf{e}_1 + \cdots + t_n\mathbf{e}_n = \mathbf{0}$ (because $\ker T = \{\mathbf{0}\}$). But then each $t_i = 0$ by the independence of the $\mathbf{e}_i$, so $\{T(\mathbf{e}_1),\ \dots,\ T(\mathbf{e}_n)\}$ is independent. To show that it spans W, choose $\mathbf{w}$ in W. Because T is onto, $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V, so write $\mathbf{v} = t_1\mathbf{e}_1 + \cdots + t_n\mathbf{e}_n$. Hence we obtain $\mathbf{w} = T(\mathbf{v}) = t_1T(\mathbf{e}_1) + \cdots + t_nT(\mathbf{e}_n)$, proving that $\{T(\mathbf{e}_1),\ \dots,\ T(\mathbf{e}_n)\}$ spans W.

(2) ⇒ (3). This is because V has a basis.

(3) ⇒ (1). If $T(\mathbf{v}) = \mathbf{0}$, write $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$ where each v_i is in $\mathbb{R}$. Then

$$\mathbf{0} = T(\mathbf{v}) = v_1T(\mathbf{e}_1) + \cdots + v_nT(\mathbf{e}_n)$$

so $v_1 = \cdots = v_n = 0$ by (3). Hence $\mathbf{v} = \mathbf{0}$, so $\ker T = \{\mathbf{0}\}$ and T is one-to-one. To show that T is onto, let $\mathbf{w}$ be any vector in W. By (3) there exist $w_1, \dots, w_n$ in $\mathbb{R}$ such that

$$\mathbf{w} = w_1 T(\mathbf{e}_1) + \cdots + w_n T(\mathbf{e}_n) = T(w_1\mathbf{e}_1 + \cdots + w_n\mathbf{e}_n)$$

Thus T is onto. □

Theorem 7.3.1 dovetails nicely with Theorem 7.1.3 as follows. Let V and W be vector spaces of dimension n, and suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ are bases of V and W, respectively. Theorem 7.1.3 asserts that there exists a linear transformation $T : V \to W$ such that

$$T(\mathbf{e}_i) = \mathbf{f}_i \quad \text{for each } i = 1, 2, \dots, n$$

Then $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)\}$ is evidently a basis of W, so T is an isomorphism by Theorem 7.3.1. Furthermore, the action of T is prescribed by

$$T(r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n) = r_1\mathbf{f}_1 + \cdots + r_n\mathbf{f}_n$$

so isomorphisms between spaces of equal dimension can be easily defined as soon as bases are known. In particular, this shows that if two vector spaces V and W have the same dimension then they are isomorphic, that is $V \cong W$. This is half of the following theorem.

Theorem 7.3.2

If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Proof. It remains to show that if $V \cong W$ then $\dim V = \dim W$. But if $V \cong W$, then there exists an isomorphism $T : V \to W$. Since V is finite dimensional, let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis of V. Then $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)\}$ is a basis of W by Theorem 7.3.1, so $\dim W = n = \dim V$. □

Corollary 7.3.1

Let U, V, and W denote vector spaces. Then:

1. *$V \cong V$ for every vector space V.*
2. *If $V \cong W$ then $W \cong V$.*
3. *If $U \cong V$ and $V \cong W$, then $U \cong W$.*

The proof is left to the reader. By virtue of these properties, the relation $\cong$ is called an *equivalence relation* on the class of finite dimensional vector spaces. Since $\dim(\mathbb{R}^n) = n$ it follows that

Corollary 7.3.2

If V is a vector space and $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$.

If V is a vector space of dimension n, note that there are important explicit isomorphisms $V \to \mathbb{R}^n$. Fix a basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ of V and write $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ for the standard basis of $\mathbb{R}^n$. By Theorem 7.1.3 there is a unique linear transformation $C_B : V \to \mathbb{R}^n$ given by

$$C_B(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n = \left[\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array}\right]$$

where each v_i is in $\mathbb{R}$. Moreover, $C_B(\mathbf{b}_i) = \mathbf{e}_i$ for each i so C_B is an isomorphism by Theorem 7.3.1, called the **coordinate isomorphism** corresponding to the basis B. These isomorphisms will play a central role in Chapter **??**.

The conclusion in the above corollary can be phrased as follows: As far as vector space properties are concerned, every n-dimensional vector space V is essentially the same as $\mathbb{R}^n$; they are the "same" vector space except for a change of symbols. This appears to make the process of abstraction seem less important—just study $\mathbb{R}^n$ and be done with it! But consider the different "feel" of the spaces $\mathbf{P}_8$ and $\mathbf{M}_{33}$ even though they are both the "same" as $\mathbb{R}^9$: For example, vectors in $\mathbf{P}_8$ can have roots, while vectors in $\mathbf{M}_{33}$ can be multiplied. So the merit in the abstraction process lies in identifying *common* properties of the vector spaces in the various examples. This is important even for finite dimensional spaces. However, the payoff from abstraction is much greater in the infinite dimensional case, particularly for spaces of functions.

Example 7.3.4

Let V denote the space of all 2×2 symmetric matrices. Find an isomorphism $T : \mathbf{P}_2 \to V$ such that $T(1) = I$, where I is the 2×2 identity matrix.

Solution. $\{1, x, x^2\}$ is a basis of $\mathbf{P}_2$, and we want a basis of V containing I. The set $\left\{\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]\right\}$ is independent in V, so it is a basis because $\dim V = 3$ (by Example 6.3.11). Hence define $T : \mathbf{P}_2 \to V$ by taking $T(1) = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$, $T(x) = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$, $T(x^2) = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$, and extending linearly as in Theorem 7.1.3. Then T is an isomorphism by Theorem 7.3.1, and its action is given by

$$T(a + bx + cx^2) = aT(1) + bT(x) + cT(x^2) = \left[\begin{array}{cc} a & b \\ b & a+c \end{array}\right]$$

The dimension theorem (Theorem 7.2.4) gives the following useful fact about isomorphisms.

Theorem 7.3.3

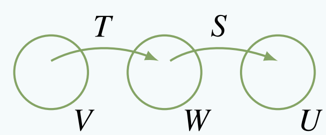
If V and W have the same dimension n, a linear transformation $T : V \to W$ is an isomorphism if it is either one-to-one or onto.

Proof. The dimension theorem asserts that $\dim(\ker T) + \dim(\operatorname{im} T) = n$, so $\dim(\ker T) = 0$ if and only if $\dim(\operatorname{im} T) = n$. Thus T is one-to-one if and only if T is onto, and the result follows. □

Composition

Suppose that $T : V \to W$ and $S : W \to U$ are linear transformations. They link together as in the diagram so, as in Section 2.3, it is possible to define a new function $V \to U$ by first applying T and then S.

Definition 7.5 Composition of Linear Transformations



Given linear transformations $V \xrightarrow{T} W \xrightarrow{S} U$, *the* ***composite*** $ST : V \to U$ *of* T *and* S *is defined by*

$$ST(\mathbf{v}) = S[T(\mathbf{v})] \quad \text{for all } \mathbf{v} \text{ in } V$$

The operation of forming the new function ST *is called* ***composition***.[1]

The action of ST can be described compactly as follows: ST means first T then S.

Not all pairs of linear transformations can be composed. For example, if $T : V \to W$ and $S : W \to U$ are linear transformations then $ST : V \to U$ is defined, but TS *cannot* be formed unless $U = V$ because $S : W \to U$ and $T : V \to W$ do not "link" in that order.[2]

Moreover, even if ST and TS can both be formed, they may not be equal. In fact, if $S : \mathbb{R}^m \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are induced by matrices A and B respectively, then ST and TS can both be formed (they are induced by AB and BA respectively), but the matrix products AB and BA may not be equal (they may not even be the same *size*). Here is another example.

Example 7.3.5

Define: $S : \mathbf{M}_{22} \to \mathbf{M}_{22}$ and $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ by $S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and $T(A) = A^T$ for $A \in \mathbf{M}_{22}$. Describe the action of ST and TS, and show that $ST \neq TS$.

Solution. $ST\begin{bmatrix} a & b \\ c & d \end{bmatrix} = S\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$, whereas
$TS\begin{bmatrix} a & b \\ c & d \end{bmatrix} = T\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$.
It is clear that $TS\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ need not equal $ST\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $TS \neq ST$.

The next theorem collects some basic properties of the composition operation.

[1] In Section 2.3 we denoted the composite as $S \circ T$. However, it is more convenient to use the simpler notation ST.
[2] Actually, all that is required is $U \subseteq V$.

Theorem 7.3.4: [3]

Let $V \xrightarrow{T} W \xrightarrow{S} U \xrightarrow{R} Z$ be linear transformations.

1. *The composite ST is again a linear transformation.*
2. *$T1_V = T$ and $1_W T = T$.*
3. *$(RS)T = R(ST)$.*

Proof. The proofs of (1) and (2) are left as Exercise 7.3.25. To prove (3), observe that, for all $\mathbf{v}$ in V:

$$\{(RS)T\}(\mathbf{v}) = (RS)[T(\mathbf{v})] = R\{S[T(\mathbf{v})]\} = R\{(ST)(\mathbf{v})\} = \{R(ST)\}(\mathbf{v})$$

□

Up to this point, composition seems to have no connection with isomorphisms. In fact, the two notions are closely related.

Theorem 7.3.5

Let V and W be finite dimensional vector spaces. The following conditions are equivalent for a linear transformation $T : V \to W$.

1. *T is an isomorphism.*
2. *There exists a linear transformation $S : W \to V$ such that $ST = 1_V$ and $TS = 1_W$.*

Moreover, in this case S is also an isomorphism and is uniquely determined by T:

$$\text{If } \mathbf{w} \text{ in } W \text{ is written as } \mathbf{w} = T(\mathbf{v}), \quad \text{then } S(\mathbf{w}) = \mathbf{v}.$$

Proof. (1) ⇒ (2). If $B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of V, then $D = \{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)\}$ is a basis of W by Theorem 7.3.1. Hence (using Theorem 7.1.3), define a linear transformation $S : W \to V$ by

$$S[T(\mathbf{e}_i)] = \mathbf{e}_i \quad \text{for each } i \tag{7.2}$$

Since $\mathbf{e}_i = 1_V(\mathbf{e}_i)$, this gives $ST = 1_V$ by Theorem 7.1.2. But applying T gives $T[S[T(\mathbf{e}_i)]] = T(\mathbf{e}_i)$ for each i, so $TS = 1_W$ (again by Theorem 7.1.2, using the basis D of W).

(2) ⇒ (1). If $T(\mathbf{v}) = T(\mathbf{v}_1)$, then $S[T(\mathbf{v})] = S[T(\mathbf{v}_1)]$. Because $ST = 1_V$ by (2), this reads $\mathbf{v} = \mathbf{v}_1$; that is, T is one-to-one. Given $\mathbf{w}$ in W, the fact that $TS = 1_W$ means that $\mathbf{w} = T[S(\mathbf{w})]$, so T is onto.

[3]Theorem 7.3.4 can be expressed by saying that vector spaces and linear transformations are an example of a category. In general a category consists of certain objects and, for any two objects X and Y, a set $\text{mor}(X, Y)$. The elements α of $\text{mor}(X, Y)$ are called morphisms from X to Y and are written $\alpha : X \to Y$. It is assumed that identity morphisms and composition are defined in such a way that Theorem 7.3.4 holds. Hence, in the category of vector spaces the objects are the vector spaces themselves and the morphisms are the linear transformations. Another example is the category of metric spaces, in which the objects are sets equipped with a distance function (called a metric), and the morphisms are continuous functions (with respect to the metric). The category of sets and functions is a very basic example.

Finally, S is uniquely determined by the condition $ST = 1_V$ because this condition implies (7.2). S is an isomorphism because it carries the basis D to B. As to the last assertion, given $\mathbf{w}$ in W, write $\mathbf{w} = r_1T(\mathbf{e}_1) + \cdots + r_nT(\mathbf{e}_n)$. Then $\mathbf{w} = T(\mathbf{v})$, where $\mathbf{v} = r_1\mathbf{e}_1 + \cdots + r_n\mathbf{e}_n$. Then $S(\mathbf{w}) = \mathbf{v}$ by (7.2). □

Given an isomorphism $T : V \to W$, the unique isomorphism $S : W \to V$ satisfying condition (2) of Theorem 7.3.5 is called the **inverse** of T and is denoted by T^{-1}. Hence $T : V \to W$ and $T^{-1} : W \to V$ are related by the **fundamental identities**:

$$T^{-1}[T(\mathbf{v})] = \mathbf{v} \text{ for all } \mathbf{v} \text{ in } V \quad \text{and} \quad T\left[T^{-1}(\mathbf{w})\right] = \mathbf{w} \text{ for all } \mathbf{w} \text{ in } W$$

In other words, each of T and T^{-1} reverses the action of the other. In particular, equation (7.2) in the proof of Theorem 7.3.5 shows how to define T^{-1} using the image of a basis under the isomorphism T. Here is an example.

Example 7.3.6

Define $T : \mathbf{P}_1 \to \mathbf{P}_1$ by $T(a+bx) = (a-b)+ax$. Show that T has an inverse, and find the action of T^{-1}.

Solution. The transformation T is linear (verify). Because $T(1) = 1+x$ and $T(x) = -1$, T carries the basis $B = \{1,\ x\}$ to the basis $D = \{1+x,\ -1\}$. Hence T is an isomorphism, and T^{-1} carries D back to B, that is,

$$T^{-1}(1+x) = 1 \quad \text{and} \quad T^{-1}(-1) = x$$

Because $a+bx = b(1+x)+(b-a)(-1)$, we obtain

$$T^{-1}(a+bx) = bT^{-1}(1+x)+(b-a)T^{-1}(-1) = b+(b-a)x$$

Sometimes the action of the inverse of a transformation is apparent.

Example 7.3.7

If $B = \{\mathbf{b}_1,\ \mathbf{b}_2,\ \dots,\ \mathbf{b}_n\}$ is a basis of a vector space V, the coordinate transformation $C_B : V \to \mathbb{R}^n$ is an isomorphism defined by

$$C_B(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = (v_1,\ v_2,\ \dots,\ v_n)^T$$

The way to reverse the action of C_B is clear: $C_B^{-1} : \mathbb{R}^n \to V$ is given by

$$C_B^{-1}(v_1,\ v_2,\ \dots,\ v_n) = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n \quad \text{for all } v_i \text{ in } V$$

Condition (2) in Theorem 7.3.5 characterizes the inverse of a linear transformation $T : V \to W$ as the (unique) transformation $S : W \to V$ that satisfies $ST = 1_V$ and $TS = 1_W$. This often determines the inverse.

Example 7.3.8

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x,\ y,\ z) = (z,\ x,\ y)$. Show that $T^3 = 1_{\mathbb{R}^3}$, and hence find T^{-1}.

Solution. $T^2(x,\ y,\ z) = T[T(x,\ y,\ z)] = T(z,\ x,\ y) = (y,\ z,\ x)$. Hence

$$T^3(x,\ y,\ z) = T\left[T^2(x,\ y,\ z)\right] = T(y,\ z,\ x) = (x,\ y,\ z)$$

Since this holds for all $(x,\ y,\ z)$, it shows that $T^3 = 1_{\mathbb{R}^3}$, so $T(T^2) = 1_{\mathbb{R}^3} = (T^2)T$. Thus $T^{-1} = T^2$ by (2) of Theorem 7.3.5.

Example 7.3.9

Define $T : \mathbf{P}_n \to \mathbb{R}^{n+1}$ by $T(p) = (p(0),\ p(1),\ \dots,\ p(n))$ for all p in $\mathbf{P}_n$. Show that T^{-1} exists.

Solution. The verification that T is linear is left to the reader. If $T(p) = 0$, then $p(k) = 0$ for $k = 0,\ 1,\ \dots,\ n$, so p has $n+1$ distinct roots. Because p has degree at most n, this implies that $p = 0$ is the zero polynomial (Theorem **??**) and hence that T is one-to-one. But $\dim \mathbf{P}_n = n+1 = \dim \mathbb{R}^{n+1}$, so this means that T is also onto and hence is an isomorphism. Thus T^{-1} exists by Theorem 7.3.5. Note that we have not given a description of the action of T^{-1}, we have merely shown that such a description exists. To give it explicitly requires some ingenuity; one method involves the Lagrange interpolation expansion (Theorem **??**).

Exercises for 7.3

Exercise 7.3.1 Verify that each of the following is an isomorphism (Theorem 7.3.3 is useful).

a. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x,\ y,\ z) = (x+y,\ y+z,\ z+x)$

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x,\ y,\ z) = (x,\ x+y,\ x+y+z)$

c. $T : \mathbb{C} \to \mathbb{C}$; $T(z) = \overline{z}$

d. $T : \mathbf{M}_{mn} \to \mathbf{M}_{mn}$; $T(X) = UXV$, U and V invertible

e. $T : \mathbf{P}_1 \to \mathbb{R}^2$; $T[p(x)] = [p(0),\ p(1)]$

f. $T : V \to V$; $T(\mathbf{v}) = k\mathbf{v}$, $k \neq 0$ a fixed number, V any vector space

g. $T : \mathbf{M}_{22} \to \mathbb{R}^4$; $T\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = (a+b,\ d,\ c,\ a-b)$

h. $T : \mathbf{M}_{mn} \to \mathbf{M}_{nm}$; $T(A) = A^T$

---

b. T is onto because $T(1,\ -1,\ 0) = (1,\ 0,\ 0)$, $T(0,\ 1,\ -1) = (0,\ 1,\ 0)$, and $T(0,\ 0,\ 1) = (0,\ 0,\ 1)$. Use Theorem 7.3.3.

d. T is one-to-one because $0 = T(X) = UXV$ implies that $X = 0$ (U and V are invertible). Use Theorem 7.3.3.

f. T is one-to-one because $\mathbf{0} = T(\mathbf{v}) = k\mathbf{v}$ implies that $\mathbf{v} = \mathbf{0}$ (because $k \neq 0$). T is onto because $T\left(\frac{1}{k}\mathbf{v}\right) = \mathbf{v}$ for all $\mathbf{v}$. [Here Theorem 7.3.3 does not apply if $\dim V$ is not finite.]

h. T is one-to-one because $T(A) = 0$ implies $A^T = 0$, whence $A = 0$. Use Theorem 7.3.3.

Exercise 7.3.2 Show that

$$\{a+bx+cx^2,\ a_1+b_1x+c_1x^2,\ a_2+b_2x+c_2x^2\}$$

is a basis of $\mathbf{P}_2$ if and only if
$\{(a,\ b,\ c),\ (a_1,\ b_1,\ c_1),\ (a_2,\ b_2,\ c_2)\}$ is a basis of $\mathbb{R}^3$.

Exercise 7.3.3 If V is any vector space, let V^n denote the space of all n-tuples $(\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n)$, where each $\mathbf{v}_i$ lies in V. (This is a vector space with component-wise operations; see Exercise 6.1.17.) If $C_j(A)$ denotes the jth column of the $m\times n$ matrix A, show that $T:\mathbf{M}_{mn}\to(\mathbb{R}^m)^n$ is an isomorphism if $T(A)=\left[\begin{array}{cccc} C_1(A) & C_2(A) & \cdots & C_n(A)\end{array}\right]$. (Here $\mathbb{R}^m$ consists of columns.)

Exercise 7.3.4 In each case, compute the action of ST and TS, and show that $ST\neq TS$.

a. $S:\mathbb{R}^2\to\mathbb{R}^2$ with $S(x,\ y)=(y,\ x)$; $T:\mathbb{R}^2\to\mathbb{R}^2$ with $T(x,\ y)=(x,\ 0)$

b. $S:\mathbb{R}^3\to\mathbb{R}^3$ with $S(x,\ y,\ z)=(x,\ 0,\ z)$;
$T:\mathbb{R}^3\to\mathbb{R}^3$ with $T(x,\ y,\ z)=(x+y,\ 0,\ y+z)$

c. $S:\mathbf{P}_2\to\mathbf{P}_2$ with $S(p)=p(0)+p(1)x+p(2)x^2$;
$T:\mathbf{P}_2\to\mathbf{P}_2$ with $T(a+bx+cx^2)=b+cx+ax^2$

d. $S:\mathbf{M}_{22}\to\mathbf{M}_{22}$ with $S\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\left[\begin{array}{cc} a & 0\\ 0 & d\end{array}\right]$;
$T:\mathbf{M}_{22}\to\mathbf{M}_{22}$ with $T\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\left[\begin{array}{cc} c & a\\ d & b\end{array}\right]$

---

b. $ST(x,\ y,\ z)=(x+y,\ 0,\ y+z)$, $TS(x,\ y,\ z)=(x,\ 0,\ z)$

d. $ST\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\left[\begin{array}{cc} c & 0\\ 0 & d\end{array}\right]$, $TS\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\left[\begin{array}{cc} 0 & a\\ d & 0\end{array}\right]$

Exercise 7.3.5 In each case, show that the linear transformation T satisfies $T^2=T$.

a. $T:\mathbb{R}^4\to\mathbb{R}^4$; $T(x,\ y,\ z,\ w)=(x,\ 0,\ z,\ 0)$

b. $T:\mathbb{R}^2\to\mathbb{R}^2$; $T(x,\ y)=(x+y,\ 0)$

c. $T:\mathbf{P}_2\to\mathbf{P}_2$;
$T(a+bx+cx^2)=(a+b-c)+cx+cx^2$

d. $T:\mathbf{M}_{22}\to\mathbf{M}_{22}$;
$T\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\frac{1}{2}\left[\begin{array}{cc} a+c & b+d\\ a+c & b+d\end{array}\right]$

---

b. $T^2(x,\ y)=T(x+y,\ 0)=(x+y,\ 0)=T(x,\ y)$. Hence $T^2=T$.

d. $T^2\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\frac{1}{2}T\left[\begin{array}{cc} a+c & b+d\\ a+c & b+d\end{array}\right]=\frac{1}{2}\left[\begin{array}{cc} a+c & b+d\\ a+c & b+d\end{array}\right]$

Exercise 7.3.6 Determine whether each of the following transformations T has an inverse and, if so, determine the action of T^{-1}.

a. $T:\mathbb{R}^3\to\mathbb{R}^3$;
$T(x,\ y,\ z)=(x+y,\ y+z,\ z+x)$

b. $T:\mathbb{R}^4\to\mathbb{R}^4$;
$T(x,\ y,\ z,\ t)=(x+y,\ y+z,\ z+t,\ t+x)$

c. $T:\mathbf{M}_{22}\to\mathbf{M}_{22}$;
$T\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\left[\begin{array}{cc} a-c & b-d\\ 2a-c & 2b-d\end{array}\right]$

d. $T:\mathbf{M}_{22}\to\mathbf{M}_{22}$;
$T\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\left[\begin{array}{cc} a+2c & b+2d\\ 3c-a & 3d-b\end{array}\right]$

e. $T:\mathbf{P}_2\to\mathbb{R}^3$; $T(a+bx+cx^2)=(a-c,\ 2b,\ a+c)$

f. $T:\mathbf{P}_2\to\mathbb{R}^3$; $T(p)=[p(0),\ p(1),\ p(-1)]$

---

b. No inverse; $(1,\ -1,\ 1,\ -1)$ is in $\ker T$.

d. $T^{-1}\left[\begin{array}{cc} a & b\\ c & d\end{array}\right]=\frac{1}{5}\left[\begin{array}{cc} 3a-2c & 3b-2d\\ a+c & b+d\end{array}\right]$

f. $T^{-1}(a,\ b,\ c)=\frac{1}{2}\left[2a+(b-c)x-(2a-b-c)x^2\right]$

Exercise 7.3.7 In each case, show that T is self-inverse, that is: $T^{-1}=T$.

a. $T:\mathbb{R}^4\to\mathbb{R}^4$; $T(x,\ y,\ z,\ w)=(x,\ -y,\ -z,\ w)$

b. $T:\mathbb{R}^2\to\mathbb{R}^2$; $T(x,\ y)=(ky-x,\ y)$, k any fixed number

c. $T:\mathbf{P}_n\to\mathbf{P}_n$; $T(p(x))=p(3-x)$

d. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = AX$ where $A = \frac{1}{4}\left[\begin{array}{rr} 5 & -3 \\ 3 & -5 \end{array}\right]$

---

b. $T^2(x,\ y) = T(ky - x,\ y) = (ky - (ky - x),\ y) = (x,\ y)$

d. $T^2(X) = A^2X = IX = X$

Exercise 7.3.8 In each case, show that $T^6 = 1_{R^4}$ and so determine T^{-1}.

a. $T : \mathbb{R}^4 \to \mathbb{R}^4$; $T(x,\ y,\ z,\ w) = (-x,\ z,\ w,\ y)$

b. $T : \mathbb{R}^4 \to \mathbb{R}^4$; $T(x,\ y,\ z,\ w) = (-y,\ x - y,\ z,\ -w)$

---

b. $T^3(x, y, z, w) = (x, y, z, -w)$ so $T^6(x, y, z, w) = T^3\left[T^3(x,\ y,\ z,\ w)\right] = (x,\ y,\ z,\ w)$. Hence $T^{-1} = T^5$. So $T^{-1}(x,\ y,\ z,\ w) = (y - x,\ -x,\ z,\ -w)$.

