1.3 Homogeneous Equations

A system of equations in the variables $x_1, x_2, \dots, x_n$ is called **homogeneous** if all the constant terms are zero—that is, if each equation of the system has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Clearly $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is a solution to such a system; it is called the **trivial solution**. Any solution in which at least one variable has a nonzero value is called a **nontrivial solution**. Our chief goal in this section is to give a useful condition for a homogeneous system to have nontrivial solutions. The following example is instructive.

Example 1.3.1

Show that the following homogeneous system has nontrivial solutions.

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & 2x_3 & - & x_4 & = & 0 \\ 2x_1 & + & 2x_2 & & & + & x_4 & = & 0 \\ 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 0 \end{array}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is outlined below.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 4 & -4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The leading variables are x_1, x_2, and x_4, so x_3 is assigned as a parameter—say $x_3 = t$. Then the general solution is $x_1 = -t$, $x_2 = t$, $x_3 = t$, $x_4 = 0$. Hence, taking $t = 1$ (say), we get a nontrivial solution: $x_1 = -1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$.

The existence of a nontrivial solution in Example 1.3.1 is ensured by the presence of a parameter in the solution. This is due to the fact that there is a *nonleading* variable (x_3 in this case). But there *must* be a nonleading variable here because there are four variables and only three equations (and hence at *most* three leading variables). This discussion generalizes to a proof of the following fundamental theorem.

Theorem 1.3.1

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Proof. Suppose there are m equations in n variables where $n > m$, and let R denote the reduced row-echelon form of the augmented matrix. If there are r leading variables, there are $n - r$ nonleading variables, and so $n - r$ parameters. Hence, it suffices to show that $r < n$. But $r \leq m$ because R has r leading 1s and m rows, and $m < n$ by hypothesis. So $r \leq m < n$, which gives $r < n$. □

Note that the converse of Theorem 1.3.1 is not true: if a homogeneous system has nontrivial solutions, it need not have more variables than equations (the system $x_1 + x_2 = 0$, $2x_1 + 2x_2 = 0$ has nontrivial solutions but $m = 2 = n$.)

Theorem 1.3.1 is very useful in applications. The next example provides an illustration from geometry.

Example 1.3.2

We call the graph of an equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ a **conic** if the numbers a, b, and c are not all zero. Show that there is at least one conic through any five points in the plane that are not all on a line.

Solution. Let the coordinates of the five points be (p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4), and (p_5, q_5). The graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ passes through (p_i, q_i) if

$$ap_i^2 + bp_iq_i + cq_i^2 + dp_i + eq_i + f = 0$$

This gives five equations, one for each i, linear in the six variables a, b, c, d, e, and f. Hence, there is a nontrivial solution by Theorem 1.3.1. If $a = b = c = 0$, the five points all lie on the line with equation $dx + ey + f = 0$, contrary to assumption. Hence, one of a, b, c is nonzero.

Linear Combinations and Basic Solutions

As for rows, two columns are regarded as **equal** if they have the same number of entries and corresponding entries are the same. Let $\mathbf{x}$ and $\mathbf{y}$ be columns with the same number of entries. As for elementary row operations, their **sum** $\mathbf{x} + \mathbf{y}$ is obtained by adding corresponding entries and, if k is a number, the **scalar product** $k\mathbf{x}$ is defined by multiplying each entry of $\mathbf{x}$ by k. More precisely:

$$\text{If } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ then } \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \text{ and } k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}.$$

A sum of scalar multiples of several columns is called a **linear combination** of these columns. For example, $s\mathbf{x} + t\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$ for any choice of numbers s and t.

Example 1.3.3

If $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then $2\mathbf{x} + 5\mathbf{y} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 1.3.4

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. If $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, determine whether $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

Solution. For $\mathbf{v}$, we must determine whether numbers r, s, and t exist such that $\mathbf{v} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$, that is, whether

$$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r+2s+3t \\ s+t \\ r+t \end{bmatrix}$$