Exercise 7.3.9 In each case, show that T is an isomorphism by defining T^{-1} explicitly.

a. $T : \mathbf{P}_n \to \mathbf{P}_n$ is given by $T[p(x)] = p(x+1)$.

b. $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ is given by $T(A) = UA$ where U is invertible in $\mathbf{M}_{nn}$.

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b. $T^{-1}(A) = U^{-1}A$.

Exercise 7.3.10 Given linear transformations $V \xrightarrow{T} W \xrightarrow{S} U$:

a. If S and T are both one-to-one, show that ST is one-to-one.

b. If S and T are both onto, show that ST is onto.

---

b. Given $\mathbf{u}$ in U, write $\mathbf{u} = S(\mathbf{w})$, $\mathbf{w}$ in W (because S is onto). Then write $\mathbf{w} = T(\mathbf{v})$, $\mathbf{v}$ in V (T is onto). Hence $\mathbf{u} = ST(\mathbf{v})$, so ST is onto.

Exercise 7.3.11 Let $T : V \to W$ be a linear transformation.

a. If T is one-to-one and $TR = TR_1$ for transformations R and $R_1 : U \to V$, show that $R = R_1$.

b. If T is onto and $ST = S_1T$ for transformations S and $S_1 : W \to U$, show that $S = S_1$.

Exercise 7.3.12 Consider the linear transformations $V \xrightarrow{T} W \xrightarrow{R} U$.

a. Show that $\ker T \subseteq \ker RT$.

b. Show that $\operatorname{im} RT \subseteq \operatorname{im} R$.

---

b. For all $\mathbf{v}$ in V, $(RT)(\mathbf{v}) = R[T(\mathbf{v})]$ is in $\operatorname{im}(R)$.

Exercise 7.3.13 Let $V \xrightarrow{T} U \xrightarrow{S} W$ be linear transformations.

a. If ST is one-to-one, show that T is one-to-one and that $\dim V \leq \dim U$.

b. If ST is onto, show that S is onto and that $\dim W \leq \dim U$.

---

b. Given $\mathbf{w}$ in W, write $\mathbf{w} = ST(\mathbf{v})$, $\mathbf{v}$ in V (ST is onto). Then $\mathbf{w} = S[T(\mathbf{v})]$, $T(\mathbf{v})$ in U, so S is onto. But then $\operatorname{im} S = W$, so $\dim U = \dim(\ker S) + \dim(\operatorname{im} S) \geq \dim(\operatorname{im} S) = \dim W$.

Exercise 7.3.14 Let $T : V \to V$ be a linear transformation. Show that $T^2 = 1_V$ if and only if T is invertible and $T = T^{-1}$.

Exercise 7.3.15 Let N be a nilpotent $n \times n$ matrix (that is, $N^k = 0$ for some k). Show that $T : \mathbf{M}_{nm} \to \mathbf{M}_{nm}$ is an isomorphism if $T(X) = X - NX$. [*Hint*: If X is in $\ker T$, show that $X = NX = N^2X = \cdots$. Then use Theorem 7.3.3.]

Exercise 7.3.16 Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{e}_1,\ \dots,\ \mathbf{e}_r,\ \mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$ be any basis of V such that $\{\mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$ is a basis of $\ker T$.

Show that $\text{im } T \cong \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$. [*Hint*: See Theorem 7.2.5.]

$\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_r)\}$ is a basis of $\text{im } T$ by Theorem 7.2.5. So $T : \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_r\} \to \text{im } T$ is an isomorphism by Theorem 7.3.1.

Exercise 7.3.17 Is every isomorphism $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ given by an invertible matrix U such that $T(X) = UX$ for all X in $\mathbf{M}_{22}$? Prove your answer.

Exercise 7.3.18 Let $\mathbf{D}_n$ denote the space of all functions f from $\{1, 2, \dots, n\}$ to $\mathbb{R}$ (see Exercise 6.3.35). If $T : \mathbf{D}_n \to \mathbb{R}^n$ is defined by

$$T(f) = (f(1), f(2), \dots, f(n)),$$

show that T is an isomorphism.

Exercise 7.3.19

a. Let V be the vector space of Exercise 6.1.3. Find an isomorphism $T : V \to \mathbb{R}^1$.

b. Let V be the vector space of Exercise 6.1.4. Find an isomorphism $T : V \to \mathbb{R}^2$.

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b. $T(x, y) = (x, y+1)$

Exercise 7.3.20 Let $V \xrightarrow{T} W \xrightarrow{S} V$ be linear transformations such that $ST = 1_V$. If $\dim V = \dim W = n$, show that $S = T^{-1}$ and $T = S^{-1}$. [*Hint*: Exercise 7.3.13 and Theorem 7.3.3, Theorem 7.3.4, and Theorem 7.3.5.]

Exercise 7.3.21 Let $V \xrightarrow{T} W \xrightarrow{S} V$ be functions such that $TS = 1_W$ and $ST = 1_V$. If T is linear, show that S is also linear.

Exercise 7.3.22 Let A and B be matrices of size $p \times m$ and $n \times q$. Assume that $mn = pq$. Define $R : \mathbf{M}_{mn} \to \mathbf{M}_{pq}$ by $R(X) = AXB$.

a. Show that $\mathbf{M}_{mn} \cong \mathbf{M}_{pq}$ by comparing dimensions.

b. Show that R is a linear transformation.

c. Show that if R is an isomorphism, then $m = p$ and $n = q$. [*Hint*: Show that $T : \mathbf{M}_{mn} \to \mathbf{M}_{pn}$ given by $T(X) = AX$ and $S : \mathbf{M}_{mn} \to \mathbf{M}_{mq}$ given by $S(X) = XB$ are both one-to-one, and use the dimension theorem.]

Exercise 7.3.23 Let $T : V \to V$ be a linear transformation such that $T^2 = 0$ is the zero transformation.

a. If $V \neq \{\mathbf{0}\}$, show that T cannot be invertible.

b. If $R : V \to V$ is defined by $R(\mathbf{v}) = \mathbf{v} + T(\mathbf{v})$ for all $\mathbf{v}$ in V, show that R is linear and invertible.

Exercise 7.3.24 Let V consist of all sequences $[x_0, x_1, x_2, \dots)$ of numbers, and define vector operations

$$[x_o, x_1, \dots) + [y_0, y_1, \dots) = [x_0 + y_0, x_1 + y_1, \dots)$$
$$r[x_0, x_1, \dots) = [rx_0, rx_1, \dots)$$

a. Show that V is a vector space of infinite dimension.

b. Define $T : V \to V$ and $S : V \to V$ by $T[x_0, x_1, \dots) = [x_1, x_2, \dots)$ and $S[x_0, x_1, \dots) = [0, x_0, x_1, \dots)$. Show that $TS = 1_V$, so TS is one-to-one and onto, but that T is not one-to-one and S is not onto.

---

b. $TS[x_0, x_1, \dots) = T[0, x_0, x_1, \dots) = [x_0, x_1, \dots)$, so $TS = 1_V$. Hence TS is both onto and one-to-one, so T is onto and S is one-to-one by Exercise 7.3.13. But $[1, 0, 0, \dots)$ is in $\ker T$ while $[1, 0, 0, \dots)$ is not in $\text{im } S$.

Exercise 7.3.25 Prove (1) and (2) of Theorem 7.3.4.

Exercise 7.3.26 Define $T : \mathbf{P}_n \to \mathbf{P}_n$ by $T(p) = p(x) + xp'(x)$ for all p in $\mathbf{P}_n$.

a. Show that T is linear.

b. Show that $\ker T = \{\mathbf{0}\}$ and conclude that T is an isomorphism. [*Hint*: Write $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and compare coefficients if $p(x) = -xp'(x)$.]

c. Conclude that each $q(x)$ in $\mathbf{P}_n$ has the form $q(x) = p(x) + xp'(x)$ for some unique polynomial $p(x)$.

d. Does this remain valid if T is defined by $T[p(x)] = p(x) - xp'(x)$? Explain.

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b. If $T(p) = 0$, then $p(x) = -xp'(x)$. We write $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, and this becomes $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = -a_1x - 2a_2x^2 - \cdots - na_nx^n$. Equating coefficients yields $a_0 = 0,\ 2a_1 = 0,\ 3a_2 = 0,\ \dots,\ (n+1)a_n = 0$, whence $p(x) = 0$. This means that $\ker T = 0$, so T is one-to-one. But then T is an isomorphism by Theorem 7.3.3.

Exercise 7.3.27 Let $T : V \to W$ be a linear transformation, where V and W are finite dimensional.

a. Show that T is one-to-one if and only if there exists a linear transformation $S : W \to V$ with $ST = 1_V$. [*Hint*: If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of V and T is one-to-one, show that W has a basis $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n), \mathbf{f}_{n+1}, \dots, \mathbf{f}_{n+k}\}$ and use Theorem 7.1.2 and Theorem 7.1.3.]

b. Show that T is onto if and only if there exists a linear transformation $S : W \to V$ with $TS = 1_W$. [*Hint*: Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$. Use Theorem 7.2.5, Theorem 7.1.2 and Theorem 7.1.3.]

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b. If $ST = 1_V$ for some S, then T is onto by Exercise 7.3.13. If T is onto, let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$. Since T is onto, $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T = W$ by Theorem 7.2.5. Thus $S : W \to V$ is an isomorphism where by $S\{T(\mathbf{e}_i)] = \mathbf{e}_i$ for $i = 1, 2, \dots, r$. Hence $TS[T(\mathbf{e}_i)] = T(\mathbf{e}_i)$ for each i, that is $TS[T(\mathbf{e}_i)] = 1_W[T(\mathbf{e}_i)]$. This means that $TS = 1_W$ because they agree on the basis $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ of W.

Exercise 7.3.28 Let S and T be linear transformations $V \to W$, where $\dim V = n$ and $\dim W = m$.

a. Show that $\ker S = \ker T$ if and only if $T = RS$ for some isomorphism $R : W \to W$. [*Hint*: Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker S = \ker T$. Use Theorem 7.2.5 to extend $\{S(\mathbf{e}_1), \dots, S(\mathbf{e}_r)\}$ and $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ to bases of W.]

b. Show that $\operatorname{im} S = \operatorname{im} T$ if and only if $T = SR$ for some isomorphism $R : V \to V$. [*Hint*: Show that $\dim(\ker S) = \dim(\ker T)$ and choose bases $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_r, \dots, \mathbf{f}_n\}$ of V where $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ are bases of $\ker S$ and $\ker T$, respectively. If $1 \le i \le r$, show that $S(\mathbf{e}_i) = T(\mathbf{g}_i)$ for some $\mathbf{g}_i$ in V, and prove that $\{\mathbf{g}_1, \dots, \mathbf{g}_r, \mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ is a basis of V.]

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b. If $T = SR$, then every vector $T(\mathbf{v})$ in $\operatorname{im} T$ has the form $T(\mathbf{v}) = S[R(\mathbf{v})]$, whence $\operatorname{im} T \subseteq \operatorname{im} S$. Since R is invertible, $S = TR^{-1}$ implies $\operatorname{im} S \subseteq \operatorname{im} T$. Conversely, assume that $\operatorname{im} S = \operatorname{im} T$. Then $\dim(\ker S) = \dim(\ker T)$ by the dimension theorem. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_r, \mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ be bases of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ are bases of $\ker S$ and $\ker T$, respectively. By Theorem 7.2.5, $\{S(\mathbf{e}_1), \dots, S(\mathbf{e}_r)\}$ and $\{T(\mathbf{f}_1), \dots, T(\mathbf{f}_r)\}$ are both bases of $\operatorname{im} S = \operatorname{im} T$. So let $\mathbf{g}_1, \dots, \mathbf{g}_r$ in V be such that $S(\mathbf{e}_i) = T(\mathbf{g}_i)$ for each $i = 1, 2, \dots, r$. Show that

$$B = \{\mathbf{g}_1, \dots, \mathbf{g}_r, \mathbf{f}_{r+1}, \dots, \mathbf{f}_n\} \text{ is a basis of } V.$$

Then define $R : V \to V$ by $R(\mathbf{g}_i) = \mathbf{e}_i$ for $i = 1, 2, \dots, r$, and $R(\mathbf{f}_j) = \mathbf{e}_j$ for $j = r+1, \dots, n$. Then R is an isomorphism by Theorem 7.3.1. Finally $SR = T$ since they have the same effect on the basis B.

Exercise 7.3.29 If $T : V \to V$ is a linear transformation where $\dim V = n$, show that $TST = T$ for some isomorphism $S : V \to V$. [*Hint*: Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ be as in Theorem 7.2.5. Extend $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ to a basis of V, and use Theorem 7.3.1, Theorem 7.1.2 and Theorem 7.1.3.]

Let $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ be a basis of V with $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ a basis of $\ker T$. If $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r), \mathbf{w}_{r+1}, \dots, \mathbf{w}_n\}$ is a basis of V, define S by $S[T(\mathbf{e}_i)] = \mathbf{e}_i$ for $1 \leq i \leq r$, and $S(\mathbf{w}_j) = \mathbf{e}_j$ for $r+1 \leq j \leq n$. Then S is an isomorphism by Theorem 7.3.1, and $TST(\mathbf{e}_i) = T(\mathbf{e}_i)$ clearly holds for $1 \leq i \leq r$. But if $i \geq r+1$, then $T(\mathbf{e}_i) = \mathbf{0} = TST(\mathbf{e}_i)$, so $T = TST$ by Theorem 7.1.2.

Exercise 7.3.30 Let A and B denote $m \times n$ matrices. In each case show that (1) and (2) are equivalent.

a. (1) A and B have the same null space. (2) $B = PA$ for some invertible $m \times m$ matrix P.

b. (1) A and B have the same range. (2) $B = AQ$ for some invertible $n \times n$ matrix Q.

[*Hint*: Use Exercise 7.3.28.]

8. Orthogonality

Contents

In Section 5.3 we introduced the dot product in $\mathbb{R}^n$ and extended the basic geometric notions of length and distance. A set $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ of nonzero vectors in $\mathbb{R}^n$ was called an **orthogonal set** if $\mathbf{f}_i \cdot \mathbf{f}_j = 0$ for all $i \neq j$, and it was proved that every orthogonal set is independent. In particular, it was observed that the expansion of a vector as a linear combination of orthogonal basis vectors is easy to obtain because formulas exist for the coefficients. Hence the orthogonal bases are the "nice" bases, and much of this chapter is devoted to extending results about bases to orthogonal bases. This leads to some very powerful methods and theorems. Our first task is to show that every subspace of $\mathbb{R}^n$ *has* an orthogonal basis.

8.1 Orthogonal Complements and Projections

If $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent in a general vector space, and if $\mathbf{v}_{m+1}$ is not in $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}\}$ is independent (Lemma 6.4.1). Here is the analog for *orthogonal* sets in $\mathbb{R}^n$.

Lemma 8.1.1: Orthogonal Lemma

Let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ be an orthogonal set in $\mathbb{R}^n$. Given $\mathbf{x}$ in $\mathbb{R}^n$, write

$$\mathbf{f}_{m+1} = \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 - \cdots - \frac{\mathbf{x}\cdot\mathbf{f}_m}{\|\mathbf{f}_m\|^2}\mathbf{f}_m$$

Then:

1. $\mathbf{f}_{m+1} \cdot \mathbf{f}_k = 0$ *for* $k = 1, 2, \dots, m$.
2. *If $\mathbf{x}$ is not in* $\text{span}\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$, *then* $\mathbf{f}_{m+1} \neq \mathbf{0}$ *and* $\{\mathbf{f}_1, \dots, \mathbf{f}_m, \mathbf{f}_{m+1}\}$ *is an orthogonal set.*

Proof. For convenience, write $t_i = (\mathbf{x}\cdot\mathbf{f}_i)/\|\mathbf{f}_i\|^2$ for each i. Given $1 \leq k \leq m$:

$$\begin{aligned}
\mathbf{f}_{m+1}\cdot\mathbf{f}_k &= (\mathbf{x} - t_1\mathbf{f}_1 - \cdots - t_k\mathbf{f}_k - \cdots - t_m\mathbf{f}_m)\cdot\mathbf{f}_k \\
&= \mathbf{x}\cdot\mathbf{f}_k - t_1(\mathbf{f}_1\cdot\mathbf{f}_k) - \cdots - t_k(\mathbf{f}_k\cdot\mathbf{f}_k) - \cdots - t_m(\mathbf{f}_m\cdot\mathbf{f}_k) \\
&= \mathbf{x}\cdot\mathbf{f}_k - t_k\|\mathbf{f}_k\|^2 \\
&= 0
\end{aligned}$$

This proves (1), and (2) follows because $\mathbf{f}_{m+1} \neq \mathbf{0}$ if $\mathbf{x}$ is not in $\text{span}\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$. □

The orthogonal lemma has three important consequences for $\mathbb{R}^n$. The first is an extension for orthogonal sets of the fundamental fact that any independent set is part of a basis (Theorem 6.4.1).

Theorem 8.1.1

Let U be a subspace of $\mathbb{R}^n$.

1. *Every orthogonal subset $\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ in U is a subset of an orthogonal basis of U.*
2. *U has an orthogonal basis.*

Proof.

1. If $\text{span}\{\mathbf{f}_1, \dots, \mathbf{f}_m\} = U$, it is *already* a basis. Otherwise, there exists $\mathbf{x}$ in U outside $\text{span}\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$. If $\mathbf{f}_{m+1}$ is as given in the orthogonal lemma, then $\mathbf{f}_{m+1}$ is in U and $\{\mathbf{f}_1, \dots, \mathbf{f}_m, \mathbf{f}_{m+1}\}$ is orthogonal. If $\text{span}\{\mathbf{f}_1, \dots, \mathbf{f}_m, \mathbf{f}_{m+1}\} = U$, we are done. Otherwise,

the process continues to create larger and larger orthogonal subsets of U. They are all independent by Theorem 5.3.5, so we have a basis when we reach a subset containing dim U vectors.

2. If $U = \{\mathbf{0}\}$, the empty basis is orthogonal. Otherwise, if $\mathbf{f} \neq \mathbf{0}$ is in U, then $\{\mathbf{f}\}$ is orthogonal, so (2) follows from (1). □

We can improve upon (2) of Theorem 8.1.1. In fact, the second consequence of the orthogonal lemma is a procedure by which *any* basis $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ of a subspace U of $\mathbb{R}^n$ can be systematically modified to yield an orthogonal basis $\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ of U. The $\mathbf{f}_i$ are constructed one at a time from the $\mathbf{x}_i$.

To start the process, take $\mathbf{f}_1 = \mathbf{x}_1$. Then $\mathbf{x}_2$ is not in $\text{span}\{\mathbf{f}_1\}$ because $\{\mathbf{x}_1, \mathbf{x}_2\}$ is independent, so take

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1$$

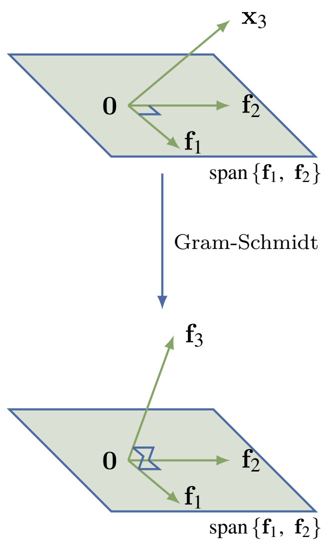
Thus $\{\mathbf{f}_1, \mathbf{f}_2\}$ is orthogonal by Lemma 8.1.1. Moreover, $\text{span}\{\mathbf{f}_1, \mathbf{f}_2\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ (verify), so $\mathbf{x}_3$ is not in $\text{span}\{\mathbf{f}_1, \mathbf{f}_2\}$. Hence $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is orthogonal where

$$\mathbf{f}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2$$

Again, $\text{span}\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, so $\mathbf{x}_4$ is not in $\text{span}\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ and the process continues. At the mth iteration we construct an orthogonal set $\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ such that

$$\text{span}\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} = U$$

Hence $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is the desired orthogonal basis of U. The procedure can be summarized as follows.



Theorem 8.1.2: Gram-Schmidt Orthogonalization Algorithm[1]

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ *is any basis of a subspace* U *of* $\mathbb{R}^n$*, construct* $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m$ *in* U *successively as follows:*

$$\begin{aligned}
\mathbf{f}_1 &= \mathbf{x}_1 \\
\mathbf{f}_2 &= \mathbf{x}_2 - \tfrac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 \\
\mathbf{f}_3 &= \mathbf{x}_3 - \tfrac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \tfrac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 \\
&\vdots \\
\mathbf{f}_k &= \mathbf{x}_k - \tfrac{\mathbf{x}_k \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \tfrac{\mathbf{x}_k \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 - \cdots - \tfrac{\mathbf{x}_k \cdot \mathbf{f}_{k-1}}{\|\mathbf{f}_{k-1}\|^2}\mathbf{f}_{k-1}
\end{aligned}$$

for each $k = 2, 3, \dots, m$. *Then*

1. $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ *is an orthogonal basis of* U.

2. $\operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k\} = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *for each* $k = 1, 2, \dots, m$.

The process (for $k = 3$) is depicted in the diagrams. Of course, the algorithm converts any basis of $\mathbb{R}^n$ itself into an orthogonal basis.

Example 8.1.1

Find an orthogonal basis of the row space of $A = \left[\begin{array}{rrrr} 1 & 1 & -1 & -1 \\ 3 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right]$.

Solution. Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ denote the rows of A and observe that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent. Take $\mathbf{f}_1 = \mathbf{x}_1$. The algorithm gives

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{x}_2 - \tfrac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = (3, 2, 0, 1) - \tfrac{4}{4}(1, 1, -1, -1) = (2, 1, 1, 2) \\
\mathbf{f}_3 &= \mathbf{x}_3 - \tfrac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \tfrac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \mathbf{x}_3 - \tfrac{0}{4}\mathbf{f}_1 - \tfrac{3}{10}\mathbf{f}_2 = \tfrac{1}{10}(4, -3, 7, -6)
\end{aligned}$$

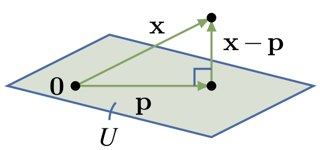
Hence $\{(1, 1, -1, -1), (2, 1, 1, 2), \frac{1}{10}(4, -3, 7, -6)\}$ is the orthogonal basis provided by the algorithm. In hand calculations it may be convenient to eliminate fractions (see the Remark below), so $\{(1, 1, -1, -1), (2, 1, 1, 2), (4, -3, 7, -6)\}$ is also an orthogonal basis for row A.

[1] Erhardt Schmidt (1876–1959) was a German mathematician who studied under the great David Hilbert and later developed the theory of Hilbert spaces. He first described the present algorithm in 1907. Jörgen Pederson Gram (1850–1916) was a Danish actuary.

Remark

Observe that the vector $\frac{\mathbf{x}\cdot\mathbf{f}_i}{\|\mathbf{f}_i\|^2}\mathbf{f}_i$ is unchanged if a nonzero scalar multiple of $\mathbf{f}_i$ is used in place of $\mathbf{f}_i$. Hence, if a newly constructed $\mathbf{f}_i$ is multiplied by a nonzero scalar at some stage of the Gram-Schmidt algorithm, the subsequent $\mathbf{f}$s will be unchanged. This is useful in actual calculations.

Projections



Suppose a point $\mathbf{x}$ and a plane U through the origin in $\mathbb{R}^3$ are given, and we want to find the point $\mathbf{p}$ in the plane that is closest to $\mathbf{x}$. Our geometric intuition assures us that such a point $\mathbf{p}$ exists. In fact (see the diagram), $\mathbf{p}$ must be chosen in such a way that $\mathbf{x}-\mathbf{p}$ is *perpendicular* to the plane.

Now we make two observations: first, the plane U is a *subspace of* $\mathbb{R}^3$ (because U contains the origin); and second, that the condition that $\mathbf{x}-\mathbf{p}$ is perpendicular to the plane U means that $\mathbf{x}-\mathbf{p}$ is *orthogonal* to every vector in U. In these terms the whole discussion makes sense in $\mathbb{R}^n$. Furthermore, the orthogonal lemma provides exactly what is needed to find $\mathbf{p}$ in this more general setting.

Definition 8.1 Orthogonal Complement of a Subspace of $\mathbb{R}^n$

*If U is a subspace of $\mathbb{R}^n$, define the **orthogonal complement** $U^\perp$ of U (pronounced "U-perp") by*

$$U^\perp = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \mathbf{x}\cdot\mathbf{y} = 0 \text{ for all } \mathbf{y} \text{ in } U\}$$

The following lemma collects some useful properties of the orthogonal complement; the proof of (1) and (2) is left as Exercise 8.1.6.

Lemma 8.1.2

Let U be a subspace of $\mathbb{R}^n$.

1. *$U^\perp$ is a subspace of $\mathbb{R}^n$.*
2. *$\{\mathbf{0}\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{\mathbf{0}\}$.*
3. *If $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$, then $U^\perp = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \mathbf{x}\cdot\mathbf{x}_i = 0 \text{ for } i = 1, 2, \dots, k\}$.*

Proof.

3. Let $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$; we must show that $U^\perp = \{\mathbf{x} \mid \mathbf{x}\cdot\mathbf{x}_i = 0 \text{ for each } i\}$. If $\mathbf{x}$ is in $U^\perp$ then $\mathbf{x}\cdot\mathbf{x}_i = 0$ for all i because each $\mathbf{x}_i$ is in U. Conversely, suppose that $\mathbf{x}\cdot\mathbf{x}_i = 0$ for all i; we must show that $\mathbf{x}$ is in $U^\perp$, that is, $\mathbf{x}\cdot\mathbf{y} = 0$ for each $\mathbf{y}$ in U. Write $\mathbf{y} = r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k$, where each r_i is in $\mathbb{R}$. Then, using Theorem 5.3.1,

$$\mathbf{x}\cdot\mathbf{y} = r_1(\mathbf{x}\cdot\mathbf{x}_1) + r_2(\mathbf{x}\cdot\mathbf{x}_2) + \cdots + r_k(\mathbf{x}\cdot\mathbf{x}_k) = r_1 0 + r_2 0 + \cdots + r_k 0 = 0$$

as required. □

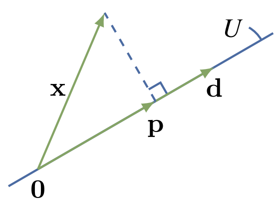
Example 8.1.2

Find $U^\perp$ if $U = \text{span}\{(1,\ -1,\ 2,\ 0),\ (1,\ 0,\ -2,\ 3)\}$ in $\mathbb{R}^4$.

Solution. By Lemma 8.1.2, $\mathbf{x} = (x,\ y,\ z,\ w)$ is in $U^\perp$ if and only if it is orthogonal to both $(1,\ -1,\ 2,\ 0)$ and $(1,\ 0,\ -2,\ 3)$; that is,

$$\begin{aligned} x - y + 2z &= 0 \\ x - 2z + 3w &= 0 \end{aligned}$$

Gaussian elimination gives $U^\perp = \text{span}\{(2,\ 4,\ 1,\ 0),\ (3,\ 3,\ 0,\ -1)\}$.



Now consider vectors $\mathbf{x}$ and $\mathbf{d} \neq \mathbf{0}$ in $\mathbb{R}^3$. The projection $\mathbf{p} = \text{proj}_{\mathbf{d}}\, \mathbf{x}$ of $\mathbf{x}$ on $\mathbf{d}$ was defined in Section 4.2 as in the diagram.

The following formula for $\mathbf{p}$ was derived in Theorem 4.2.4

$$\mathbf{p} = \text{proj}_{\mathbf{d}}\, \mathbf{x} = \left(\frac{\mathbf{x}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\right)\mathbf{d}$$

where it is shown that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{d}$. Now observe that the line $U = \mathbb{R}\mathbf{d} = \{t\mathbf{d} \mid t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$, that $\{\mathbf{d}\}$ is an orthogonal basis of U, and that $\mathbf{p} \in U$ and $\mathbf{x} - \mathbf{p} \in U^\perp$ (by Theorem 4.2.4).

In this form, this makes sense for any vector $\mathbf{x}$ in $\mathbb{R}^n$ and any subspace U of $\mathbb{R}^n$, so we generalize it as follows. If $\{\mathbf{f}_1,\ \mathbf{f}_2,\ \dots,\ \mathbf{f}_m\}$ is an orthogonal basis of U, we define the projection $\mathbf{p}$ of $\mathbf{x}$ on U by the formula

$$\mathbf{p} = \left(\frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\right)\mathbf{f}_1 + \left(\frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\right)\mathbf{f}_2 + \cdots + \left(\frac{\mathbf{x}\cdot\mathbf{f}_m}{\|\mathbf{f}_m\|^2}\right)\mathbf{f}_m \tag{8.1}$$

Then $\mathbf{p} \in U$ and (by the orthogonal lemma) $\mathbf{x} - \mathbf{p} \in U^\perp$, so it looks like we have a generalization of Theorem 4.2.4.

However there is a potential problem: the formula (8.1) for $\mathbf{p}$ must be shown to be independent of the choice of the orthogonal basis $\{\mathbf{f}_1,\ \mathbf{f}_2,\ \dots,\ \mathbf{f}_m\}$. To verify this, suppose that $\{\mathbf{f}'_1,\ \mathbf{f}'_2,\ \dots,\ \mathbf{f}'_m\}$ is another orthogonal basis of U, and write

$$\mathbf{p}' = \left(\frac{\mathbf{x}\cdot\mathbf{f}'_1}{\|\mathbf{f}'_1\|^2}\right)\mathbf{f}'_1 + \left(\frac{\mathbf{x}\cdot\mathbf{f}'_2}{\|\mathbf{f}'_2\|^2}\right)\mathbf{f}'_2 + \cdots + \left(\frac{\mathbf{x}\cdot\mathbf{f}'_m}{\|\mathbf{f}'_m\|^2}\right)\mathbf{f}'_m$$

As before, $\mathbf{p}' \in U$ and $\mathbf{x} - \mathbf{p}' \in U^\perp$, and we must show that $\mathbf{p}' = \mathbf{p}$. To see this, write the vector $\mathbf{p} - \mathbf{p}'$ as follows:

$$\mathbf{p} - \mathbf{p}' = (\mathbf{x} - \mathbf{p}') - (\mathbf{x} - \mathbf{p})$$

This vector is in U (because $\mathbf{p}$ and $\mathbf{p}'$ are in U) and it is in $U^\perp$ (because $\mathbf{x} - \mathbf{p}'$ and $\mathbf{x} - \mathbf{p}$ are in $U^\perp$), and so it must be zero (it is orthogonal to itself!). This means $\mathbf{p}' = \mathbf{p}$ as desired.

Hence, the vector $\mathbf{p}$ in equation (8.1) depends only on $\mathbf{x}$ and the subspace U, and *not* on the choice of orthogonal basis $\{\mathbf{f}_1,\ \dots,\ \mathbf{f}_m\}$ of U used to compute it. Thus, we are entitled to make the following definition:

Definition 8.2 Projection onto a Subspace of $\mathbb{R}^n$

Let U be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$. If $\mathbf{x}$ is in $\mathbb{R}^n$, the vector

$$\text{proj}_U \mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \cdots + \frac{\mathbf{x}\cdot\mathbf{f}_m}{\|\mathbf{f}_m\|^2}\mathbf{f}_m$$

*is called the **orthogonal projection** of $\mathbf{x}$ on U. For the zero subspace $U = \{\mathbf{0}\}$, we define*

$$\text{proj}_{\{\mathbf{0}\}} \mathbf{x} = \mathbf{0}$$

The preceding discussion proves (1) of the following theorem.

Theorem 8.1.3: Projection Theorem

If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in $\mathbb{R}^n$, write $\mathbf{p} = \text{proj}_U \mathbf{x}$. Then:

1. *$\mathbf{p}$ is in U and $\mathbf{x} - \mathbf{p}$ is in $U^\perp$.*
2. *$\mathbf{p}$ is the vector in U closest to $\mathbf{x}$ in the sense that*

$$\|\mathbf{x} - \mathbf{p}\| < \|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{y} \in U, \ \mathbf{y} \neq \mathbf{p}$$

Proof.

1. This is proved in the preceding discussion (it is clear if $U = \{\mathbf{0}\}$).
2. Write $\mathbf{x} - \mathbf{y} = (\mathbf{x} - \mathbf{p}) + (\mathbf{p} - \mathbf{y})$. Then $\mathbf{p} - \mathbf{y}$ is in U and so is orthogonal to $\mathbf{x} - \mathbf{p}$ by (1). Hence, the Pythagorean theorem gives

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{p}\|^2 + \|\mathbf{p} - \mathbf{y}\|^2 > \|\mathbf{x} - \mathbf{p}\|^2$$

because $\mathbf{p} - \mathbf{y} \neq \mathbf{0}$. This gives (2). □

Example 8.1.3

Let $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ in $\mathbb{R}^4$ where $\mathbf{x}_1 = (1, 1, 0, 1)$ and $\mathbf{x}_2 = (0, 1, 1, 2)$. If $\mathbf{x} = (3, -1, 0, 2)$, find the vector in U closest to $\mathbf{x}$ and express $\mathbf{x}$ as the sum of a vector in U and a vector orthogonal to U.

Solution. $\{\mathbf{x}_1, \mathbf{x}_2\}$ is independent but not orthogonal. The Gram-Schmidt process gives an orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ of U where $\mathbf{f}_1 = \mathbf{x}_1 = (1, 1, 0, 1)$ and

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = \mathbf{x}_2 - \frac{3}{3}\mathbf{f}_1 = (-1, 0, 1, 1)$$

Hence, we can compute the projection using $\{\mathbf{f}_1, \mathbf{f}_2\}$:

$$\mathbf{p} = \text{proj}_U \mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \frac{4}{3}\mathbf{f}_1 + \frac{-1}{3}\mathbf{f}_2 = \frac{1}{3}\left[\begin{array}{cccc} 5 & 4 & -1 & 3 \end{array}\right]$$

Thus, $\mathbf{p}$ is the vector in U closest to $\mathbf{x}$, and $\mathbf{x}-\mathbf{p} = \frac{1}{3}(4,\ -7,\ 1,\ 3)$ is orthogonal to every vector in U. (This can be verified by checking that it is orthogonal to the generators $\mathbf{x}_1$ and $\mathbf{x}_2$ of U.) The required decomposition of $\mathbf{x}$ is thus

$$\mathbf{x} = \mathbf{p} + (\mathbf{x}-\mathbf{p}) = \tfrac{1}{3}(5,\ 4,\ -1,\ 3) + \tfrac{1}{3}(4,\ -7,\ 1,\ 3)$$

Example 8.1.4

Find the point in the plane with equation $2x+y-z=0$ that is closest to the point $(2,\ -1,\ -3)$.

Solution. We write $\mathbb{R}^3$ as rows. The plane is the subspace U whose points $(x,\ y,\ z)$ satisfy $z = 2x+y$. Hence

$$U = \{(s,\ t,\ 2s+t) \mid s,\ t \text{ in } \mathbb{R}\} = \text{span}\,\{(0,\ 1,\ 1),\ (1,\ 0,\ 2)\}$$

The Gram-Schmidt process produces an orthogonal basis $\{\mathbf{f}_1,\ \mathbf{f}_2\}$ of U where $\mathbf{f}_1 = (0,\ 1,\ 1)$ and $\mathbf{f}_2 = (1,\ -1,\ 1)$. Hence, the vector in U closest to $\mathbf{x} = (2,\ -1,\ -3)$ is

$$\text{proj}_U\ \mathbf{x} = \tfrac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \tfrac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = -2\mathbf{f}_1 + 0\mathbf{f}_2 = (0,\ -2,\ -2)$$

Thus, the point in U closest to $(2,\ -1,\ -3)$ is $(0,\ -2,\ -2)$.

The next theorem shows that projection on a subspace of $\mathbb{R}^n$ is actually a linear operator $\mathbb{R}^n \to \mathbb{R}^n$.

Theorem 8.1.4

Let U be a fixed subspace of $\mathbb{R}^n$. If we define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\mathbf{x}) = \text{proj}_U\ \mathbf{x} \quad \textit{for all } \mathbf{x} \textit{ in } \mathbb{R}^n$$

1. *T is a linear operator.*
2. *$\text{im}\ T = U$ and $\ker T = U^\perp$.*
3. *$\dim U + \dim U^\perp = n$.*

Proof. If $U = \{\mathbf{0}\}$, then $U^\perp = \mathbb{R}^n$, and so $T(\mathbf{x}) = \text{proj}_{\{\mathbf{0}\}}\ \mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$. Thus $T = 0$ is the zero (linear) operator, so (1), (2), and (3) hold. Hence assume that $U \neq \{\mathbf{0}\}$.

1. If $\{\mathbf{f}_1,\ \mathbf{f}_2,\ \dots,\ \mathbf{f}_m\}$ is an orthonormal basis of U, then

$$T(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{f}_1)\mathbf{f}_1 + (\mathbf{x}\cdot\mathbf{f}_2)\mathbf{f}_2 + \cdots + (\mathbf{x}\cdot\mathbf{f}_m)\mathbf{f}_m \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{8.2}$$

by the definition of the projection. Thus T is linear because

$$(\mathbf{x}+\mathbf{y})\cdot\mathbf{f}_i = \mathbf{x}\cdot\mathbf{f}_i + \mathbf{y}\cdot\mathbf{f}_i \quad \text{and} \quad (r\mathbf{x})\cdot\mathbf{f}_i = r(\mathbf{x}\cdot\mathbf{f}_i) \quad \text{for each } i$$

2. We have $\text{im } T \subseteq U$ by (8.2) because each $\mathbf{f}_i$ is in U. But if $\mathbf{x}$ is in U, then $\mathbf{x} = T(\mathbf{x})$ by (8.2) and the expansion theorem applied to the space U. This shows that $U \subseteq \text{im } T$, so $\text{im } T = U$. Now suppose that $\mathbf{x}$ is in $U^\perp$. Then $\mathbf{x} \cdot \mathbf{f}_i = 0$ for each i (again because each $\mathbf{f}_i$ is in U) so $\mathbf{x}$ is in $\ker T$ by (8.2). Hence $U^\perp \subseteq \ker T$. On the other hand, Theorem 8.1.3 shows that $\mathbf{x} - T(\mathbf{x})$ is in $U^\perp$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and it follows that $\ker T \subseteq U^\perp$. Hence $\ker T = U^\perp$, proving (2).

3. This follows from (1), (2), and the dimension theorem (Theorem 7.2.4). □

Exercises for 8.1

Exercise 8.1.1 In each case, use the Gram-Schmidt algorithm to convert the given basis B of V into an orthogonal basis.

a. $V = \mathbb{R}^2$, $B = \{(1, -1), (2, 1)\}$

b. $V = \mathbb{R}^2$, $B = \{(2, 1), (1, 2)\}$

c. $V = \mathbb{R}^3$, $B = \{(1, -1, 1), (1, 0, 1), (1, 1, 2)\}$

d. $V = \mathbb{R}^3$, $B = \{(0, 1, 1), (1, 1, 1), (1, -2, 2)\}$

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b. $\{(2, 1), \frac{3}{5}(-1, 2)\}$

d. $\{(0, 1, 1), (1, 0, 0), (0, -2, 2)\}$

Exercise 8.1.2 In each case, write $\mathbf{x}$ as the sum of a vector in U and a vector in $U^\perp$.

a. $\mathbf{x} = (1, 5, 7)$, $U = \text{span}\{(1, -2, 3), (-1, 1, 1)\}$

b. $\mathbf{x} = (2, 1, 6)$, $U = \text{span}\{(3, -1, 2), (2, 0, -3)\}$

c. $\mathbf{x} = (3, 1, 5, 9)$,
$U = \text{span}\{(1, 0, 1, 1), (0, 1, -1, 1), (-2, 0, 1, 1)\}$

d. $\mathbf{x} = (2, 0, 1, 6)$,
$U = \text{span}\{(1, 1, 1, 1), (1, 1, -1, -1), (1, -1, 1, -1)\}$

e. $\mathbf{x} = (a, b, c, d)$,
$U = \text{span}\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$

f. $\mathbf{x} = (a, b, c, d)$,
$U = \text{span}\{(1, -1, 2, 0), (-1, 1, 1, 1)\}$

---

b. $\mathbf{x} = \frac{1}{182}(271, -221, 1030) + \frac{1}{182}(93, 403, 62)$

d. $\mathbf{x} = \frac{1}{4}(1, 7, 11, 17) + \frac{1}{4}(7, -7, -7, 7)$

f. $\mathbf{x} = \frac{1}{12}(5a - 5b + c - 3d, -5a + 5b - c + 3d, a - b + 11c + 3d, -3a + 3b + 3c + 3d) + \frac{1}{12}(7a + 5b - c + 3d, 5a + 7b + c - 3d, -a + b + c - 3d, 3a - 3b - 3c + 9d)$

Exercise 8.1.3 Let $\mathbf{x} = (1, -2, 1, 6)$ in $\mathbb{R}^4$, and let $U = \text{span}\{(2, 1, 3, -4), (1, 2, 0, 1)\}$.

a. Compute $\text{proj}_U \mathbf{x}$.

b. Show that $\{(1, 0, 2, -3), (4, 7, 1, 2)\}$ is another orthogonal basis of U.

c. Use the basis in part (b) to compute $\text{proj}_U \mathbf{x}$.

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a. $\frac{1}{10}(-9, 3, -21, 33) = \frac{3}{10}(-3, 1, -7, 11)$

c. $\frac{1}{70}(-63, 21, -147, 231) = \frac{3}{10}(-3, 1, -7, 11)$

Exercise 8.1.4 In each case, use the Gram-Schmidt algorithm to find an orthogonal basis of the subspace U, and find the vector in U closest to $\mathbf{x}$.

a. $U = \text{span}\{(1, 1, 1), (0, 1, 1)\}$, $\mathbf{x} = (-1, 2, 1)$

b. $U = \text{span}\{(1, -1, 0), (-1, 0, 1)\}$, $\mathbf{x} = (2, 1, 0)$

c. $U = \text{span}\{(1, 0, 1, 0), (1, 1, 1, 0), (1, 1, 0, 0)\}$, $\mathbf{x} = (2, 0, -1, 3)$

d. $U = \text{span}\{(1, -1, 0, 1), (1, 1, 0, 0), (1, 1, 0, 1)\}$, $\mathbf{x} = (2, 0, 3, 1)$

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b. $\{(1, -1, 0), \frac{1}{2}(-1, -1, 2)\}$; $\text{proj}_U \mathbf{x} = (1, 0, -1)$

d. $\{(1, -1, 0, 1), (1, 1, 0, 0), \frac{1}{3}(-1, 1, 0, 2)\}$; $\text{proj}_U \mathbf{x} = (2, 0, 0, 1)$

Exercise 8.1.5 Let $U = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, $\mathbf{v}_i$ in $\mathbb{R}^n$, and let A be the $k \times n$ matrix with the $\mathbf{v}_i$ as rows.

a. Show that $U^\perp = \{\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n, A\mathbf{x}^T = \mathbf{0}\}$.

b. Use part (a) to find $U^\perp$ if $U = \text{span}\{(1, -1, 2, 1), (1, 0, -1, 1)\}$.

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b. $U^\perp = \text{span}\{(1, 3, 1, 0), (-1, 0, 0, 1)\}$

Exercise 8.1.6

a. Prove part 1 of Lemma 8.1.2.

b. Prove part 2 of Lemma 8.1.2.

Exercise 8.1.7 Let U be a subspace of $\mathbb{R}^n$. If $\mathbf{x}$ in $\mathbb{R}^n$ can be written in any way at all as $\mathbf{x} = \mathbf{p} + \mathbf{q}$ with $\mathbf{p}$ in U and $\mathbf{q}$ in $U^\perp$, show that necessarily $\mathbf{p} = \text{proj}_U \mathbf{x}$.

Exercise 8.1.8 Let U be a subspace of $\mathbb{R}^n$ and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Using Exercise 8.1.7, or otherwise, show that $\mathbf{x}$ is in U if and only if $\mathbf{x} = \text{proj}_U \mathbf{x}$.

Write $\mathbf{p} = \text{proj}_U \mathbf{x}$. Then $\mathbf{p}$ is in U by definition. If $\mathbf{x}$ is U, then $\mathbf{x} - \mathbf{p}$ is in U. But $\mathbf{x} - \mathbf{p}$ is also in $U^\perp$ by Theorem 8.1.3, so $\mathbf{x} - \mathbf{p}$ is in $U \cap U^\perp = \{\mathbf{0}\}$. Thus $\mathbf{x} = \mathbf{p}$.

Exercise 8.1.9 Let U be a subspace of $\mathbb{R}^n$.

a. Show that $U^\perp = \mathbb{R}^n$ if and only if $U = \{\mathbf{0}\}$.

b. Show that $U^\perp = \{\mathbf{0}\}$ if and only if $U = \mathbb{R}^n$.

Exercise 8.1.10 If U is a subspace of $\mathbb{R}^n$, show that $\text{proj}_U \mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in U.

Let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ be an orthonormal basis of U. If $\mathbf{x}$ is in U the expansion theorem gives $\mathbf{x} = (\mathbf{x} \cdot \mathbf{f}_1)\mathbf{f}_1 + (\mathbf{x} \cdot \mathbf{f}_2)\mathbf{f}_2 + \cdots + (\mathbf{x} \cdot \mathbf{f}_m)\mathbf{f}_m = \text{proj}_U \mathbf{x}$.

Exercise 8.1.11 If U is a subspace of $\mathbb{R}^n$, show that $\mathbf{x} = \text{proj}_U \mathbf{x} + \text{proj}_{U^\perp} \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 8.1.12 If $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ is an orthogonal basis of $\mathbb{R}^n$ and $U = \text{span}\{\mathbf{f}_1, \dots, \mathbf{f}_m\}$, show that $U^\perp = \text{span}\{\mathbf{f}_{m+1}, \dots, \mathbf{f}_n\}$.

Exercise 8.1.13 If U is a subspace of $\mathbb{R}^n$, show that $U^{\perp\perp} = U$. [*Hint*: Show that $U \subseteq U^{\perp\perp}$, then use Theorem 8.1.4 (3) twice.]

Exercise 8.1.14 If U is a subspace of $\mathbb{R}^n$, show how to find an $n \times n$ matrix A such that $U = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$. [*Hint*: Exercise 8.1.13.]

Let $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m\}$ be a basis of $U^\perp$, and let A be the $n \times n$ matrix with rows $\mathbf{y}_1^T, \mathbf{y}_2^T, \dots, \mathbf{y}_m^T, 0, \dots, 0$. Then $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{y}_i \cdot \mathbf{x} = 0$ for each $i = 1, 2, \dots, m$; if and only if $\mathbf{x}$ is in $U^{\perp\perp} = U$.

Exercise 8.1.15 Write $\mathbb{R}^n$ as rows. If A is an $n \times n$ matrix, write its null space as $\text{null } A = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x}^T = \mathbf{0}\}$. Show that:

a) $\text{null } A = (\text{row } A)^\perp$; b) $\text{null } A^T = (\text{col } A)^\perp$.

Exercise 8.1.16 If U and W are subspaces, show that $(U + W)^\perp = U^\perp \cap W^\perp$. [See Exercise 5.1.22.]

Exercise 8.1.17 Think of $\mathbb{R}^n$ as consisting of rows.

a. Let E be an $n \times n$ matrix, and let $U = \{\mathbf{x}E \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$. Show that the following are equivalent.

   i. $E^2 = E = E^T$ (E is a **projection matrix**).

   ii. $(\mathbf{x} - \mathbf{x}E) \cdot (\mathbf{y}E) = 0$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

   iii. $\text{proj}_U \mathbf{x} = \mathbf{x}E$ for all $\mathbf{x}$ in $\mathbb{R}^n$. [*Hint*: For (ii) implies (iii): Write $\mathbf{x} = \mathbf{x}E + (\mathbf{x} - \mathbf{x}E)$ and use the uniqueness argument preceding the definition of $\text{proj}_U \mathbf{x}$. For (iii) implies (ii): $\mathbf{x} - \mathbf{x}E$ is in $U^\perp$ for all $\mathbf{x}$ in $\mathbb{R}^n$.]

b. If E is a projection matrix, show that $I - E$ is also a projection matrix.

c. If $EF = 0 = FE$ and E and F are projection matrices, show that $E+F$ is also a projection matrix.

d. If A is $m \times n$ and AA^T is invertible, show that $E = A^T(AA^T)^{-1}A$ is a projection matrix.

---

d. $E^T = A^T[(AA^T)^{-}1]^T(A^T)^T = A^T[(AA^T)^T]^{-1}A = A^T[AA^T]^{-1}A = E$ $E^2 = A^T(AA^T)^{-1}AA^T(AA^T)^{-1}A = A^T(AA^T)^{-1}A = E$

Exercise 8.1.18 Let A be an $n \times n$ matrix of rank r. Show that there is an invertible $n \times n$ matrix U such that UA is a row-echelon matrix with the property that the first r rows are orthogonal. [*Hint*: Let R be the row-echelon form of A, and use the Gram-Schmidt process on the nonzero rows of R from the bottom up. Use Lemma 2.4.1.]

Exercise 8.1.19 Let A be an $(n-1) \times n$ matrix with rows $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-1}$ and let A_i denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting column i. Define the vector $\mathbf{y}$ in $\mathbb{R}^n$ by

$$\mathbf{y} = \left[\begin{array}{cccc} \det A_1 & -\det A_2 & \det A_3 & \cdots & (-1)^{n+1}\det A_n \end{array}\right]$$

Show that:

a. $\mathbf{x}_i \cdot \mathbf{y} = 0$ for all $i = 1, 2, \dots, n-1$. [*Hint*: Write $B_i = \left[\begin{array}{c} x_i \\ A \end{array}\right]$ and show that $\det B_i = 0$.]

b. $\mathbf{y} \neq \mathbf{0}$ if and only if $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-1}\}$ is linearly independent. [*Hint*: If some $\det A_i \neq 0$, the rows of A_i are linearly independent. Conversely, if the $\mathbf{x}_i$ are independent, consider $A = UR$ where R is in reduced row-echelon form.]

c. If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-1}\}$ is linearly independent, use Theorem 8.1.3(3) to show that all solutions to the system of $n-1$ homogeneous equations

$$A\mathbf{x}^T = \mathbf{0}$$

are given by $t\mathbf{y}$, t a parameter.

8.2 Orthogonal Diagonalization

Recall (Theorem 5.5.3) that an $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. Moreover, the matrix P with these eigenvectors as columns is a diagonalizing matrix for A, that is

$$P^{-1}AP \text{ is diagonal.}$$

As we have seen, the really nice bases of $\mathbb{R}^n$ are the orthogonal ones, so a natural question is: which $n \times n$ matrices have an *orthogonal* basis of eigenvectors? These turn out to be precisely the symmetric matrices, and this is the main result of this section.

Before proceeding, recall that an orthogonal set of vectors is called *orthonormal* if $\|\mathbf{v}\| = 1$ for each vector $\mathbf{v}$ in the set, and that any orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ can be "*normalized*", that is converted into an orthonormal set $\{\frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1, \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2, \dots, \frac{1}{\|\mathbf{v}_k\|}\mathbf{v}_k\}$. In particular, if a matrix A has n orthogonal eigenvectors, they can (by normalizing) be taken to be orthonormal. The corresponding diagonalizing matrix P has orthonormal columns, and such matrices are very easy to invert.

Theorem 8.2.1

The following conditions are equivalent for an $n \times n$ matrix P.

1. *P is invertible and $P^{-1} = P^T$.*
2. *The rows of P are orthonormal.*
3. *The columns of P are orthonormal.*

Proof. First recall that condition (1) is equivalent to $PP^T = I$ by Corollary 2.4.1 of Theorem 2.4.5. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ denote the rows of P. Then $\mathbf{x}_j^T$ is the jth column of P^T, so the (i, j)-entry of PP^T is $\mathbf{x}_i \cdot \mathbf{x}_j$. Thus $PP^T = I$ means that $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ if $i \neq j$ and $\mathbf{x}_i \cdot \mathbf{x}_j = 1$ if $i = j$. Hence condition (1) is equivalent to (2). The proof of the equivalence of (1) and (3) is similar. □

Definition 8.3 Orthogonal Matrices

An $n \times n$ matrix P is called an ***orthogonal matrix***[2] *if it satisfies one (and hence all) of the conditions in Theorem 8.2.1.*

Example 8.2.1

The rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal for any angle θ.

These orthogonal matrices have the virtue that they are easy to invert—simply take the transpose. But they have many other important properties as well. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator,

[2] In view of (2) and (3) of Theorem 8.2.1, *orthonormal matrix* might be a better name. But *orthogonal matrix* is standard.

we will prove (Theorem **??**) that T is distance preserving if and only if its matrix is orthogonal. In particular, the matrices of rotations and reflections about the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$ are all orthogonal (see Example 8.2.1).

It is not enough that the rows of a matrix A are merely orthogonal for A to be an orthogonal matrix. Here is an example.

Example 8.2.2

The matrix $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ has orthogonal rows but the columns are not orthogonal.

However, if the rows are normalized, the resulting matrix $\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal (so the columns are now orthonormal as the reader can verify).

Example 8.2.3

If P and Q are orthogonal matrices, then PQ is also orthogonal, as is $P^{-1} = P^T$.

Solution. P and Q are invertible, so PQ is also invertible and

$$(PQ)^{-1} = Q^{-1}P^{-1} = Q^T P^T = (PQ)^T$$

Hence PQ is orthogonal. Similarly,

$$(P^{-1})^{-1} = P = (P^T)^T = (P^{-1})^T$$

shows that P^{-1} is orthogonal.

Definition 8.4 Orthogonally Diagonalizable Matrices

An $n \times n$ matrix A is said to be ***orthogonally diagonalizable*** *when an orthogonal matrix P can be found such that $P^{-1}AP = P^T AP$ is diagonal.*

This condition turns out to characterize the symmetric matrices.

Theorem 8.2.2: Principal Axes Theorem

The following conditions are equivalent for an $n \times n$ matrix A.

1. *A has an orthonormal set of n eigenvectors.*
2. *A is orthogonally diagonalizable.*

3. *A is symmetric.*

Proof. (1) $\Leftrightarrow$ (2). Given (1), let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be orthonormal eigenvectors of A. Then $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix}$ is orthogonal, and $P^{-1}AP$ is diagonal by Theorem 3.3.4. This proves (2). Conversely, given (2) let $P^{-1}AP$ be diagonal where P is orthogonal. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the columns of P then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ that consists of eigenvectors of A by Theorem 3.3.4. This proves (1).

(2) $\Rightarrow$ (3). If $P^TAP = D$ is diagonal, where $P^{-1} = P^T$, then $A = PDP^T$. But $D^T = D$, so this gives $A^T = P^{TT}D^TP^T = PDP^T = A$.

(3) $\Rightarrow$ (2). If A is an $n \times n$ symmetric matrix, we proceed by induction on n. If $n = 1$, A is already diagonal. If $n > 1$, assume that (3) $\Rightarrow$ (2) for $(n-1)\times(n-1)$ symmetric matrices. By Theorem 5.5.7 let λ_1 be a (real) eigenvalue of A, and let $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, where $\|\mathbf{x}_1\| = 1$. Use the Gram-Schmidt algorithm to find an orthonormal basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ for $\mathbb{R}^n$. Let $P_1 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix}$, so P_1 is an orthogonal matrix and $P_1^TAP_1 = \begin{bmatrix} \lambda_1 & B \\ 0 & A_1 \end{bmatrix}$ in block form by Lemma 5.5.2. But $P_1^TAP_1$ is symmetric (A is), so it follows that $B = 0$ and A_1 is symmetric. Then, by induction, there exists an $(n-1)\times(n-1)$ orthogonal matrix Q such that $Q^TA_1Q = D_1$ is diagonal. Observe that $P_2 = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$ is orthogonal, and compute:

$$\begin{aligned}(P_1P_2)^TA(P_1P_2) &= P_2^T(P_1^TAP_1)P_2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & D_1 \end{bmatrix}\end{aligned}$$

is diagonal. Because P_1P_2 is orthogonal, this proves (2). □

A set of orthonormal eigenvectors of a symmetric matrix A is called a set of **principal axes** for A. The name comes from geometry, and this is discussed in Section ??. Because the eigenvalues of a (real) symmetric matrix are real, Theorem 8.2.2 is also called the **real spectral theorem**, and the set of distinct eigenvalues is called the **spectrum** of the matrix. In full generality, the spectral theorem is a similar result for matrices with complex entries (Theorem ??).

Example 8.2.4

Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix}$.

Solution. The characteristic polynomial of A is (adding twice row 1 to row 2):

$$c_A(x) = \det \begin{bmatrix} x-1 & 0 & 1 \\ 0 & x-1 & -2 \\ 1 & -2 & x-5 \end{bmatrix} = x(x-1)(x-6)$$

Thus the eigenvalues are $\lambda = 0$, 1, and 6, and corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

respectively. Moreover, by what appears to be remarkably good luck, these eigenvectors are *orthogonal*. We have $\|\mathbf{x}_1\|^2 = 6$, $\|\mathbf{x}_2\|^2 = 5$, and $\|\mathbf{x}_3\|^2 = 30$, so

$$P = \left[\begin{array}{ccc} \frac{1}{\sqrt{6}}\mathbf{x}_1 & \frac{1}{\sqrt{5}}\mathbf{x}_2 & \frac{1}{\sqrt{30}}\mathbf{x}_3 \end{array} \right] = \frac{1}{\sqrt{30}} \begin{bmatrix} \sqrt{5} & 2\sqrt{6} & -1 \\ -2\sqrt{5} & \sqrt{6} & 2 \\ \sqrt{5} & 0 & 5 \end{bmatrix}$$

is an orthogonal matrix. Thus $P^{-1} = P^T$ and

$$P^TAP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

by the diagonalization algorithm.

Actually, the fact that the eigenvectors in Example 8.2.4 are orthogonal is no coincidence. Theorem 5.5.4 guarantees they are linearly independent (they correspond to distinct eigenvalues); the fact that the matrix is *symmetric* implies that they are orthogonal. To prove this we need the following useful fact about symmetric matrices.

Theorem 8.2.3

If A is an $n \times n$ symmetric matrix, then

$$(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$$

for all columns $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.[3]

Proof. Recall that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$ for all columns $\mathbf{x}$ and $\mathbf{y}$. Because $A^T = A$, we get

$$(A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^TA^T\mathbf{y} = \mathbf{x}^TA\mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$$

□

Theorem 8.2.4

If A is a symmetric matrix, then eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

[3] The converse also holds (Exercise 8.2.15).

Proof. Let $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$, where $\lambda \neq \mu$. Using Theorem 8.2.3, we compute

$$\lambda(\mathbf{x}\cdot\mathbf{y}) = (\lambda\mathbf{x})\cdot\mathbf{y} = (A\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y}) = \mathbf{x}\cdot(\mu\mathbf{y}) = \mu(\mathbf{x}\cdot\mathbf{y})$$

Hence $(\lambda-\mu)(\mathbf{x}\cdot\mathbf{y}) = 0$, and so $\mathbf{x}\cdot\mathbf{y} = 0$ because $\lambda \neq \mu$. □

Now the procedure for diagonalizing a symmetric $n\times n$ matrix is clear. Find the distinct eigenvalues (all real by Theorem 5.5.7) and find orthonormal bases for each eigenspace (the Gram-Schmidt algorithm may be needed). Then the set of all these basis vectors is orthonormal (by Theorem 8.2.4) and contains n vectors. Here is an example.

Example 8.2.5

Orthogonally diagonalize the symmetric matrix $A = \left[\begin{array}{rrr} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{array}\right]$.

Solution. The characteristic polynomial is

$$c_A(x) = \det \left[\begin{array}{ccc} x-8 & 2 & -2 \\ 2 & x-5 & -4 \\ -2 & -4 & x-5 \end{array}\right] = x(x-9)^2$$

Hence the distinct eigenvalues are 0 and 9 of multiplicities 1 and 2, respectively, so $\dim(E_0) = 1$ and $\dim(E_9) = 2$ by Theorem 5.5.6 (A is diagonalizable, being symmetric). Gaussian elimination gives

$$E_0(A) = \operatorname{span}\{\mathbf{x}_1\}, \quad \mathbf{x}_1 = \left[\begin{array}{r} 1 \\ 2 \\ -2 \end{array}\right], \quad \text{and} \quad E_9(A) = \operatorname{span}\left\{\left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} 2 \\ 0 \\ 1 \end{array}\right]\right\}$$

The eigenvectors in E_9 are both orthogonal to $\mathbf{x}_1$ as Theorem 8.2.4 guarantees, but not to each other. However, the Gram-Schmidt process yields an orthogonal basis

$$\{\mathbf{x}_2,\ \mathbf{x}_3\} \text{ of } E_9(A) \quad \text{where} \quad \mathbf{x}_2 = \left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array}\right] \text{ and } \mathbf{x}_3 = \left[\begin{array}{r} 2 \\ 4 \\ 5 \end{array}\right]$$

Normalizing gives orthonormal vectors $\{\frac{1}{3}\mathbf{x}_1,\ \frac{1}{\sqrt{5}}\mathbf{x}_2,\ \frac{1}{3\sqrt{5}}\mathbf{x}_3\}$, so

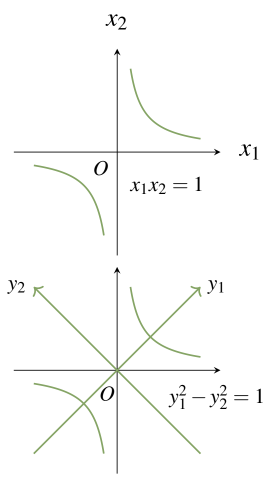
$$P = \left[\begin{array}{ccc} \frac{1}{3}\mathbf{x}_1 & \frac{1}{\sqrt{5}}\mathbf{x}_2 & \frac{1}{3\sqrt{5}}\mathbf{x}_3 \end{array}\right] = \frac{1}{3\sqrt{5}}\left[\begin{array}{rrr} \sqrt{5} & -6 & 2 \\ 2\sqrt{5} & 3 & 4 \\ -2\sqrt{5} & 0 & 5 \end{array}\right]$$

is an orthogonal matrix such that $P^{-1}AP$ is diagonal.
It is worth noting that other, more convenient, diagonalizing matrices P exist. For example, $\mathbf{y}_2 = \left[\begin{array}{r} 2 \\ 1 \\ 2 \end{array}\right]$ and $\mathbf{y}_3 = \left[\begin{array}{r} -2 \\ 2 \\ 1 \end{array}\right]$ lie in $E_9(A)$ and they are orthogonal. Moreover, they both

have norm 3 (as does $\mathbf{x}_1$), so

$$Q = \left[\begin{array}{ccc} \frac{1}{3}\mathbf{x}_1 & \frac{1}{3}\mathbf{y}_2 & \frac{1}{3}\mathbf{y}_3 \end{array} \right] = \frac{1}{3} \left[\begin{array}{rrr} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{array} \right]$$

is a nicer orthogonal matrix with the property that $Q^{-1}AQ$ is diagonal.



If A is symmetric and a set of orthogonal eigenvectors of A is given, the eigenvectors are called principal axes of A. The name comes from geometry. An expression $q = ax_1^2 + bx_1x_2 + cx_2^2$ is called a **quadratic form** in the variables x_1 and x_2, and the graph of the equation $q = 1$ is called a **conic** in these variables. For example, if $q = x_1x_2$, the graph of $q = 1$ is given in the first diagram.

But if we introduce new variables y_1 and y_2 by setting $x_1 = y_1 + y_2$ and $x_2 = y_1 - y_2$, then q becomes $q = y_1^2 - y_2^2$, a diagonal form with no cross term y_1y_2 (see the second diagram). Because of this, the y_1 and y_2 axes are called the principal axes for the conic (hence the name). Orthogonal diagonalization provides a systematic method for finding principal axes. Here is an illustration.

Example 8.2.6

Find principal axes for the quadratic form $q = x_1^2 - 4x_1x_2 + x_2^2$.

Solution. In order to utilize diagonalization, we first express q in matrix form. Observe that

$$q = \left[\begin{array}{cc} x_1 & x_2 \end{array} \right] \left[\begin{array}{rr} 1 & -4 \\ 0 & 1 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]$$

The matrix here is not symmetric, but we can remedy that by writing

$$q = x_1^2 - 2x_1x_2 - 2x_2x_1 + x_2^2$$

Then we have

$$q = \left[\begin{array}{cc} x_1 & x_2 \end{array} \right] \left[\begin{array}{rr} 1 & -2 \\ -2 & 1 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \mathbf{x}^T A \mathbf{x}$$

where $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]$ and $A = \left[\begin{array}{rr} 1 & -2 \\ -2 & 1 \end{array} \right]$ is symmetric. The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$, with corresponding (orthogonal) eigenvectors $\mathbf{x}_1 = \left[\begin{array}{r} 1 \\ -1 \end{array} \right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$. Since $\|\mathbf{x}_1\| = \|\mathbf{x}_2\| = \sqrt{2}$, so

$$P = \frac{1}{\sqrt{2}} \left[\begin{array}{rr} 1 & 1 \\ -1 & 1 \end{array} \right] \text{ is orthogonal and } P^TAP = D = \left[\begin{array}{rr} 3 & 0 \\ 0 & -1 \end{array} \right]$$

Now define new variables $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{y}$ by $\mathbf{y} = P^T\mathbf{x}$, equivalently $\mathbf{x} = P\mathbf{y}$ (since $P^{-1} = P^T$). Hence

$$y_1 = \frac{1}{\sqrt{2}}(x_1 - x_2) \quad \text{and} \quad y_2 = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

In terms of y_1 and y_2, q takes the form

$$q = \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T(P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 3y_1^2 - y_2^2$$

Note that $\mathbf{y} = P^T\mathbf{x}$ is obtained from $\mathbf{x}$ by a counterclockwise rotation of $\frac{\pi}{4}$ (see Theorem 2.4.6).

Observe that the quadratic form q in Example 8.2.6 can be diagonalized in other ways. For example

$$q = x_1^2 - 4x_1x_2 + x_2^2 = z_1^2 - \frac{1}{3}z_2^2$$

where $z_1 = x_1 - 2x_2$ and $z_2 = 3x_2$. We examine this more carefully in Section ??.

If we are willing to replace "diagonal" by "upper triangular" in the principal axes theorem, we can weaken the requirement that A is symmetric to insisting only that A has real eigenvalues.

Theorem 8.2.5: Triangulation Theorem

If A is an $n \times n$ matrix with n real eigenvalues, an orthogonal matrix P exists such that $P^T AP$ is upper triangular.[4]

Proof. We modify the proof of Theorem 8.2.2. If $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ where $\|\mathbf{x}_1\| = 1$, let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be an orthonormal basis of $\mathbb{R}^n$, and let $P_1 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$. Then P_1 is orthogonal and $P_1^T AP_1 = \begin{bmatrix} \lambda_1 & B \\ 0 & A_1 \end{bmatrix}$ in block form. By induction, let $Q^T A_1 Q = T_1$ be upper triangular where Q is of size $(n-1) \times (n-1)$ and orthogonal. Then $P_2 = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$ is orthogonal, so $P = P_1P_2$ is also orthogonal and $P^T AP = \begin{bmatrix} \lambda_1 & BQ \\ 0 & T_1 \end{bmatrix}$ is upper triangular. □

The proof of Theorem 8.2.5 gives no way to construct the matrix P. However, an algorithm will be given in Section ?? where an improved version of Theorem 8.2.5 is presented. In a different direction, a version of Theorem 8.2.5 holds for an arbitrary matrix with complex entries (Schur's theorem in Section ??).

As for a diagonal matrix, the eigenvalues of an upper triangular matrix are displayed along the main diagonal. Because A and $P^T AP$ have the same determinant and trace whenever P is orthogonal, Theorem 8.2.5 gives:

[4]There is also a lower triangular version.

Corollary 8.2.1

If A is an $n \times n$ matrix with real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (possibly not all distinct), then $\det A = \lambda_1 \lambda_2 \dots \lambda_n$ *and* $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

This corollary remains true even if the eigenvalues are not real (using Schur's theorem).

Exercises for 8.2

Exercise 8.2.1 Normalize the rows to make each of the following matrices orthogonal.

a) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ b) $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix}$

d) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $(a, b) \neq (0, 0)$

e) $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 2 \end{bmatrix}$

f) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

g) $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

h) $A = \begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ -6 & 3 & 2 \end{bmatrix}$

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b. $\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

d. $\frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

f. $\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

h. $\frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ -6 & 3 & 2 \end{bmatrix}$

Exercise 8.2.2 is diagonal and that all diagonal entries are 1 or -1.

We have $P^T = P^{-1}$; this matrix is lower triangular (left side) and also upper triangular (right side–see Lemma 2.7.1), and so is diagonal. But then $P = P^T = P^{-1}$, so $P^2 = I$. This implies that the diagonal entries of P are all ± 1.

Exercise 8.2.3 If P is orthogonal, show that kP is orthogonal if and only if $k = 1$ or $k = -1$.

Exercise 8.2.4 If the first two rows of an orthogonal matrix are $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3})$, find all possible third rows.

Exercise 8.2.5 For each matrix A, find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ d) $A = \begin{bmatrix} 3 & 0 & 7 \\ 0 & 5 & 0 \\ 7 & 0 & 3 \end{bmatrix}$

e) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ f) $A = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}$

g) $A = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

h) $A = \begin{bmatrix} 3 & 5 & -1 & 1 \\ 5 & 3 & 1 & -1 \\ -1 & 1 & 3 & 5 \\ 1 & -1 & 5 & 3 \end{bmatrix}$

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b. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

d. $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

f. $\frac{1}{3\sqrt{2}}\begin{bmatrix} 2\sqrt{2} & 3 & 1 \\ \sqrt{2} & 0 & -4 \\ 2\sqrt{2} & -3 & 1 \end{bmatrix}$ or $\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

h. $\frac{1}{2}\begin{bmatrix} 1 & -1 & \sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} & 0 \\ -1 & -1 & 0 & \sqrt{2} \\ 1 & 1 & 0 & \sqrt{2} \end{bmatrix}$

Exercise 8.2.6 Consider $A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & c \\ 0 & c & 0 \end{bmatrix}$ where one of $a, c \neq 0$. Show that $c_A(x) = x(x-k)(x+k)$, where $k = \sqrt{a^2+c^2}$ and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

$P = \frac{1}{\sqrt{2}k}\begin{bmatrix} c\sqrt{2} & a & a \\ 0 & k & -k \\ -a\sqrt{2} & c & c \end{bmatrix}$

Exercise 8.2.7 Consider $A = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}$. Show that $c_A(x) = (x-b)(x-a)(x+a)$ and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

Exercise 8.2.8 Given $A = \begin{bmatrix} b & a \\ a & b \end{bmatrix}$, show that $c_A(x) = (x-a-b)(x+a-b)$ and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

Exercise 8.2.9 Consider $A = \begin{bmatrix} b & 0 & a \\ 0 & b & 0 \\ a & 0 & b \end{bmatrix}$. Show that $c_A(x) = (x-b)(x-b-a)(x-b+a)$ and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

Exercise 8.2.10 In each case find new variables y_1 and y_2 that diagonalize the quadratic form q.

a) $q = x_1^2 + 6x_1x_2 + x_2^2$ b) $q = x_1^2 + 4x_1x_2 - 2x_2^2$

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b. $y_1 = \frac{1}{\sqrt{5}}(-x_1 + 2x_2)$ and $y_2 = \frac{1}{\sqrt{5}}(2x_1 + x_2)$; $q = -3y_1^2 + 2y_2^2$.

Exercise 8.2.11 Show that the following are equivalent for a symmetric matrix A.

a) A is orthogonal. b) $A^2 = I$.

c) All eigenvalues of A are ± 1.

[Hint: For (b) if and only if (c), use Theorem 8.2.2.]

---

c. $\Rightarrow$ a. By Theorem 8.2.1 let $P^{-1}AP = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ where the λ_i are the eigenvalues of A. By c. we have $\lambda_i = \pm 1$ for each i, whence $D^2 = I$. But then $A^2 = (PDP^{-1})^2 = PD^2P^{-1} = I$. Since A is symmetric this is $AA^T = I$, proving a.

Exercise 8.2.12 We call matrices A and B **orthogonally similar** (and write $A \overset{\circ}{\sim} B$) if $B = P^TAP$ for an orthogonal matrix P.

a. Show that $A \overset{\circ}{\sim} A$ for all A; $A \overset{\circ}{\sim} B \Rightarrow B \overset{\circ}{\sim} A$; and $A \overset{\circ}{\sim} B$ and $B \overset{\circ}{\sim} C \Rightarrow A \overset{\circ}{\sim} C$.

b. Show that the following are equivalent for two symmetric matrices A and B.

   i. A and B are similar.

   ii. A and B are orthogonally similar.

   iii. A and B have the same eigenvalues.

Exercise 8.2.13 Assume that A and B are orthogonally similar (Exercise 8.2.12).

a. If A and B are invertible, show that A^{-1} and B^{-1} are orthogonally similar.

b. Show that A^2 and B^2 are orthogonally similar.

c. Show that, if A is symmetric, so is B.

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b. If $B = P^T AP = P^{-1}$, then $B^2 = P^T APP^T AP = P^T A^2 P$.

Exercise 8.2.14 If A is symmetric, show that every eigenvalue of A is nonnegative if and only if $A = B^2$ for some symmetric matrix B.

Exercise 8.2.15 Prove the converse of Theorem 8.2.3: If $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all n-columns $\mathbf{x}$ and $\mathbf{y}$, then A is symmetric.

If $\mathbf{x}$ and $\mathbf{y}$ are respectively columns i and j of I_n, then $\mathbf{x}^T A^T \mathbf{y} = \mathbf{x}^T A\mathbf{y}$ shows that the $(i,\ j)$-entries of A^T and A are equal.

Exercise 8.2.16 Show that every eigenvalue of A is zero if and only if A is nilpotent ($A^k = 0$ for some $k \geq 1$).

Exercise 8.2.17 If A has real eigenvalues, show that $A = B + C$ where B is symmetric and C is nilpotent.
[*Hint*: Theorem 8.2.5.]

Exercise 8.2.18 Let P be an orthogonal matrix.

a. Show that $\det P = 1$ or $\det P = -1$.

b. Give 2×2 examples of P such that $\det P = 1$ and $\det P = -1$.

c. If $\det P = -1$, show that $I + P$ has no inverse. [*Hint*: $P^T(I + P) = (I + P)^T$.]

d. If P is $n \times n$ and $\det P \neq (-1)^n$, show that $I - P$ has no inverse. [*Hint*: $P^T(I - P) = -(I - P)^T$.]

---

b. $\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1$ and $\det \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = -1$ [*Remark*: These are the *only* 2×2 examples.]

d. Use the fact that $P^{-1} = P^T$ to show that $P^T(I - P) = -(I - P)^T$. Now take determinants and use the hypothesis that $\det P \neq (-1)^n$.

Exercise 8.2.19 We call a square matrix E a **projection matrix** if $E^2 = E = E^T$. (See Exercise 8.1.17.)

a. If E is a projection matrix, show that $P = I - 2E$ is orthogonal and symmetric.

b. If P is orthogonal and symmetric, show that $E = \frac{1}{2}(I - P)$ is a projection matrix.

c. If U is $m \times n$ and $U^T U = I$ (for example, a unit column in $\mathbb{R}^n$), show that $E = UU^T$ is a projection matrix.

Exercise 8.2.20 A matrix that we obtain from the identity matrix by writing its rows in a different order is called a **permutation matrix**. Show that every permutation matrix is orthogonal.

Exercise 8.2.21 If the rows $\mathbf{r}_1, \ldots, \mathbf{r}_n$ of the $n \times n$ matrix $A = [a_{ij}]$ are orthogonal, show that the $(i,\ j)$-entry of A^{-1} is $\frac{a_{ji}}{\|\mathbf{r}_j\|^2}$.

We have $AA^T = D$, where D is diagonal with main diagonal entries $\|R_1\|^2, \ldots, \|R_n\|^2$. Hence $A^{-1} = A^T D^{-1}$, and the result follows because D^{-1} has diagonal entries $1/\|R_1\|^2, \ldots, 1/\|R_n\|^2$.

Exercise 8.2.22

a. Let A be an $m \times n$ matrix. Show that the following are equivalent.

   i. A has orthogonal rows.

   ii. A can be factored as $A = DP$, where D is invertible and diagonal and P has orthonormal rows.

   iii. AA^T is an invertible, diagonal matrix.

b. Show that an $n \times n$ matrix A has orthogonal rows if and only if A can be factored as $A = DP$, where P is orthogonal and D is diagonal and invertible.

Exercise 8.2.23 Let A be a skew-symmetric matrix; that is, $A^T = -A$. Assume that A is an $n \times n$ matrix.

a. Show that $I + A$ is invertible. [*Hint*: By Theorem 2.4.5, it suffices to show that $(I + A)\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, implies $\mathbf{x} = \mathbf{0}$. Compute $\mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T\mathbf{x}$, and use the fact that $A\mathbf{x} = -\mathbf{x}$ and $A^2\mathbf{x} = \mathbf{x}$.]

b. Show that $P = (I - A)(I + A)^{-1}$ is orthogonal.

c. Show that every orthogonal matrix P such that $I + P$ is invertible arises as in part (b) from some skew-symmetric matrix A.
[*Hint*: Solve $P = (I - A)(I + A)^{-1}$ for A.]

b. Because $I-A$ and $I+A$ commute, $PP^T = (I-A)(I+A)^{-1}[(I+A)^{-1}]^T(I-A)^T = (I-A)(I+A)^{-1}(I-A)^{-1}(I+A) = I$.

Exercise 8.2.24 Show that the following are equivalent for an $n \times n$ matrix P.

a. P is orthogonal.

b. $\|P\mathbf{x}\| = \|\mathbf{x}\|$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$.

c. $\|P\mathbf{x} - P\mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$ for all columns $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

d. $(P\mathbf{x}) \cdot (P\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all columns $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.
[*Hints*: For (c) $\Rightarrow$ (d), see Exercise 5.3.14(a). For (d) $\Rightarrow$ (a), show that column i of P equals $P\mathbf{e}_i$, where $\mathbf{e}_i$ is column i of the identity matrix.]

Exercise 8.2.25 Show that every 2×2 orthogonal matrix has the form $\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$ or $\left[\begin{array}{cc} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{array}\right]$ for some angle θ.
[*Hint*: If $a^2 + b^2 = 1$, then $a = \cos\theta$ and $b = \sin\theta$ for some angle θ.]

Exercise 8.2.26 Use Theorem 8.2.5 to show that every symmetric matrix is orthogonally diagonalizable.

8.3 Positive Definite Matrices

All the eigenvalues of any symmetric matrix are real; this section is about the case in which the eigenvalues are positive. These matrices, which arise whenever optimization (maximum and minimum) problems are encountered, have countless applications throughout science and engineering. They also arise in statistics (for example, in factor analysis used in the social sciences) and in geometry (see Section **??**). We will encounter them again in Chapter **??** when describing all inner products in $\mathbb{R}^n$.

Definition 8.5 Positive Definite Matrices

A square matrix is called ***positive definite*** *if it is symmetric and all its eigenvalues* λ *are positive, that is* $\lambda > 0$.

Because these matrices are symmetric, the principal axes theorem plays a central role in the theory.

Theorem 8.3.1

If A *is positive definite, then it is invertible and* $\det A > 0$.

Proof. If A is $n \times n$ and the eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\det A = \lambda_1 \lambda_2 \cdots \lambda_n > 0$ by the principal axes theorem (or the corollary to Theorem 8.2.5). □

If $\mathbf{x}$ is a column in $\mathbb{R}^n$ and A is any real $n \times n$ matrix, we view the 1×1 matrix $\mathbf{x}^T A \mathbf{x}$ as a real number. With this convention, we have the following characterization of positive definite matrices.

Theorem 8.3.2

A symmetric matrix A *is positive definite if and only if* $\mathbf{x}^T A \mathbf{x} > 0$ *for every column* $\mathbf{x} \neq \mathbf{0}$ *in* $\mathbb{R}^n$.

Proof. A is symmetric so, by the principal axes theorem, let $P^T A P = D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where $P^{-1} = P^T$ and the λ_i are the eigenvalues of A. Given a column $\mathbf{x}$ in $\mathbb{R}^n$, write $\mathbf{y} = P^T \mathbf{x} = \left[\begin{array}{cccc} y_1 & y_2 & \dots & y_n \end{array} \right]^T$. Then

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (PDP^T) \mathbf{x} = \mathbf{y}^T D \mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \tag{8.3}$$

If A is positive definite and $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^T A \mathbf{x} > 0$ by (8.3) because some $y_j \neq 0$ and every $\lambda_i > 0$. Conversely, if $\mathbf{x}^T A \mathbf{x} > 0$ whenever $\mathbf{x} \neq \mathbf{0}$, let $\mathbf{x} = P\mathbf{e}_j \neq \mathbf{0}$ where $\mathbf{e}_j$ is column j of I_n. Then $\mathbf{y} = \mathbf{e}_j$, so (8.3) reads $\lambda_j = \mathbf{x}^T A \mathbf{x} > 0$. □

Note that Theorem 8.3.2 shows that the positive definite matrices are exactly the symmetric matrices A for which the quadratic form $q = \mathbf{x}^T A \mathbf{x}$ takes only positive values.

Example 8.3.1

If U is any invertible $n \times n$ matrix, show that $A = U^T U$ is positive definite.

Solution. If $\mathbf{x}$ is in $\mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$, then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T (U^T U)\mathbf{x} = (U\mathbf{x})^T (U\mathbf{x}) = \|U\mathbf{x}\|^2 > 0$$

because $U\mathbf{x} \neq \mathbf{0}$ (U is invertible). Hence Theorem 8.3.2 applies.

It is remarkable that the converse to Example 8.3.1 is also true. In fact every positive definite matrix A can be factored as $A = U^T U$ where U is an upper triangular matrix with positive elements on the main diagonal. However, before verifying this, we introduce another concept that is central to any discussion of positive definite matrices.

If A is any $n \times n$ matrix, let ${}^{(r)}A$ denote the $r \times r$ submatrix in the upper left corner of A; that is, ${}^{(r)}A$ is the matrix obtained from A by deleting the last $n-r$ rows and columns. The matrices ${}^{(1)}A,\ {}^{(2)}A,\ {}^{(3)}A,\ \dots,\ {}^{(n)}A = A$ are called the **principal submatrices** of A.

Example 8.3.2

If $A = \left[\begin{array}{rrr} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{array} \right]$ then ${}^{(1)}A = [10]$, ${}^{(2)}A = \left[\begin{array}{rr} 10 & 5 \\ 5 & 3 \end{array} \right]$ and ${}^{(3)}A = A$.

Lemma 8.3.1

If A is positive definite, so is each principal submatrix ${}^{(r)}A$ for $r = 1, 2, \dots, n$.

Proof. Write $A = \left[\begin{array}{cc} {}^{(r)}A & P \\ Q & R \end{array} \right]$ in block form. If $\mathbf{y} \neq \mathbf{0}$ in $\mathbb{R}^r$, write $\mathbf{x} = \left[\begin{array}{c} \mathbf{y} \\ \mathbf{0} \end{array} \right]$ in $\mathbb{R}^n$.

Then $\mathbf{x} \neq \mathbf{0}$, so the fact that A is positive definite gives

$$0 < \mathbf{x}^T A\mathbf{x} = \left[\begin{array}{cc} \mathbf{y}^T & \mathbf{0} \end{array} \right] \left[\begin{array}{cc} {}^{(r)}A & P \\ Q & R \end{array} \right] \left[\begin{array}{c} \mathbf{y} \\ \mathbf{0} \end{array} \right] = \mathbf{y}^T ({}^{(r)}A)\mathbf{y}$$

This shows that ${}^{(r)}A$ is positive definite by Theorem 8.3.2.[5] □

If A is positive definite, Lemma 8.3.1 and Theorem 8.3.1 show that $\det({}^{(r)}A) > 0$ for every r. This proves part of the following theorem which contains the converse to Example 8.3.1, and characterizes the positive definite matrices among the symmetric ones.

[5]A similar argument shows that, if B is any matrix obtained from a positive definite matrix A by deleting certain rows and deleting the *same* columns, then B is also positive definite.

Theorem 8.3.3

The following conditions are equivalent for a symmetric $n \times n$ matrix A:

1. *A is positive definite.*
2. *$\det({}^{(r)}A) > 0$ for each $r = 1, 2, \ldots, n$.*
3. *$A = U^T U$ where U is an upper triangular matrix with positive entries on the main diagonal.*

*Furthermore, the factorization in (3) is unique (called the **Cholesky factorization**[6] of A).*

Proof. First, (3) $\Rightarrow$ (1) by Example 8.3.1, and (1) $\Rightarrow$ (2) by Lemma 8.3.1 and Theorem 8.3.1.

(2) $\Rightarrow$ (3). Assume (2) and proceed by induction on n. If $n = 1$, then $A = [a]$ where $a > 0$ by (2), so take $U = [\sqrt{a}]$. If $n > 1$, write $B = {}^{(n-1)}A$. Then B is symmetric and satisfies (2) so, by induction, we have $B = U^T U$ as in (3) where U is of size $(n-1) \times (n-1)$. Then, as A is symmetric, it has block form $A = \left[\begin{array}{cc} B & \mathbf{p} \\ \mathbf{p}^T & b \end{array}\right]$ where $\mathbf{p}$ is a column in $\mathbb{R}^{n-1}$ and b is in $\mathbb{R}$. If we write $\mathbf{x} = (U^T)^{-1}\mathbf{p}$ and $c = b - \mathbf{x}^T\mathbf{x}$, block multiplication gives

$$A = \left[\begin{array}{cc} U^T U & \mathbf{p} \\ \mathbf{p}^T & b \end{array}\right] = \left[\begin{array}{cc} U^T & 0 \\ \mathbf{x}^T & 1 \end{array}\right] \left[\begin{array}{cc} U & \mathbf{x} \\ 0 & c \end{array}\right]$$

as the reader can verify. Taking determinants and applying Theorem 3.1.5 gives $\det A = \det(U^T) \det U \cdot c = c(\det U)^2$. Hence $c > 0$ because $\det A > 0$ by (2), so the above factorization can be written

$$A = \left[\begin{array}{cc} U^T & 0 \\ \mathbf{x}^T & \sqrt{c} \end{array}\right] \left[\begin{array}{cc} U & \mathbf{x} \\ 0 & \sqrt{c} \end{array}\right]$$

Since U has positive diagonal entries, this proves (3).

As to the uniqueness, suppose that $A = U^T U = U_1^T U_1$ are two Cholesky factorizations. Now write $D = UU_1^{-1} = (U^T)^{-1}U_1^T$. Then D is upper triangular, because $D = UU_1^{-1}$, and lower triangular, because $D = (U^T)^{-1}U_1^T$, and so it is a diagonal matrix. Thus $U = DU_1$ and $U_1 = DU$, so it suffices to show that $D = I$. But eliminating U_1 gives $U = D^2 U$, so $D^2 = I$ because U is invertible. Since the diagonal entries of D are positive (this is true of U and U_1), it follows that $D = I$. □

The remarkable thing is that the matrix U in the Cholesky factorization is easy to obtain from A using row operations. The key is that Step 1 of the following algorithm is *possible* for any positive definite matrix A. A proof of the algorithm is given following Example 8.3.3.

Algorithm for the Cholesky Factorization

If A is a positive definite matrix, the Cholesky factorization $A = U^T U$ can be obtained as follows:

Step 1. Carry A to an upper triangular matrix U_1 with positive diagonal entries using row

[6] Andre-Louis Cholesky (1875–1918), was a French mathematician who died in World War I. His factorization was published in 1924 by a fellow officer.

operations each of which adds a multiple of a row to a lower row.

Step 2. Obtain U from U_1 by dividing each row of U_1 by the square root of the diagonal entry in that row.

Example 8.3.3

Find the Cholesky factorization of $A = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution. The matrix A is positive definite by Theorem 8.3.3 because $\det{}^{(1)}A = 10 > 0$, $\det{}^{(2)}A = 5 > 0$, and $\det{}^{(3)}A = \det A = 3 > 0$. Hence Step 1 of the algorithm is carried out as follows:

$$A = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 10 & 5 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{13}{5} \end{bmatrix} \to \begin{bmatrix} 10 & 5 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{3}{5} \end{bmatrix} = U_1$$

Now carry out Step 2 on U_1 to obtain $U = \begin{bmatrix} \sqrt{10} & \frac{5}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{5}} \end{bmatrix}$.

The reader can verify that $U^T U = A$.

Proof of the Cholesky Algorithm. If A is positive definite, let $A = U^T U$ be the Cholesky factorization, and let $D = \mathbf{diag}(d_1, \dots, d_n)$ be the common diagonal of U and U^T. Then $U^T D^{-1}$ is lower triangular with ones on the diagonal (call such matrices LT-1). Hence $L = (U^T D^{-1})^{-1}$ is also LT-1, and so $I_n \to L$ by a sequence of row operations each of which adds a multiple of a row to a lower row (verify; modify columns right to left). But then $A \to LA$ by the same sequence of row operations (see the discussion preceding Theorem 2.5.1). Since $LA = [D(U^T)^{-1}][U^T U] = DU$ is upper triangular with positive entries on the diagonal, this shows that Step 1 of the algorithm is possible.

Turning to Step 2, let $A \to U_1$ as in Step 1 so that $U_1 = L_1 A$ where L_1 is LT-1. Since A is symmetric, we get

$$L_1 U_1^T = L_1 (L_1 A)^T = L_1 A^T L_1^T = L_1 A L_1^T = U_1 L_1^T \tag{8.4}$$

Let $D_1 = \mathbf{diag}(e_1, \dots, e_n)$ denote the diagonal of U_1. Then (8.4) gives $L_1(U_1^T D_1^{-1}) = U_1 L_1^T D_1^{-1}$. This is both upper triangular (right side) and LT-1 (left side), and so must equal I_n. In particular, $U_1^T D_1^{-1} = L_1^{-1}$. Now let $D_2 = \mathbf{diag}(\sqrt{e_1}, \dots, \sqrt{e_n})$, so that $D_2^2 = D_1$. If we write $U = D_2^{-1} U_1$ we have

$$U^T U = (U_1^T D_2^{-1})(D_2^{-1} U_1) = U_1^T (D_2^2)^{-1} U_1 = (U_1^T D_1^{-1}) U_1 = (L_1^{-1}) U_1 = A$$

This proves Step 2 because $U = D_2^{-1} U_1$ is formed by dividing each row of U_1 by the square root of its diagonal entry (verify). □

Exercises for 8.3

Exercise 8.3.1 Find the Cholesky decomposition of each of the following matrices.

a) $\begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 12 & 4 & 3 \\ 4 & 2 & -1 \\ 3 & -1 & 7 \end{bmatrix}$ d) $\begin{bmatrix} 20 & 4 & 5 \\ 4 & 2 & 3 \\ 5 & 3 & 5 \end{bmatrix}$

---

b. $U = \frac{\sqrt{2}}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

d. $U = \frac{1}{30}\begin{bmatrix} 60\sqrt{5} & 12\sqrt{5} & 15\sqrt{5} \\ 0 & 6\sqrt{30} & 10\sqrt{30} \\ 0 & 0 & 5\sqrt{15} \end{bmatrix}$

Exercise 8.3.2

a. If A is positive definite, show that A^k is positive definite for all $k \geq 1$.

b. Prove the converse to (a) when k is odd.

c. Find a symmetric matrix A such that A^2 is positive definite but A is not.

---

b. If $\lambda^k > 0$, k odd, then $\lambda > 0$.

Exercise 8.3.3 Let $A = \begin{bmatrix} 1 & a \\ a & b \end{bmatrix}$. If $a^2 < b$, show that A is positive definite and find the Cholesky factorization.

Exercise 8.3.4 If A and B are positive definite and $r > 0$, show that $A+B$ and rA are both positive definite.

If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^T A\mathbf{x} > 0$ and $\mathbf{x}^T B\mathbf{x} > 0$. Hence $\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} > 0$ and $\mathbf{x}^T(rA)\mathbf{x} = r(\mathbf{x}^T A\mathbf{x}) > 0$, as $r > 0$.

Exercise 8.3.5 If A and B are positive definite, show that $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is positive definite.

Exercise 8.3.6 If A is an $n \times n$ positive definite matrix and U is an $n \times m$ matrix of rank m, show that $U^T AU$ is positive definite.

Let $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$. Then $\mathbf{x}^T(U^T AU)\mathbf{x} = (U\mathbf{x})^T A(U\mathbf{x}) > 0$ provided $U\mathbf{x} \neq 0$. But if $U = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{bmatrix}$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$, then $U\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n \neq \mathbf{0}$ because $\mathbf{x} \neq \mathbf{0}$ and the $\mathbf{c}_i$ are independent.

Exercise 8.3.7 If A is positive definite, show that each diagonal entry is positive.

Exercise 8.3.8 Let A_0 be formed from A by deleting rows 2 and 4 and deleting columns 2 and 4. If A is positive definite, show that A_0 is positive definite.

Exercise 8.3.9 If A is positive definite, show that $A = CC^T$ where C has orthogonal columns.

Exercise 8.3.10 If A is positive definite, show that $A = C^2$ where C is positive definite.

Let $P^T AP = D = \text{diag}(\lambda_1, \dots, \lambda_n)$ where $P^T = P$. Since A is positive definite, each eigenvalue $\lambda_i > 0$. If $B = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$ then $B^2 = D$, so $A = PB^2P^T = (PBP^T)^2$. Take $C = PBP^T$. Since C has eigenvalues $\sqrt{\lambda_i} > 0$, it is positive definite.

Exercise 8.3.11 Let A be a positive definite matrix. If a is a real number, show that aA is positive definite if and only if $a > 0$.

Exercise 8.3.12

a. Suppose an invertible matrix A can be factored in $\mathbf{M}_{nn}$ as $A = LDU$ where L is lower triangular with 1s on the diagonal, U is upper triangular with 1s on the diagonal, and D is diagonal with positive diagonal entries. Show that the factorization is unique: If $A = L_1D_1U_1$ is another such factorization, show that $L_1 = L$, $D_1 = D$, and $U_1 = U$.

b. Show that a matrix A is positive definite if and only if A is symmetric and admits a factorization $A = LDU$ as in (a).

---

b. If A is positive definite, use Theorem 8.3.1 to write $A = U^T U$ where U is upper triangular with positive diagonal D. Then $A = (D^{-1}U)^T D^2 (D^{-1}U)$ so $A = L_1 D_1 U_1$ is such a factorization if $U_1 = D^{-1}U$, $D_1 = D^2$, and $L_1 = U_1^T$. Conversely, let $A^T = A = LDU$ be such a factorization. Then $U^T D^T L^T = A^T = A = LDU$, so $L = U^T$ by **(a)**. Hence $A = LDL^T = V^T V$ where $V = LD_0$ and D_0 is diagonal with $D_0^2 = D$ (the matrix D_0 exists because D has positive diagonal entries). Hence A is symmetric, and it is positive definite by Example 8.3.1.

Exercise 8.3.13 Let A be positive definite and write $d_r = \det {}^{(r)}A$ for each $r = 1, 2, \dots, n$. If U is the upper triangular matrix obtained in step 1 of the algorithm, show that the diagonal elements $u_{11}, u_{22}, \dots, u_{nn}$ of U are given by $u_{11} = d_1$, $u_{jj} = d_j / d_{j-1}$ if $j > 1$. [*Hint*: If $LA = U$ where L is lower triangular with 1s on the diagonal, use block multiplication to show that $\det {}^{(r)}A = \det {}^{(r)}U$ for each r.]

8.4 QR-Factorization[7]

One of the main virtues of orthogonal matrices is that they can be easily inverted—the transpose is the inverse. This fact, combined with the factorization theorem in this section, provides a useful way to simplify many matrix calculations (for example, in least squares approximation).

Definition 8.6 QR-factorization

*Let A be an $m \times n$ matrix with independent columns. A **QR-factorization** of A expresses it as $A = QR$ where Q is $m \times n$ with orthonormal columns and R is an invertible and upper triangular matrix with positive diagonal entries.*

The importance of the factorization lies in the fact that there are computer algorithms that accomplish it with good control over round-off error, making it particularly useful in matrix calculations. The factorization is a matrix version of the Gram-Schmidt process.

Suppose $A = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array} \right]$ is an $m \times n$ matrix with linearly independent columns $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$. The Gram-Schmidt algorithm can be applied to these columns to provide orthogonal columns $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ where $\mathbf{f}_1 = \mathbf{c}_1$ and

$$\mathbf{f}_k = \mathbf{c}_k - \frac{\mathbf{c}_k \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{c}_k \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 - \cdots - \frac{\mathbf{c}_k \cdot \mathbf{f}_{k-1}}{\|\mathbf{f}_{k-1}\|^2}\mathbf{f}_{k-1}$$

for each $k = 2, 3, \dots, n$. Now write $\mathbf{q}_k = \frac{1}{\|\mathbf{f}_k\|}\mathbf{f}_k$ for each k. Then $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ are orthonormal columns, and the above equation becomes

$$\|\mathbf{f}_k\|\mathbf{q}_k = \mathbf{c}_k - (\mathbf{c}_k \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{c}_k \cdot \mathbf{q}_2)\mathbf{q}_2 - \cdots - (\mathbf{c}_k \cdot \mathbf{q}_{k-1})\mathbf{q}_{k-1}$$

Using these equations, express each $\mathbf{c}_k$ as a linear combination of the $\mathbf{q}_i$:

$$\begin{array}{rcl}
\mathbf{c}_1 & = & \|\mathbf{f}_1\|\mathbf{q}_1 \\
\mathbf{c}_2 & = & (\mathbf{c}_2 \cdot \mathbf{q}_1)\mathbf{q}_1 + \|\mathbf{f}_2\|\mathbf{q}_2 \\
\mathbf{c}_3 & = & (\mathbf{c}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{c}_3 \cdot \mathbf{q}_2)\mathbf{q}_2 + \|\mathbf{f}_3\|\mathbf{q}_3 \\
\vdots & & \vdots \\
\mathbf{c}_n & = & (\mathbf{c}_n \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{c}_n \cdot \mathbf{q}_2)\mathbf{q}_2 + (\mathbf{c}_n \cdot \mathbf{q}_3)\mathbf{q}_3 + \cdots + \|\mathbf{f}_n\|\mathbf{q}_n
\end{array}$$

These equations have a matrix form that gives the required factorization:

$$\begin{aligned}
A &= \left[\begin{array}{ccccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \cdots & \mathbf{c}_n \end{array} \right] \\
&= \left[\begin{array}{ccccc} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \cdots & \mathbf{q}_n \end{array} \right] \left[\begin{array}{ccccc}
\|\mathbf{f}_1\| & \mathbf{c}_2 \cdot \mathbf{q}_1 & \mathbf{c}_3 \cdot \mathbf{q}_1 & \cdots & \mathbf{c}_n \cdot \mathbf{q}_1 \\
0 & \|\mathbf{f}_2\| & \mathbf{c}_3 \cdot \mathbf{q}_2 & \cdots & \mathbf{c}_n \cdot \mathbf{q}_2 \\
0 & 0 & \|\mathbf{f}_3\| & \cdots & \mathbf{c}_n \cdot \mathbf{q}_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \|\mathbf{f}_n\|
\end{array} \right]
\end{aligned} \tag{8.5}$$

Here the first factor $Q = \left[\begin{array}{ccccc} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \cdots & \mathbf{q}_n \end{array} \right]$ has orthonormal columns, and the second factor is an $n \times n$ upper triangular matrix R with positive diagonal entries (and so is invertible). We record this in the following theorem.

[7]This section is not used elsewhere in the book

Theorem 8.4.1: QR-Factorization

Every $m \times n$ matrix A with linearly independent columns has a QR-factorization $A = QR$ where Q has orthonormal columns and R is upper triangular with positive diagonal entries.

The matrices Q and R in Theorem 8.4.1 are uniquely determined by A; we return to this below.

Example 8.4.1

Find the QR-factorization of $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Denote the columns of A as $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$, and observe that $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is independent. If we apply the Gram-Schmidt algorithm to these columns, the result is:

$$\mathbf{f}_1 = \mathbf{c}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_2 = \mathbf{c}_2 - \tfrac{1}{2}\mathbf{f}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{f}_3 = \mathbf{c}_3 + \tfrac{1}{2}\mathbf{f}_1 - \mathbf{f}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Write $\mathbf{q}_j = \frac{1}{\|\mathbf{f}_j\|}\mathbf{f}_j$ for each j, so $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is orthonormal. Then equation (8.5) preceding Theorem 8.4.1 gives $A = QR$ where

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & 1 & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} \|\mathbf{f}_1\| & \mathbf{c}_2 \cdot \mathbf{q}_1 & \mathbf{c}_3 \cdot \mathbf{q}_1 \\ 0 & \|\mathbf{f}_2\| & \mathbf{c}_3 \cdot \mathbf{q}_2 \\ 0 & 0 & \|\mathbf{f}_3\| \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & -1 \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

The reader can verify that indeed $A = QR$.

If a matrix A has independent rows and we apply QR-factorization to A^T, the result is:

Corollary 8.4.1

If A has independent rows, then A factors uniquely as $A = LP$ where P has orthonormal rows and L is an invertible lower triangular matrix with positive main diagonal entries.

Since a square matrix with orthonormal columns is orthogonal, we have

Theorem 8.4.2

Every square, invertible matrix A has factorizations $A = QR$ and $A = LP$ where Q and P are orthogonal, R is upper triangular with positive diagonal entries, and L is lower triangular with positive diagonal entries.

Remark

In Section **??** we found how to find a best approximation $\mathbf{z}$ to a solution of a (possibly inconsistent) system $A\mathbf{x} = \mathbf{b}$ of linear equations: take $\mathbf{z}$ to be any solution of the "normal" equations $(A^TA)\mathbf{z} = A^T\mathbf{b}$. If A has independent columns this $\mathbf{z}$ is unique (A^TA is invertible by Theorem 5.4.3), so it is often desirable to compute $(A^TA)^{-1}$. This is particularly useful in least squares approximation (Section **??**). This is simplified if we have a QR-factorization of A (and is one of the main reasons for the importance of Theorem 8.4.1). For if $A = QR$ is such a factorization, then $Q^TQ = I_n$ because Q has orthonormal columns (verify), so we obtain

$$A^TA = R^TQ^TQR = R^TR$$

Hence computing $(A^TA)^{-1}$ amounts to finding R^{-1}, and this is a routine matter because R is upper triangular. Thus the difficulty in computing $(A^TA)^{-1}$ lies in obtaining the QR-factorization of A.

We conclude by proving the uniqueness of the QR-factorization.

Theorem 8.4.3

Let A be an $m \times n$ matrix with independent columns. If $A = QR$ and $A = Q_1R_1$ are QR-factorizations of A, then $Q_1 = Q$ and $R_1 = R$.

Proof. Write $Q = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$ and $Q_1 = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \cdots & \mathbf{d}_n \end{bmatrix}$ in terms of their columns, and observe first that $Q^TQ = I_n = Q_1^TQ_1$ because Q and Q_1 have orthonormal columns. Hence it suffices to show that $Q_1 = Q$ (then $R_1 = Q_1^TA = Q^TA = R$). Since $Q_1^TQ_1 = I_n$, the equation $QR = Q_1R_1$ gives $Q_1^TQ = R_1R^{-1}$; for convenience we write this matrix as

$$Q_1^TQ = R_1R^{-1} = \begin{bmatrix} t_{ij} \end{bmatrix}$$

This matrix is upper triangular with positive diagonal elements (since this is true for R and R_1), so $t_{ii} > 0$ for each i and $t_{ij} = 0$ if $i > j$. On the other hand, the $(i,\ j)$-entry of Q_1^TQ is $\mathbf{d}_i^T\mathbf{c}_j = \mathbf{d}_i \cdot \mathbf{c}_j$, so we have $\mathbf{d}_i \cdot \mathbf{c}_j = t_{ij}$ for all i and j. But each $\mathbf{c}_j$ is in $\text{span}\{\mathbf{d}_1,\ \mathbf{d}_2,\ \dots,\ \mathbf{d}_n\}$ because $Q = Q_1(R_1R^{-1})$. Hence the expansion theorem gives

$$\mathbf{c}_j = (\mathbf{d}_1 \cdot \mathbf{c}_j)\mathbf{d}_1 + (\mathbf{d}_2 \cdot \mathbf{c}_j)\mathbf{d}_2 + \cdots + (\mathbf{d}_n \cdot \mathbf{c}_j)\mathbf{d}_n = t_{1j}\mathbf{d}_1 + t_{2j}\mathbf{d}_2 + \cdots + t_{jj}\mathbf{d}_i$$

because $\mathbf{d}_i \cdot \mathbf{c}_j = t_{ij} = 0$ if $i > j$. The first few equations here are

$$\begin{array}{lcl} \mathbf{c}_1 & = & t_{11}\mathbf{d}_1 \\ \mathbf{c}_2 & = & t_{12}\mathbf{d}_1 + t_{22}\mathbf{d}_2 \\ \mathbf{c}_3 & = & t_{13}\mathbf{d}_1 + t_{23}\mathbf{d}_2 + t_{33}\mathbf{d}_3 \\ \mathbf{c}_4 & = & t_{14}\mathbf{d}_1 + t_{24}\mathbf{d}_2 + t_{34}\mathbf{d}_3 + t_{44}\mathbf{d}_4 \\ \vdots & & \vdots \end{array}$$

The first of these equations gives $1 = \|\mathbf{c}_1\| = \|t_{11}\mathbf{d}_1\| = |t_{11}|\|\mathbf{d}_1\| = t_{11}$, whence $\mathbf{c}_1 = \mathbf{d}_1$. But then we have $t_{12} = \mathbf{d}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_2 = 0$, so the second equation becomes $\mathbf{c}_2 = t_{22}\mathbf{d}_2$. Now a similar argument gives $\mathbf{c}_2 = \mathbf{d}_2$, and then $t_{13} = 0$ and $t_{23} = 0$ follows in the same way. Hence $\mathbf{c}_3 = t_{33}\mathbf{d}_3$ and $\mathbf{c}_3 = \mathbf{d}_3$. Continue in this way to get $\mathbf{c}_i = \mathbf{d}_i$ for all i. This means that $Q_1 = Q$, which is what we wanted. □

Exercises for 8.4

Exercise 8.4.1 In each case find the QR-factorization of A.

a) $A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ d) $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

---

b. $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $R = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$

d. $Q = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$,

$R = \frac{1}{\sqrt{3}}\begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Exercise 8.4.2 Let A and B denote matrices.

a. If A and B have independent columns, show that AB has independent columns. [*Hint*: Theorem 5.4.3.]

b. Show that A has a QR-factorization if and only if A has independent columns.

c. If AB has a QR-factorization, show that the same is true of B but not necessarily A. [*Hint*: Consider AA^T where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.]

If A has a QR-factorization, use (a). For the converse use Theorem 8.4.1.

Exercise 8.4.3 If R is upper triangular and invertible, show that there exists a diagonal matrix D with diagonal entries ± 1 such that $R_1 = DR$ is invertible, upper triangular, and has positive diagonal entries.

Exercise 8.4.4 If A has independent columns, let $A = QR$ where Q has orthonormal columns and R is invertible and upper triangular. [Some authors call *this* a QR-factorization of A.] Show that there is a diagonal matrix D with diagonal entries ± 1 such that $A = (QD)(DR)$ is the QR-factorization of A. [*Hint*: Preceding exercise.]

8.5 Computing Eigenvalues

In practice, the problem of finding eigenvalues of a matrix is virtually never solved by finding the roots of the characteristic polynomial. This is difficult for large matrices and iterative methods are much better. Two such methods are described briefly in this section.

The Power Method

In Chapter 3 our initial rationale for diagonalizing matrices was to be able to compute the powers of a square matrix, and the eigenvalues were needed to do this. In this section, we are interested in efficiently computing eigenvalues, and it may come as no surprise that the first method we discuss uses the powers of a matrix.

Recall that an eigenvalue λ of an $n \times n$ matrix A is called a **dominant eigenvalue** if λ has multiplicity 1, and

$$|\lambda| > |\mu| \quad \text{for all eigenvalues } \mu \neq \lambda$$

Any corresponding eigenvector is called a **dominant eigenvector** of A. When such an eigenvalue exists, one technique for finding it is as follows: Let $\mathbf{x}_0$ in $\mathbb{R}^n$ be a first approximation to a dominant eigenvector λ, and compute successive approximations $\mathbf{x}_1$, $\mathbf{x}_2$, $\dots$ as follows:

$$\mathbf{x}_1 = A\mathbf{x}_0 \quad \mathbf{x}_2 = A\mathbf{x}_1 \quad \mathbf{x}_3 = A\mathbf{x}_2 \quad \cdots$$

In general, we define

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for each } k \geq 0$$

If the first estimate $\mathbf{x}_0$ is good enough, these vectors $\mathbf{x}_n$ will approximate the dominant eigenvector λ (see below). This technique is called the **power method** (because $\mathbf{x}_k = A^k\mathbf{x}_0$ for each $k \geq 1$). Observe that if $\mathbf{z}$ is any eigenvector corresponding to λ, then

$$\frac{\mathbf{z}\cdot(A\mathbf{z})}{\|\mathbf{z}\|^2} = \frac{\mathbf{z}\cdot(\lambda\mathbf{z})}{\|\mathbf{z}\|^2} = \lambda$$

Because the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\dots$, $\mathbf{x}_n$, $\dots$ approximate dominant eigenvectors, this suggests that we define the **Rayleigh quotients** as follows:

$$r_k = \frac{\mathbf{x}_k\cdot\mathbf{x}_{k+1}}{\|\mathbf{x}_k\|^2} \quad \text{for } k \geq 1$$

Then the numbers r_k approximate the dominant eigenvalue λ.

Example 8.5.1

Use the power method to approximate a dominant eigenvector and eigenvalue of $A = \left[\begin{array}{rr} 1 & 1 \\ 2 & 0 \end{array} \right]$.

Solution. The eigenvalues of A are 2 and -1, with eigenvectors $\left[\begin{array}{r} 1 \\ 1 \end{array} \right]$ and $\left[\begin{array}{r} 1 \\ -2 \end{array} \right]$. Take

$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the first approximation and compute $\mathbf{x}_1, \mathbf{x}_2, \dots,$ successively, from $\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1, \dots$. The result is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 11 \\ 10 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 21 \\ 22 \end{bmatrix}, \quad \dots$$

These vectors are approaching scalar multiples of the dominant eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Moreover, the Rayleigh quotients are

$$r_1 = \tfrac{7}{5}, \; r_2 = \tfrac{27}{13}, \; r_3 = \tfrac{115}{61}, \; r_4 = \tfrac{451}{221}, \; \dots$$

and these are approaching the dominant eigenvalue 2.

To see why the power method works, let $\lambda_1, \lambda_2, \dots, \lambda_m$ be eigenvalues of A with λ_1 dominant and let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m$ be corresponding eigenvectors. What is required is that the first approximation $\mathbf{x}_0$ be a linear combination of these eigenvectors:

$$\mathbf{x}_0 = a_1\mathbf{y}_1 + a_2\mathbf{y}_2 + \cdots + a_m\mathbf{y}_m \quad \text{with } a_1 \neq 0$$

If $k \geq 1$, the fact that $\mathbf{x}_k = A^k\mathbf{x}_0$ and $A^k\mathbf{y}_i = \lambda_i^k\mathbf{y}_i$ for each i gives

$$\mathbf{x}_k = a_1\lambda_1^k\mathbf{y}_1 + a_2\lambda_2^k\mathbf{y}_2 + \cdots + a_m\lambda_m^k\mathbf{y}_m \quad \text{for } k \geq 1$$

Hence

$$\tfrac{1}{\lambda_1^k}\mathbf{x}_k = a_1\mathbf{y}_1 + a_2\left(\tfrac{\lambda_2}{\lambda_1}\right)^k\mathbf{y}_2 + \cdots + a_m\left(\tfrac{\lambda_m}{\lambda_1}\right)^k\mathbf{y}_m$$

The right side approaches $a_1\mathbf{y}_1$ as k increases because λ_1 is dominant $\left(\left|\tfrac{\lambda_i}{\lambda_1}\right| < 1 \text{ for each } i > 1\right)$. Because $a_1 \neq 0$, this means that $\mathbf{x}_k$ approximates the dominant eigenvector $a_1\lambda_1^k\mathbf{y}_1$.

The power method requires that the first approximation $\mathbf{x}_0$ be a linear combination of eigenvectors. (In Example 8.5.1 the eigenvectors form a basis of $\mathbb{R}^2$.) But even in this case the method fails if $a_1 = 0$, where a_1 is the coefficient of the dominant eigenvector (try $\mathbf{x}_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in Example 8.5.1). In general, the rate of convergence is quite slow if any of the ratios $\left|\tfrac{\lambda_i}{\lambda_1}\right|$ is near 1. Also, because the method requires repeated multiplications by A, it is not recommended unless these multiplications are easy to carry out (for example, if most of the entries of A are zero).

QR-Algorithm

A much better method for approximating the eigenvalues of an invertible matrix A depends on the factorization (using the Gram-Schmidt algorithm) of A in the form

$$A = QR$$

where Q is orthogonal and R is invertible and upper triangular (see Theorem 8.4.2). The **QR-algorithm** uses this repeatedly to create a sequence of matrices $A_1 = A, A_2, A_3, \dots,$ as follows:

1. Define $A_1 = A$ and factor it as $A_1 = Q_1R_1$.
2. Define $A_2 = R_1Q_1$ and factor it as $A_2 = Q_2R_2$.
3. Define $A_3 = R_2Q_2$ and factor it as $A_3 = Q_3R_3$.

$\vdots$

In general, A_k is factored as $A_k = Q_kR_k$ and we define $A_{k+1} = R_kQ_k$. Then A_{k+1} is similar to A_k [in fact, $A_{k+1} = R_kQ_k = (Q_k^{-1}A_k)Q_k$], and hence each A_k has the same eigenvalues as A. If the eigenvalues of A are real and have distinct absolute values, the remarkable thing is that the sequence of matrices $A_1, A_2, A_3, \dots$ converges to an upper triangular matrix with these eigenvalues on the main diagonal. [See below for the case of complex eigenvalues.]

Example 8.5.2

If $A = \left[\begin{array}{rr} 1 & 1 \\ 2 & 0 \end{array} \right]$ as in Example 8.5.1, use the QR-algorithm to approximate the eigenvalues.

Solution. The matrices A_1, A_2, and A_3 are as follows:

$$A_1 = \left[\begin{array}{rr} 1 & 1 \\ 2 & 0 \end{array} \right] = Q_1R_1 \quad \text{where } Q_1 = \frac{1}{\sqrt{5}}\left[\begin{array}{rr} 1 & 2 \\ 2 & -1 \end{array} \right] \text{ and } R_1 = \frac{1}{\sqrt{5}}\left[\begin{array}{rr} 5 & 1 \\ 0 & 2 \end{array} \right]$$

$$A_2 = \frac{1}{5}\left[\begin{array}{rr} 7 & 9 \\ 4 & -2 \end{array} \right] = \left[\begin{array}{rr} 1.4 & -1.8 \\ -0.8 & -0.4 \end{array} \right] = Q_2R_2$$

$$\text{where } Q_2 = \frac{1}{\sqrt{65}}\left[\begin{array}{rr} 7 & 4 \\ 4 & -7 \end{array} \right] \text{ and } R_2 = \frac{1}{\sqrt{65}}\left[\begin{array}{rr} 13 & 11 \\ 0 & 10 \end{array} \right]$$

$$A_3 = \frac{1}{13}\left[\begin{array}{rr} 27 & -5 \\ 8 & -14 \end{array} \right] = \left[\begin{array}{rr} 2.08 & -0.38 \\ 0.62 & -1.08 \end{array} \right]$$

This is converging to $\left[\begin{array}{rr} 2 & * \\ 0 & -1 \end{array} \right]$ and so is approximating the eigenvalues 2 and -1 on the main diagonal.

It is beyond the scope of this book to pursue a detailed discussion of these methods. The reader is referred to J. M. Wilkinson, *The Algebraic Eigenvalue Problem* (Oxford, England: Oxford University

Press, 1965) or G. W. Stewart, *Introduction to Matrix Computations* (New York: Academic Press, 1973). We conclude with some remarks on the QR-algorithm.

Shifting. Convergence is accelerated if, at stage k of the algorithm, a number s_k is chosen and $A_k - s_k I$ is factored in the form $Q_k R_k$ rather than A_k itself. Then

$$Q_k^{-1} A_k Q_k = Q_k^{-1}(Q_k R_k + s_k I) Q_k = R_k Q_k + s_k I$$

so we take $A_{k+1} = R_k Q_k + s_k I$. If the shifts s_k are carefully chosen, convergence can be greatly improved.

Preliminary Preparation. A matrix such as

$$\left[\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{array} \right]$$

is said to be in **upper Hessenberg** form, and the QR-factorizations of such matrices are greatly simplified. Given an $n \times n$ matrix A, a series of orthogonal matrices $H_1, H_2, \dots, H_m$ (called **Householder matrices**) can be easily constructed such that

$$B = H_m^T \cdots H_1^T A H_1 \cdots H_m$$

is in upper Hessenberg form. Then the QR-algorithm can be efficiently applied to B and, because B is similar to A, it produces the eigenvalues of A.

Complex Eigenvalues. If some of the eigenvalues of a real matrix A are not real, the QR-algorithm converges to a block upper triangular matrix where the diagonal blocks are either 1×1 (the real eigenvalues) or 2×2 (each providing a pair of conjugate complex eigenvalues of A).

Exercises for 8.5

Exercise 8.5.1 In each case, find the exact eigenvalues and determine corresponding eigenvectors. Then start with $\mathbf{x}_0 = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$ and compute $\mathbf{x}_4$ and r_3 using the power method.

a) $A = \left[\begin{array}{rr} 2 & -4 \\ -3 & 3 \end{array} \right]$

b) $A = \left[\begin{array}{rr} 5 & 2 \\ -3 & -2 \end{array} \right]$

c) $A = \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right]$

d) $A = \left[\begin{array}{cc} 3 & 1 \\ 1 & 0 \end{array} \right]$

---

b. Eigenvalues 4, -1; eigenvectors $\left[\begin{array}{r} 2 \\ -1 \end{array} \right]$, $\left[\begin{array}{r} 1 \\ -3 \end{array} \right]$; $\mathbf{x}_4 = \left[\begin{array}{r} 409 \\ -203 \end{array} \right]$; $r_3 = 3.94$

d. Eigenvalues $\lambda_1 = \frac{1}{2}(3 + \sqrt{13})$, $\lambda_2 = \frac{1}{2}(3 - \sqrt{13})$; eigenvectors $\left[\begin{array}{c} \lambda_1 \\ 1 \end{array} \right]$, $\left[\begin{array}{c} \lambda_2 \\ 1 \end{array} \right]$; $\mathbf{x}_4 = \left[\begin{array}{c} 142 \\ 43 \end{array} \right]$; $r_3 = 3.3027750$ (The true value is $\lambda_1 = 3.3027756$, to seven decimal places.)

Exercise 8.5.2 In each case, find the exact eigenvalues and then approximate them using the QR-algorithm.

a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

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b. Eigenvalues $\lambda_1 = \frac{1}{2}(3+\sqrt{13}) = 3.302776$, $\lambda_2 = \frac{1}{2}(3-\sqrt{13}) = -0.302776$ $A_1 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$, $Q_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$, $R_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 3 \\ 0 & -1 \end{bmatrix}$

$A_2 = \frac{1}{10}\begin{bmatrix} 33 & -1 \\ -1 & -3 \end{bmatrix}$,

$Q_2 = \frac{1}{\sqrt{1090}}\begin{bmatrix} 33 & 1 \\ -1 & 33 \end{bmatrix}$,

$R_2 = \frac{1}{\sqrt{1090}}\begin{bmatrix} 109 & -3 \\ 0 & -10 \end{bmatrix}$

$A_3 = \frac{1}{109}\begin{bmatrix} 360 & 1 \\ 1 & -33 \end{bmatrix}$

$= \begin{bmatrix} 3.302775 & 0.009174 \\ 0.009174 & -0.302775 \end{bmatrix}$

Exercise 8.5.3 Apply the power method to $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, starting at $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Does it converge? Explain.

Exercise 8.5.4 If A is symmetric, show that each matrix A_k in the QR-algorithm is also symmetric. Deduce that they converge to a diagonal matrix.

Use induction on k. If $k = 1$, $A_1 = A$. In general $A_{k+1} = Q_k^{-1}A_kQ_k = Q_k^TA_kQ_k$, so the fact that $A_k^T = A_k$ implies $A_{k+1}^T = A_{k+1}$. The eigenvalues of A are all real (Theorem 5.5.5), so the A_k converge to an upper triangular matrix T. But T must also be symmetric (it is the limit of symmetric matrices), so it is diagonal.

Exercise 8.5.5 Apply the QR-algorithm to $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$. Explain.

Exercise 8.5.6 Given a matrix A, let A_k, Q_k, and R_k, $k \geq 1$, be the matrices constructed in the QR-algorithm. Show that $A_k = (Q_1Q_2\cdots Q_k)(R_k\cdots R_2R_1)$ for each $k \geq 1$ and hence that this is a QR-factorization of A_k.

[*Hint*: Show that $Q_kR_k = R_{k-1}Q_{k-1}$ for each $k \geq 2$, and use this equality to compute $(Q_1Q_2\cdots Q_k)(R_k\cdots R_2R_1)$ "from the centre out." Use the fact that $(AB)^{n+1} = A(BA)^nB$ for any square matrices A and B.]

8.6 The Singular Value Decomposition

When working with a square matrix A it is clearly useful to be able to "diagonalize" A, that is to find a factorization $A = Q^{-1}DQ$ where Q is invertible and D is diagonal. Unfortunately such a factorization may not exist for A. However, even if A is not square gaussian elimination provides a factorization of the form $A = PDQ$ where P and Q are invertible and D is diagonal—the Smith Normal form (Theorem 2.5.3). However, if A is real we can choose P and Q to be *orthogonal* real matrices and D to be real. Such a factorization is called a **singular value decomposition (SVD)** for A, one of the most useful tools in applied linear algebra. In this Section we show how to explicitly compute an SVD for any real matrix A, and illustrate some of its many applications.

We need a fact about two subspaces associated with an $m \times n$ matrix A:

$$\text{im}\, A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} \qquad \text{and} \qquad \text{col}\, A = \text{span}\, \{\mathbf{a} \mid \mathbf{a} \text{ is a column of } A\}$$

Then $\text{im}\, A$ is called the **image** of A (so named because of the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ with $\mathbf{x} \mapsto A\mathbf{x}$); and $\text{col}\, A$ is called the **column space** of A (Definition 5.10). Surprisingly, these spaces are equal:

Lemma 8.6.1

For any $m \times n$ matrix A, $\text{im}\, A = \text{col}\, A$.

Proof. Let $A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array} \right]$ in terms of its columns. Let $\mathbf{x} \in \text{im}\, A$, say $\mathbf{x} = A\mathbf{y}$, $\mathbf{y}$ in $\mathbb{R}^n$. If $\mathbf{y} = \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \right]^T$, then $A\mathbf{y} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \cdots + y_n\mathbf{a}_n \in \text{col}\, A$ by Definition 2.5. This shows that $\text{im}\, A \subseteq \text{col}\, A$. For the other inclusion, each $\mathbf{a}_k = A\mathbf{e}_k$ where $\mathbf{e}_k$ is column k of I_n. □

8.6.1. Singular Value Decompositions

We know a lot about any real symmetric matrix: Its eigenvalues are real (Theorem 5.5.7), and it is orthogonally diagonalizable by the Principal Axes Theorem (Theorem 8.2.2). So for any real matrix A (square or not), the fact that both A^TA and AA^T are real and symmetric suggests that we can learn a lot about A by studying them. This section shows just how true this is.

The following Lemma reveals some similarities between A^TA and AA^T which simplify the statement and the proof of the SVD we are constructing.

Lemma 8.6.2

Let A be a real $m \times n$ matrix. Then:

1. *The eigenvalues of A^TA and AA^T are real and non-negative.*
2. *A^TA and AA^T have the same set of positive eigenvalues.*

Proof.

1. Let λ be an eigenvalue of A^TA, with eigenvector $\mathbf{0} \neq \mathbf{q} \in \mathbb{R}^n$. Then:

$$\|A\mathbf{q}\|^2 = (A\mathbf{q})^T(A\mathbf{q}) = \mathbf{q}^T(A^TA\mathbf{q}) = \mathbf{q}^T(\lambda\mathbf{q}) = \lambda(\mathbf{q}^T\mathbf{q}) = \lambda\|\mathbf{q}\|^2$$

   Then (1.) follows for A^TA, and the case AA^T follows by replacing A by A^T.

2. Write $N(B)$ for the set of positive eigenvalues of a matrix B. We must show that $N(A^TA) = N(AA^T)$. If $\lambda \in N(A^TA)$ with eigenvector $\mathbf{0} \neq \mathbf{q} \in \mathbb{R}^n$, then $A\mathbf{q} \in \mathbb{R}^m$ and

$$AA^T(A\mathbf{q}) = A[(A^TA)\mathbf{q}] = A(\lambda\mathbf{q}) = \lambda(A\mathbf{q})$$

   Moreover, $A\mathbf{q} \neq \mathbf{0}$ since $A^TA\mathbf{q} = \lambda\mathbf{q} \neq \mathbf{0}$ and both $\lambda \neq 0$ and $\mathbf{q} \neq \mathbf{0}$. Hence λ is an eigenvalue of AA^T, proving $N(A^TA) \subseteq N(AA^T)$. For the other inclusion replace A by A^T.

□

To analyze an $m \times n$ matrix A we have two symmetric matrices to work with: A^TA and AA^T. In view of Lemma 8.6.2, we choose A^TA (sometimes called the **Gram** matrix of A), and derive a series of facts which we will need. This narrative is a bit long, but trust that it will be worth the effort. We parse it out in several steps:

1. The $n \times n$ matrix A^TA is real and symmetric so, by the Principal Axes Theorem 8.2.2, let $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\} \subseteq \mathbb{R}^n$ be an orthonormal basis of eigenvectors of A^TA, with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. By Lemma 8.6.2(1), λ_i is real for each i and $\lambda_i \geq 0$. By re-ordering the $\mathbf{q}_i$ we may (and do) assume that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \quad \text{and}^{8} \quad \lambda_i = 0 \text{ if } i > r \qquad \textbf{(i)}$$

By Theorems 8.2.1 and 3.3.4, the matrix

$$Q = \left[\begin{array}{cccc} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array}\right] \text{ is orthogonal} \quad \text{and} \quad \text{orthogonally diagonalizes } A^TA \qquad \textbf{(ii)}$$

2. Even though the λ_i are the eigenvalues of A^TA, the number r in (**i**) turns out to be $\operatorname{rank} A$. To understand why, consider the vectors $A\mathbf{q}_i \in \operatorname{im} A$. For all i, j:

$$A\mathbf{q}_i \cdot A\mathbf{q}_j = (A\mathbf{q}_i)^T A\mathbf{q}_j = \mathbf{q}_i^T(A^TA)\mathbf{q}_j = \mathbf{q}_i^T(\lambda_j\mathbf{q}_j) = \lambda_j(\mathbf{q}_i^T\mathbf{q}_j) = \lambda_j(\mathbf{q}_i \cdot \mathbf{q}_j)$$

Because $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ is an orthonormal set, this gives

$$A\mathbf{q}_i \cdot A\mathbf{q}_j = 0 \text{ if } i \neq j \quad \text{and} \quad \|A\mathbf{q}_i\|^2 = \lambda_i\|\mathbf{q}_i\|^2 = \lambda_i \text{ for each } i \qquad \textbf{(iii)}$$

We can extract two conclusions from (**iii**) and (**i**):

$$\{A\mathbf{q}_1, A\mathbf{q}_2, \dots, A\mathbf{q}_r\} \subseteq \operatorname{im} A \text{ is an orthogonal set} \quad \text{and} \quad A\mathbf{q}_i = \mathbf{0} \text{ if } i > r \qquad \textbf{(iv)}$$

With this write $U = \operatorname{span}\{A\mathbf{q}_1, A\mathbf{q}_2, \dots, A\mathbf{q}_r\} \subseteq \operatorname{im} A$; we claim that $U = \operatorname{im} A$, that is $\operatorname{im} A \subseteq U$. For this we must show that $A\mathbf{x} \in U$ for each $\mathbf{x} \in \mathbb{R}^n$. Since $\{\mathbf{q}_1, \dots, \mathbf{q}_r, \dots, \mathbf{q}_n\}$ is a basis of

[8] Of course they could *all* be positive ($r = n$) or *all* zero (so $A^TA = 0$, and hence $A = 0$ by Exercise 5.3.9).

$\mathbb{R}^n$ (it is orthonormal), we can write $\mathbf{x}_k = t_1\mathbf{q}_1 + \cdots + t_r\mathbf{q}_r + \cdots + t_n\mathbf{q}_n$ where each $t_j \in \mathbb{R}$. Then, using (**iv**) we obtain

$$A\mathbf{x} = t_1A\mathbf{q}_1 + \cdots + t_rA\mathbf{q}_r + \cdots + t_nA\mathbf{q}_n = t_1A\mathbf{q}_1 + \cdots + t_rA\mathbf{q}_r \in U$$

This shows that $U = \text{im}\,A$, and so

$$\{A\mathbf{q}_1,\ A\mathbf{q}_2,\ \ldots,\ A\mathbf{q}_r\} \text{ is an } \textit{orthogonal} \text{ basis of } \text{im}\,(A) \qquad (\mathbf{v})$$

But $\text{col}\,A = \text{im}\,A$ by Lemma 8.6.1, and $\text{rank}\,A = \dim\,(\text{col}\,A)$ by Theorem 5.4.1, so

$$\text{rank}\,A = \dim\,(\text{col}\,A) = \dim\,(\text{im}\,A) \overset{(\mathbf{v})}{=} r \qquad (\mathbf{vi})$$

3. Before proceeding, some definitions are in order:

Definition 8.7

The real numbers $\sigma_i = \sqrt{\lambda_i} \overset{(iii)}{=} \|A\bar{\mathbf{q}}_i\|$ *for* $i = 1,\ 2,\ \ldots,\ n$*, are called the* ***singular values*** *of the matrix* A*.*

Clearly $\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_r$ are the *positive* singular values of A. By (**i**) we have

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \qquad \text{and} \qquad \sigma_i = 0 \text{ if } i > r \qquad (\mathbf{vii})$$

With (**vi**) this makes the following definitions depend only upon A.

Definition 8.8

Let A *be a real,* $m \times n$ *matrix of rank* r*, with positive singular values* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ *and* $\sigma_i = 0$ *if* $i > r$*. Define:*

$$D_A = \text{diag}\,(\sigma_1,\ \ldots,\ \sigma_r) \qquad \text{and} \qquad \Sigma_A = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$$

Here Σ_A *is in block form and is called the* ***singular matrix*** *of* A*.*

The singular values σ_i and the matrices D_A and Σ_A will be referred to frequently below.

4. Returning to our narrative, normalize the vectors $A\mathbf{q}_1,\ A\mathbf{q}_2,\ \ldots,\ A\mathbf{q}_r$, by defining

$$\mathbf{p}_i = \frac{1}{\|A\mathbf{q}_i\|}A\mathbf{q}_i \in \mathbb{R}^m \quad \text{for each } i = 1,\ 2,\ \ldots,\ r \qquad (\mathbf{viii})$$

By (**v**) and Lemma 8.6.1, we conclude that

$$\{\mathbf{p}_1,\ \mathbf{p}_2,\ \ldots,\ \mathbf{p}_r\} \text{ is an } \textit{orthonormal} \text{ basis of } \text{col}\,A \subseteq \mathbb{R}^m \qquad (\mathbf{ix})$$

Employing the Gram-Schmidt algorithm (or otherwise), construct $\mathbf{p}_{r+1},\ \ldots,\ \mathbf{p}_m$ so that

$$\{\mathbf{p}_1,\ \ldots,\ \mathbf{p}_r,\ \ldots,\ \mathbf{p}_m\} \text{ is an orthonormal basis of } \mathbb{R}^m \qquad (\mathbf{x})$$

5. By (**x**) and (**ii**) we have *two* orthogonal matrices

$$P = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_r & \cdots & \mathbf{p}_m \end{bmatrix} \text{ of size } m \times m \quad \text{and} \quad Q = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_r & \cdots & \mathbf{q}_n \end{bmatrix} \text{ of size } n \times n$$

These matrices are related. In fact we have:

$$\sigma_i \mathbf{p}_i = \sqrt{\lambda_i}\mathbf{p}_i \overset{\text{(iii)}}{=} \|A\mathbf{q}_i\|\mathbf{p}_i \overset{\text{(viii)}}{=} A\mathbf{q}_i \quad \text{for each } i = 1, 2, \dots, r \tag{xi}$$

This yields the following expression for AQ in terms of its columns:

$$AQ = \begin{bmatrix} A\mathbf{q}_1 & \cdots & A\mathbf{q}_r & A\mathbf{q}_{r+1} & \cdots & A\mathbf{q}_n \end{bmatrix} \overset{\text{(iv)}}{=} \begin{bmatrix} \sigma_1\mathbf{p}_1 & \cdots & \sigma_r\mathbf{p}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{xii}$$

Then we compute:

$$\begin{aligned} P\Sigma_A &= \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_r & \mathbf{p}_{r+1} & \cdots & \mathbf{p}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \sigma_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1\mathbf{p}_1 & \cdots & \sigma_r\mathbf{p}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \\ &\overset{\text{(xii)}}{=} AQ \end{aligned}$$

Finally, as $Q^{-1} = Q^T$ it follows that $A = P\Sigma_A Q^T$.

With this we can state the main theorem of this Section.

Theorem 8.6.1

Let A be a real $m \times n$ matrix, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ be the positive singular values of A. Then r is the rank of A and we have the factorization

$$A = P\Sigma_A Q^T \qquad \textit{where } P \textit{ and } Q \textit{ are orthogonal matrices}$$

The factorization $A = P\Sigma_A Q^T$ in Theorem 8.6.1, where P and Q are orthogonal matrices, is called a *Singular Value Decomposition* (*SVD*) of A. This decomposition is not unique. For example if $r < m$ then the vectors $\mathbf{p}_{r+1}, \dots, \mathbf{p}_m$ can be *any* extension of $\{\mathbf{p}_1, \dots, \mathbf{p}_r\}$ to an orthonormal basis of $\mathbb{R}^m$, and each will lead to a different matrix P in the decomposition. For a more dramatic example, if $A = I_n$ then $\Sigma_A = I_n$, and $A = P\Sigma_A P^T$ is a SVD of A for *any* orthogonal $n \times n$ matrix P.

Example 8.6.1

Find a singular value decomposition for $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

Solution. We have $A^TA = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, so the characteristic polynomial is

$$c_{A^TA}(x) = \det \begin{bmatrix} x-2 & 1 & -1 \\ 1 & x-1 & 0 \\ -1 & 0 & x-1 \end{bmatrix} = (x-3)(x-1)x$$

Hence the eigenvalues of A^TA (in descending order) are $\lambda_1 = 3$, $\lambda_2 = 1$ and $\lambda_3 = 0$ with, respectively, unit eigenvectors

$$\mathbf{q}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{q}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

It follows that the orthogonal matrix Q in Theorem 8.6.1 is

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 0 & -\sqrt{2} \\ -1 & \sqrt{3} & -\sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \end{bmatrix}$$

The singular values here are $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$ and $\sigma_3 = 0$, so $\operatorname{rank}(A) = 2$—clear in this case—and the singular matrix is

$$\Sigma_A = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

So it remains to find the 2×2 orthogonal matrix P in Theorem 8.6.1. This involves the vectors

$$A\mathbf{q}_1 = \frac{\sqrt{6}}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A\mathbf{q}_2 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad A\mathbf{q}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Normalize $A\mathbf{q}_1$ and $A\mathbf{q}_2$ to get

$$\mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In this case, $\{\mathbf{p}_1, \mathbf{p}_2\}$ is *already* a basis of $\mathbb{R}^2$ (so the Gram-Schmidt algorithm is not needed), and we have the 2×2 orthogonal matrix

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Finally (by Theorem 8.6.1) the singular value decomposition for A is

$$A = P\Sigma_A Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & -1 & 1 \\ 0 & \sqrt{3} & \sqrt{3} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

Of course this can be confirmed by direct matrix multiplication.

Thus, computing an SVD for a real matrix A is a routine matter, and we now describe a systematic procedure for doing so.

SVD Algorithm

Given a real $m \times n$ matrix A, find an SVD $A = P\Sigma_A Q^T$ as follows:

1. *Use the Diagonalization Algorithm (see page 188) to find the (real and non-negative) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A^TA with corresponding (orthonormal) eigenvectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$. Reorder the $\mathbf{q}_i$ (if necessary) to ensure that the nonzero eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\lambda_i = 0$ if $i > r$.*
2. *The integer r is the rank of the matrix A.*
3. *The $n \times n$ orthogonal matrix Q in the SVD is $Q = \left[\begin{array}{cccc} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array} \right]$.*
4. *Define $\mathbf{p}_i = \frac{1}{\|A\mathbf{q}_i\|}A\mathbf{q}_i$ for $i = 1, 2, \dots, r$ (where r is as in step 1). Then $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_r\}$ is orthonormal in $\mathbb{R}^m$ so (using Gram-Schmidt or otherwise) extend it to an orthonormal basis $\{\mathbf{p}_1, \dots, \mathbf{p}_r, \dots, \mathbf{p}_m\}$ in $\mathbb{R}^m$.*
5. *The $m \times m$ orthogonal matrix P in the SVD is $P = \left[\begin{array}{ccccc} \mathbf{p}_1 & \cdots & \mathbf{p}_r & \cdots & \mathbf{p}_m \end{array} \right]$.*
6. *The singular values for A are $\sigma_1, \sigma_2, \dots, \sigma_n$ where $\sigma_i = \sqrt{\lambda_i}$ for each i. Hence the nonzero singular values are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and so the singular matrix of A in the SVD is $\Sigma_A = \left[\begin{array}{cc} \operatorname{diag}(\sigma_1, \dots, \sigma_r) & 0 \\ 0 & 0 \end{array} \right]_{m\times n}$.*
7. *Thus $A = P\Sigma Q^T$ is a SVD for A.*

In practise the singular values σ_i, the matrices P and Q, and even the rank of an $m \times n$ matrix are not calculated this way. There are sophisticated numerical algorithms for calculating them to a high degree of accuracy. The reader is referred to books on numerical linear algebra.

So the main virtue of Theorem 8.6.1 is that it provides a way of *constructing* an SVD for every real matrix A. In particular it shows that every real matrix A *has* a singular value decomposition[9] in the following, more general, sense:

Definition 8.9

*A **Singular Value Decomposition (SVD)** of an $m \times n$ matrix A of rank r is a factorization $A = U\Sigma V^T$ where U and V are orthogonal and $\Sigma = \left[\begin{array}{cc} D & 0 \\ 0 & 0 \end{array} \right]_{m\times n}$ in block form where $D = \operatorname{diag}(d_1, d_2, \dots, d_r)$ where each $d_i > 0$, and $r \leq m$ and $r \leq n$.*

Note that for *any* SVD $A = U\Sigma V^T$ we immediately obtain some information about A:

[9] In fact every complex matrix has an SVD [J.T. Scheick, Linear Algebra with Applications, McGraw-Hill, 1997]

Lemma 8.6.3

If $A = U\Sigma V^T$ is any SVD for A as in Definition 8.9, then:

1. $r = \text{rank}\, A$.
2. *The numbers $d_1, d_2, \dots, d_r$ are the singular values of A^TA in some order.*

Proof. Use the notation of Definition 8.9. We have

$$A^TA = (V\Sigma^TU^T)(U\Sigma V^T) = V(\Sigma^T\Sigma)V^T$$

so $\Sigma^T\Sigma$ and A^TA are similar $n \times n$ matrices (Definition 5.11). Hence $r = \text{rank}\, A$ by Corollary 5.4.3, proving (1.). Furthermore, $\Sigma^T\Sigma$ and A^TA have the same eigenvalues by Theorem 5.5.1; that is (using (1.)):

$$\{d_1^2,\ d_2^2,\ \dots,\ d_r^2\} = \{\lambda_1,\ \lambda_2,\ \dots,\ \lambda_r\} \quad \text{are equal as sets}$$

where $\lambda_1, \lambda_2, \dots, \lambda_r$ are the positive eigenvalues of A^TA. Hence there is a permutation τ of $\{1, 2, \cdots, r\}$ such that $d_i^2 = \lambda_{i\tau}$ for each $i = 1, 2, \dots, r$. Hence $d_i = \sqrt{\lambda_{i\tau}} = \sigma_{i\tau}$ for each i by Definition 8.7. This proves (2.). □

We note in passing that more is true. Let A be $m \times n$ of rank r, and let $A = U\Sigma V^T$ be any SVD for A. Using the proof of Lemma 8.6.3 we have $d_i = \sigma_{i\tau}$ for some permutation τ of $\{1, 2, \dots, r\}$. In fact, it can be shown that there exist orthogonal matrices U_1 and V_1 obtained from U and V by τ-permuting columns and rows respectively, such that $A = U_1\Sigma_A V_1^T$ is an SVD of A.

8.6.2. Fundamental Subspaces

It turns out that any singular value decomposition contains a great deal of information about an $m \times n$ matrix A and the subspaces associated with A. For example, in addition to Lemma 8.6.3, the set $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_r\}$ of vectors constructed in the proof of Theorem 8.6.1 is an orthonormal basis of $\text{col}\, A$ (by (**v**) and (**viii**) in the proof). There are more such examples, which is the thrust of this subsection. In particular, there are four subspaces associated to a real $m \times n$ matrix A that have come to be called fundamental:

Definition 8.10

The ***fundamental subspaces*** *of an $m \times n$ matrix A are:*

$$\text{row}\, A = \text{span}\,\{\mathbf{x} \mid \mathbf{x} \text{ is a row of } A\}$$

$$\text{col}\, A = \text{span}\,\{\mathbf{x} \mid \mathbf{x} \text{ is a column of } A\}$$

$$\text{null}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

$$\text{null}\, A^T = \{\mathbf{x} \in \mathbb{R}^n \mid A^T\mathbf{x} = \mathbf{0}\}$$

If $A = U\Sigma V^T$ is *any* SVD for the real $m \times n$ matrix A, any orthonormal bases of U and V provide orthonormal bases for each of these fundamental subspaces. We are going to prove this, but first we need three properties related to the *orthogonal complement* $U^\perp$ of a subspace U of $\mathbb{R}^n$, where (Definition 8.1):

$$U^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u} \cdot \mathbf{x} = 0 \text{ for all } \mathbf{u} \in U\}$$

The orthogonal complement plays an important role in the Projection Theorem (Theorem 8.1.3), and we return to it in Section ??. For now we need:

Lemma 8.6.4

If A is any matrix then:

1. $(\text{row } A)^\perp = \text{null } A$ *and* $(\text{col } A)^\perp = \text{null } A^T$.
2. *If U is any subspace of $\mathbb{R}^n$ then $U^{\perp\perp} = U$.*
3. *Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ be an orthonormal basis of $\mathbb{R}^m$. If $U = \text{span}\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$, then*

$$U^\perp = \text{span}\{\mathbf{f}_{k+1}, \ldots, \mathbf{f}_m\}$$

Proof.

1. Assume A is $m \times n$, and let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be the rows of A. If $\mathbf{x}$ is a column in $\mathbb{R}^n$, then entry i of $A\mathbf{x}$ is $\mathbf{b}_i \cdot \mathbf{x}$, so $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{b}_i \cdot \mathbf{x} = 0$ for each i. Thus:

$$\mathbf{x} \in \text{null } A \quad \Leftrightarrow \quad \mathbf{b}_i \cdot \mathbf{x} = 0 \text{ for each } i \quad \Leftrightarrow \quad \mathbf{x} \in (\text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_m\})^\perp = (\text{row } A)^\perp$$

   Hence $\text{null } A = (\text{row } A)^\perp$. Now replace A by A^T to get $\text{null } A^T = (\text{row } A^T)^\perp = (\text{col } A)^\perp$, which is the other identity in (1).

2. If $\mathbf{x} \in U$ then $\mathbf{y} \cdot \mathbf{x} = 0$ for all $\mathbf{y} \in U^\perp$, that is $\mathbf{x} \in U^{\perp\perp}$. This proves that $U \subseteq U^{\perp\perp}$, so it is enough to show that $\dim U = \dim U^{\perp\perp}$. By Theorem 8.1.4 we see that $\dim V^\perp = n - \dim V$ for any subspace $V \subseteq \mathbb{R}^n$. Hence

$$\dim U^{\perp\perp} = n - \dim U^\perp = n - (n - \dim U) = \dim U, \quad \text{as required}$$

3. We have $\text{span}\{\mathbf{f}_{k+1}, \ldots, \mathbf{f}_m\} \subseteq U^\perp$ because $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ is orthogonal. For the other inclusion, let $\mathbf{x} \in U^\perp$ so $\mathbf{f}_i \cdot \mathbf{x} = 0$ for $i = 1, 2, \ldots, k$. By the Expansion Theorem 5.3.6:

$$\begin{array}{rcccccccccccc} \mathbf{x} & = & (\mathbf{f}_1 \cdot \mathbf{x})\mathbf{f}_1 & + & \cdots & + & (\mathbf{f}_k \cdot \mathbf{x})\mathbf{f}_k & + & (\mathbf{f}_{k+1} \cdot \mathbf{x})\mathbf{f}_{k+1} & + & \cdots & + & (\mathbf{f}_m \cdot \mathbf{x})\mathbf{f}_m \\ & = & \mathbf{0} & + & \cdots & + & \mathbf{0} & + & (\mathbf{f}_{k+1} \cdot \mathbf{x})\mathbf{f}_{k+1} & + & \cdots & + & (\mathbf{f}_m \cdot \mathbf{x})\mathbf{f}_m \end{array}$$

   Hence $U^\perp \subseteq \text{span}\{\mathbf{f}_{k+1}, \ldots, \mathbf{f}_m\}$.

□

With this we can see how *any* SVD for a matrix A provides orthonormal bases for each of the four fundamental subspaces of A.

Theorem 8.6.2

Let A be an $m\times n$ real matrix, let $A=U\Sigma V^T$ be any SVD for A where U and V are orthogonal of size $m\times m$ and $n\times n$ respectively, and let

$$\Sigma=\left[\begin{array}{cc} D & 0 \\ 0 & 0 \end{array}\right]_{m\times n} \qquad \text{where} \qquad D=\text{diag}\,(\lambda_1,\ \lambda_2,\ \dots,\ \lambda_r),\quad \text{with each } \lambda_i>0$$

Write $U=\left[\begin{array}{ccccc} \mathbf{u}_1 & \cdots & \mathbf{u}_r & \cdots & \mathbf{u}_m \end{array}\right]$ and $V=\left[\begin{array}{ccccc} \mathbf{v}_1 & \cdots & \mathbf{v}_r & \cdots & \mathbf{v}_n \end{array}\right]$, so $\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_r,\ \dots,\ \mathbf{u}_m\}$ and $\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_r,\ \dots,\ \mathbf{v}_n\}$ are orthonormal bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then

1. *$r=\text{rank}\,A$, and the singular values of A are $\sqrt{\lambda_1},\ \sqrt{\lambda_2},\ \dots,\ \sqrt{\lambda_r}$.*
2. *The fundamental spaces are described as follows:*
   a. *$\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_r\}$ is an orthonormal basis of $\text{col}\,A$.*
   b. *$\{\mathbf{u}_{r+1},\ \dots,\ \mathbf{u}_m\}$ is an orthonormal basis of $\text{null}\,A^T$.*
   c. *$\{\mathbf{v}_{r+1},\ \dots,\ \mathbf{v}_n\}$ is an orthonormal basis of $\text{null}\,A$.*
   d. *$\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_r\}$ is an orthonormal basis of $\text{row}\,A$.*

Proof.

1. This is Lemma 8.6.3.
2. a. As $\text{col}\,A=\text{col}\,(AV)$ by Lemma 5.4.3 and $AV=U\Sigma$, (a.) follows from

$$U\Sigma=\left[\begin{array}{ccccc} \mathbf{u}_1 & \cdots & \mathbf{u}_r & \cdots & \mathbf{u}_m \end{array}\right]\left[\begin{array}{cc} \text{diag}\,(\lambda_1,\ \lambda_2,\ \dots,\ \lambda_r) & 0 \\ 0 & 0 \end{array}\right]=\left[\begin{array}{cccccc} \lambda_1\mathbf{u}_1 & \cdots & \lambda_r\mathbf{u}_r & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right]$$

   b. We have $(\text{col}\,A)^\perp \overset{\text{(a.)}}{=} (\text{span}\,\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_r\})^\perp=\text{span}\,\{\mathbf{u}_{r+1},\ \dots,\ \mathbf{u}_m\}$ by Lemma 8.6.4(3). This proves (b.) because $(\text{col}\,A)^\perp=\text{null}\,A^T$ by Lemma 8.6.4(1).

   c. We have $\dim\,(\text{null}\,A)+\dim\,(\text{im}\,A)=n$ by the Dimension Theorem 7.2.4, applied to $T:\mathbb{R}^n\to\mathbb{R}^m$ where $T(\mathbf{x})=A\mathbf{x}$. Since also $\text{im}\,A=\text{col}\,A$ by Lemma 8.6.1, we obtain

$$\dim\,(\text{null}\,A)=n-\dim\,(\text{col}\,A)=n-r=\dim\,(\text{span}\,\{\mathbf{v}_{r+1},\ \dots,\ \mathbf{v}_n\})$$

   So to prove (c.) it is enough to show that $\mathbf{v}_j\in\text{null}\,A$ whenever $j>r$. To this end write

$$\lambda_{r+1}=\cdots=\lambda_n=0, \qquad \text{so} \quad E^TE=\text{diag}\,(\lambda_1^2,\ \dots,\ \lambda_r^2,\ \lambda_{r+1}^2,\ \dots,\ \lambda_n^2)$$

   Observe that each λ_j is an eigenvalue of $\Sigma^T\Sigma$ with eigenvector $\mathbf{e}_j=$ column j of I_n. Thus $\mathbf{v}_j=V\mathbf{e}_j$ for each j. As $A^TA=V\Sigma^T\Sigma V^T$ (proof of Lemma 8.6.3), we obtain

$$(A^TA)\mathbf{v}_j=(V\Sigma^T\Sigma V^T)(V\mathbf{e}_j)=V(\Sigma^T\Sigma\mathbf{e}_j)=V\left(\lambda_j^2\mathbf{e}_j\right)=\lambda_j^2V\mathbf{e}_j=\lambda_j^2\mathbf{v}_j$$

   for $1\leq j\leq n$. Thus each $\mathbf{v}_j$ is an eigenvector of A^TA corresponding to λ_j^2. But then

$$\|A\mathbf{v}_j\|^2=(A\mathbf{v}_j)^TA\mathbf{v}_j=\mathbf{v}_j^T(A^TA\mathbf{v}_j)=\mathbf{v}_j^T(\lambda_j^2\mathbf{v}_j)=\lambda_j^2\|\mathbf{v}_j\|^2=\lambda_j^2 \quad \text{for } i=1,\ \dots,\ n$$

   In particular, $A\mathbf{v}_j=\mathbf{0}$ whenever $j>r$, so $\mathbf{v}_j\in\text{null}\,A$ if $j>r$, as desired. This proves (c).

d. Observe that $\text{span}\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\} \overset{(c.)}{=} \text{null}\, A = (\text{row}\, A)^\perp$ by Lemma 8.6.4(1). But then parts (2) and (3) of Lemma 8.6.4 show

$$\text{row}\, A = \left((\text{row}\ A)^\perp\right)^\perp = (\text{span}\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\})^\perp = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$$

This proves (d.), and hence Theorem 8.6.2.

□

Example 8.6.2

Consider the homogeneous linear system

$$A\mathbf{x} = \mathbf{0} \text{ of } m \text{ equations in } n \text{ variables}$$

Then the set of all solutions is $\text{null}\, A$. Hence if $A = U\Sigma V^T$ is any SVD for A then (in the notation of Theorem 8.6.2) $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal basis of the set of solutions for the system. As such they are a set of **basic solutions** for the system, the most basic notion in Chapter 1.

8.6.3. The Polar Decomposition of a Real Square Matrix

If A is real and $n \times n$ the factorization in the title is related to the polar decomposition A. Unlike the SVD, in this case the decomposition is *uniquely* determined by A.

Recall (Section 8.3) that a symmetric matrix A is called positive definite if and only if $\mathbf{x}^T A\mathbf{x} > 0$ for every column $\mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$. Before proceeding, we must explore the following weaker notion:

Definition 8.11

A real $n \times n$ matrix G is called ***positive***[10] *if it is symmetric and*

$$\mathbf{x}^T G\mathbf{x} \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

Clearly every positive definite matrix is positive, but the converse fails. Indeed, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is positive because, if $\mathbf{x} = \begin{bmatrix} a & b \end{bmatrix}^T$ in $\mathbb{R}^2$, then $\mathbf{x}^T A\mathbf{x} = (a+b)^2 \geq 0$. But $\mathbf{y}^T A\mathbf{y} = 0$ if $\mathbf{y} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, so A is not positive definite.

Lemma 8.6.5

Let G denote an $n \times n$ positive matrix.

1. *If A is any $m \times n$ matrix and G is positive, then $A^T GA$ is positive (and $m \times m$).*

[10] Also called **positive semi-definite.**

2. *If* $G = \text{diag}(d_1, d_2, \cdots, d_n)$ *and each* $d_i \geq 0$ *then* G *is positive.*

Proof.

1. $\mathbf{x}^T(A^T GA)\mathbf{x} = (A\mathbf{x})^T G(A\mathbf{x}) \geq 0$ because G is positive.

2. If $\mathbf{x} = \left[\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array}\right]^T$, then

$$\mathbf{x}^T G\mathbf{x} = d_1 x_1^2 + d_2 x_2^2 + \cdots + d_n x_n^2 \geq 0$$

because $d_i \geq 0$ for each i.

□

Definition 8.12

If A *is a real* $n \times n$ *matrix, a factorization*

$$A = GQ \text{ where } G \text{ is positive and } Q \text{ is orthogonal}$$

is called a ***polar decomposition*** *for* A*.*

Any SVD for a real square matrix A yields a polar form for A.

Theorem 8.6.3

Every square real matrix has a polar form.

Proof. Let $A = U\Sigma V^T$ be a SVD for A with Σ as in Definition 8.9 and $m = n$. Since $U^T U = I_n$ here we have

$$A = U\Sigma V^T = (U\Sigma)(U^T U)V^T = (U\Sigma U^T)(UV^T)$$

So if we write $G = U\Sigma U^T$ and $Q = UV^T$, then Q is orthogonal, and it remains to show that G is positive. But this follows from Lemma 8.6.5. □

The SVD for a square matrix A is not unique ($I_n = PI_nP^T$ for any orthogonal matrix P). But given the proof of Theorem 8.6.3 it is surprising that the polar decomposition *is* unique.[11] We omit the proof.

The name "polar form" is reminiscent of the same form for complex numbers (see Appendix **??**). This is no coincidence. To see why, we represent the complex numbers as real 2×2 matrices. Write $\mathbf{M}_2(\mathbb{R})$ for the set of all real 2×2 matrices, and define

$$\sigma : \mathbb{C} \to \mathbf{M}_2(\mathbb{R}) \quad \text{by} \quad \sigma(a + bi) = \left[\begin{array}{cc} a & -b \\ b & a \end{array}\right] \text{ for all } a + bi \text{ in } \mathbb{C}$$

[11] See J.T. Scheick, Linear Algebra with Applications, McGraw-Hill, 1997, page 379.

One verifies that σ preserves addition and multiplication in the sense that

$$\sigma(zw) = \sigma(z)\sigma(w) \qquad \text{and} \qquad \sigma(z+w) = \sigma(z) + \sigma(w)$$

for all complex numbers z and w. Since θ is one-to-one we may *identify* each complex number $a+bi$ with the matrix $\theta(a+bi)$, that is we write

$$a+bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{for all } a+bi \text{ in } \mathbb{C}$$

Thus $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $r = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ if r is real.

If $z = a+bi$ is nonzero then the *absolute value* $r = |z| = \sqrt{a^2+b^2} \neq 0$. If θ is the *angle* of z in standard position, then $\cos\theta = a/r$ and $\sin\theta = b/r$. Observe:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = GQ \qquad \text{(xiii)}$$

where $G = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ is positive and $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal. But in $\mathbb{C}$ we have $G = r$ and $Q = \cos\theta + i\sin\theta$ so (**xiii**) reads $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ which is the *classical polar form* for the complex number $a+bi$. This is why (**xiii**) is called the polar form of the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$; Definition 8.12 simply adopts the terminology for $n \times n$ matrices.

8.6.4. The Pseudoinverse of a Matrix

It is impossible for a non-square matrix A to have an inverse (see the footnote to Definition 2.11). Nonetheless, one candidate for an "inverse" of A is an $m \times n$ matrix B such that

$$ABA = A \qquad \text{and} \qquad BAB = B$$

Such a matrix B is called a *middle inverse* for A. If A is invertible then A^{-1} is the unique middle inverse for A, but a middle inverse is not unique in general, even for square matrices. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $B = \begin{bmatrix} 1 & 0 & 0 \\ b & 0 & 0 \end{bmatrix}$ is a middle inverse for A for any b.

If $ABA = A$ and $BAB = B$ it is easy to see that AB and BA are both idempotent matrices. In 1955 Roger Penrose observed that the middle inverse is unique if both AB and BA are symmetric. We omit the proof.

Theorem 8.6.4: Penrose' Theorem[12]

Given any real $m \times n$ matrix A, there is exactly one $n \times m$ matrix B such that A and B satisfy the following conditions:

P1 *$ABA = A$ and $BAB = B$.*

P2 *Both AB and BA are symmetric.*

Definition 8.13

Let A be a real $m \times n$ matrix. The ***pseudoinverse*** *of A is the unique $n \times m$ matrix A^+ such that A and A^+ satisfy* ***P1*** *and* ***P2****, that is:*

$$AA^+A = A, \qquad A^+AA^+ = A^+, \qquad \text{and both } AA^+ \text{ and } A^+A \text{ are symmetric}^{13}$$

If A is invertible then $A^+ = A^{-1}$ as expected. In general, the symmetry in conditions P1 and P2 shows that A is the pseudoinverse of A^+, that is $A^{++} = A$.

[12] R. Penrose, *A generalized inverse for matrices*, Proceedings of the Cambridge Philosophical Society **51** (1955), 406-413. In fact Penrose proved this for any complex matrix, where AB and BA are both required to be hermitian (see Definition **??** in the following section).

[13] Penrose called the matrix A^+ the generalized inverse of A, but the term pseudoinverse is now commonly used. The matrix A^+ is also called the **Moore-Penrose** inverse after E.H. Moore who had the idea in 1935 as part of a larger work on "General Analysis". Penrose independently re-discovered it 20 years later.

Theorem 8.6.5

Let A be an $m \times n$ matrix.

1. *If* $\operatorname{rank} A = m$ *then* AA^T *is invertible and* $A^+ = A^T(AA^T)^{-1}$.
2. *If* $\operatorname{rank} A = n$ *then* A^TA *is invertible and* $A^+ = (A^TA)^{-1}A^T$.

Proof. Here AA^T (respectively A^TA) is invertible by Theorem 5.4.4 (respectively Theorem 5.4.3). The rest is a routine verification. □

In general, given an $m \times n$ matrix A, the pseudoinverse A^+ can be computed from any SVD for A. To see how, we need some notation. Let $A = U\Sigma V^T$ be an SVD for A (as in Definition 8.9) where U and V are orthogonal and $\Sigma = \left[\begin{array}{cc} D & 0 \\ 0 & 0 \end{array}\right]_{m \times n}$ in block form where $D = \operatorname{diag}(d_1, d_2, \dots, d_r)$ where each $d_i > 0$. Hence D is invertible, so we make:

Definition 8.14

$$\Sigma' = \left[\begin{array}{cc} D^{-1} & 0 \\ 0 & 0 \end{array}\right]_{n \times m}.$$

A routine calculation gives:

Lemma 8.6.6

- $\Sigma\Sigma'\Sigma = \Sigma$
- $\Sigma\Sigma' = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{m \times m}$
- $\Sigma'\Sigma\Sigma' = \Sigma'$
- $\Sigma'\Sigma = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]_{n \times n}$

That is, Σ' is the pseudoinverse of Σ.

Now given $A = U\Sigma V^T$, define $B = V\Sigma' U^T$. Then

$$ABA = (U\Sigma V^T)(V\Sigma' U^T)(U\Sigma V^T) = U(\Sigma\Sigma'\Sigma)V^T = U\Sigma V^T = A$$

by Lemma 8.6.6. Similarly $BAB = B$. Moreover $AB = U(\Sigma\Sigma')U^T$ and $BA = V(\Sigma'\Sigma)V^T$ are both symmetric again by Lemma 8.6.6. This proves

Theorem 8.6.6

Let A be real and $m \times n$, and let $A = U\Sigma V^T$ is any SVD for A as in Definition 8.9. Then $A^+ = V\Sigma' U^T$.

Of course we can always use the SVD constructed in Theorem 8.6.1 to find the pseudoinverse. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, we observed above that $B = \begin{bmatrix} 1 & 0 & 0 \\ b & 0 & 0 \end{bmatrix}$ is a middle inverse for A for any b. Furthermore AB is symmetric but BA is not, so $B \neq A^+$.

Example 8.6.3

Find A^+ if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution. $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$ and corresponding eigenvectors $\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence $Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} = I_2$. Also A has rank 1 with singular values $\sigma_1 = 1$ and $\sigma_2 = 0$, so $\Sigma_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = A$ and $\Sigma_A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^T$ in this case.

Since $A\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A\mathbf{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we have $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ which extends to an orthonormal basis $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ of $\mathbb{R}^3$ where (say) $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = I$, so the SVD for A is $A = P\Sigma_A Q^T$. Finally, the pseudoinverse of A is $A^+ = Q\Sigma_A' P^T = \Sigma_A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that $A^+ = A^T$ in this case.

The following Lemma collects some properties of the pseudoinverse that mimic those of the inverse. The verifications are left as exercises.

Lemma 8.6.7

Let A be an $m \times n$ matrix of rank r.

1. $A^{++} = A$.
2. *If A is invertible then* $A^+ = A^{-1}$.
3. $(A^T)^+ = (A^+)^T$.
4. $(kA)^+ = kA^+$ *for any real* k.
5. $(UAV)^+ = U^T(A^+)V^T$ *whenever U and V are orthogonal.*

Exercises for 8.6

Exercise 8.6.1 If $ACA = A$ show that $B = CAC$ is a middle inverse for A.

Exercise 8.6.2 For any matrix A show that

$$\Sigma_{A^T} = (\Sigma_A)^T$$

Exercise 8.6.3 If A is $m \times n$ with all singular values positive, what is $\operatorname{rank} A$?

Exercise 8.6.4 If A has singular values $\sigma_1, \dots, \sigma_r$, what are the singular values of:

a) A^T

b) tA where $t > 0$ is real

c) A^{-1} assuming A is invertible.

b. $t\sigma_1, \dots, t\sigma_r$.

Exercise 8.6.5 If A is square show that $\det A$ is the product of the singular values of A.

Exercise 8.6.6 If A is square and real, show that $A = 0$ if and only if every eigenvalue of A is 0.

Exercise 8.6.7 Given a SVD for an invertible matrix A, find one for A^{-1}. How are Σ_A and $\Sigma_{A^{-1}}$ related?

If $A = U\Sigma V^T$ then Σ is invertible, so $A^{-1} = V\Sigma^{-1}U^T$ is a SVD.

Exercise 8.6.8 Let $A^{-1} = A = A^T$ where A is $n \times n$. Given any orthogonal $n \times n$ matrix U, find an orthogonal matrix V such that $A = U\Sigma_A V^T$ is an SVD for A. If $A = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right]$ do this for:

a) $U = \frac{1}{5}\left[\begin{array}{rr} 3 & -4 \\ 4 & 3 \end{array}\right]$

b) $U = \frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & -1 \\ 1 & 1 \end{array}\right]$

b. First $A^TA = I_n$ so $\Sigma_A = I_n$.

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & 1 \\ 1 & -1 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right]\frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & 1 \\ -1 & 1 \end{array}\right] \\ &= \frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & -1 \\ 1 & 1 \end{array}\right]\frac{1}{\sqrt{2}}\left[\begin{array}{rr} -1 & 1 \\ 1 & 1 \end{array}\right] \\ &= \left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right] \end{aligned}$$

Exercise 8.6.9 Find a SVD for the following matrices:

a) $A = \left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{array}\right]$

b) $\left[\begin{array}{rrr} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{array}\right]$

b.

$$A = F$$

$$= \frac{1}{5}\left[\begin{array}{rr} 3 & 4 \\ 4 & -3 \end{array}\right]\left[\begin{array}{rrrr} 20 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{array}\right]\frac{1}{2}\left[\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

Exercise 8.6.10 Find an SVD for $A = \left[\begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array}\right]$.

Exercise 8.6.11 If $A = U\Sigma V^T$ is an SVD for A, find an SVD for A^T.

Exercise 8.6.12 Let A be a real, $m \times n$ matrix with positive singular values $\sigma_1, \sigma_2, \dots, \sigma_r$, and write

$$s(x) = (x - \sigma_1)(x - \sigma_2)\cdots(x - \sigma_r)$$

a. Show that $c_{A^TA}(x) = s(x)x^{n-r}$ and $c_{A^TA}(c) = s(x)x^{m-r}$.

b. If $m \leq n$ conclude that $c_{A^TA}(x) = s(x)x^{n-m}$.

Exercise 8.6.13 If G is positive show that:

a. rG is positive if $r \geq 0$

b. $G + H$ is positive for any positive H.

b. If $\mathbf{x} \in \mathbb{R}^n$ then $\mathbf{x}^T(G+H)\mathbf{x} = \mathbf{x}^TG\mathbf{x} + \mathbf{x}^TH\mathbf{x} \geq 0 + 0 = 0$.

Exercise 8.6.14 If G is positive and λ is an eigenvalue, show that $\lambda \geq 0$.

Exercise 8.6.15 If G is positive show that $G = H^2$ for some positive matrix H. [Hint: Preceding exercise and Lemma 8.6.5]

Exercise 8.6.16 If A is $n \times n$ show that AA^T and A^TA are similar. [Hint: Start with an SVD for A.]

Exercise 8.6.17 Find A^+ if:

a. $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix}$

Exercise 8.6.18 Show that $(A^+)^T = (A^T)^+$.