Equating corresponding entries gives a system of linear equations $r+2s+3t = 0$, $s+t = -1$, and $r+t = 2$ for r, s, and t. By gaussian elimination, the solution is $r = 2-k$, $s = -1-k$, and $t = k$ where k is a parameter. Taking $k = 0$, we see that $\mathbf{v} = 2\mathbf{x} - \mathbf{y}$ is a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Turning to $\mathbf{w}$, we again look for r, s, and t such that $\mathbf{w} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$; that is,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r+2s+3t \\ s+t \\ r+t \end{bmatrix}$$

leading to equations $r+2s+3t = 1$, $s+t = 1$, and $r+t = 1$ for real numbers r, s, and t. But this time there is *no* solution as the reader can verify, so $\mathbf{w}$ is *not* a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Our interest in linear combinations comes from the fact that they provide one of the best ways to describe the general solution of a homogeneous system of linear equations. When solving such a system with n variables $x_1, x_2, \dots, x_n$, write the variables as a column[6] matrix: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The trivial solution is denoted $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. As an illustration, the general solution in Example 1.3.1 is $x_1 = -t$, $x_2 = t$, $x_3 = t$, and $x_4 = 0$, where t is a parameter, and we would now express this by saying that the general solution is $\mathbf{x} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix}$, where t is arbitrary.

Now let $\mathbf{x}$ and $\mathbf{y}$ be two solutions to a homogeneous system with n variables. Then any linear combination $s\mathbf{x} + t\mathbf{y}$ of these solutions turns out to be again a solution to the system. More generally:

$$\textit{Any linear combination of solutions to a homogeneous system is again a solution.} \tag{1.1}$$

[6] The reason for using columns will be apparent later.

In fact, suppose that a typical equation in the system is $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, and suppose that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ are solutions. Then $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $a_1y_1 + a_2y_2 + \cdots + a_ny_n = 0$. Hence $s\mathbf{x} + t\mathbf{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ \vdots \\ sx_n + ty_n \end{bmatrix}$ is also a solution because

$$\begin{aligned} a_1(sx_1 + ty_1) &+ a_2(sx_2 + ty_2) + \cdots + a_n(sx_n + ty_n) \\ &= [a_1(sx_1) + a_2(sx_2) + \cdots + a_n(sx_n)] + [a_1(ty_1) + a_2(ty_2) + \cdots + a_n(ty_n)] \\ &= s(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + t(a_1y_1 + a_2y_2 + \cdots + a_ny_n) \\ &= s(0) + t(0) \\ &= 0 \end{aligned}$$

A similar argument shows that Statement 1.1 is true for linear combinations of more than two solutions.

The remarkable thing is that *every* solution to a homogeneous system is a linear combination of certain particular solutions and, in fact, these solutions are easily computed using the gaussian algorithm. Here is an example.

Example 1.3.5

Solve the homogeneous system with coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 0 \\ -2 & 4 & 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the solutions are $x_1 = 2s + \frac{1}{5}t$, $x_2 = s$, $x_3 = \frac{3}{5}$, and $x_4 = t$ by gaussian elimination. Hence we can write the general solution $\mathbf{x}$ in the matrix form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + \frac{1}{5}t \\ s \\ \frac{3}{5}t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s\mathbf{x}_1 + t\mathbf{x}_2.$$

Here $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix}$ are particular solutions determined by the gaussian algorithm.

The solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ in Example 1.3.5 are denoted as follows:

Definition 1.5 Basic Solutions

*The gaussian algorithm systematically produces solutions to any homogeneous linear system, called **basic solutions**, one for every parameter.*

Moreover, the algorithm gives a routine way to express *every* solution as a linear combination of basic solutions as in Example 1.3.5, where the general solution $\mathbf{x}$ becomes

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{5}t \begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \end{bmatrix}$$

Hence by introducing a new parameter $r = t/5$ we can multiply the original basic solution $\mathbf{x}_2$ by 5 and so eliminate fractions. For this reason:

Convention:
Any nonzero scalar multiple of a basic solution will still be called a basic solution.

In the same way, the gaussian algorithm produces basic solutions to *every* homogeneous system, one for each parameter (there are *no* basic solutions if the system has only the trivial solution). Moreover every solution is given by the algorithm as a linear combination of these basic solutions (as in Example 1.3.5). If A has rank r, Theorem 1.2.2 shows that there are exactly $n-r$ parameters, and so $n-r$ basic solutions. This proves:

Theorem 1.3.2

Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. *The system has exactly $n-r$ basic solutions, one for each parameter.*
2. *Every solution is a linear combination of these basic solutions.*

Example 1.3.6

Find basic solutions of the homogeneous system with coefficient matrix A, and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ -2 & 6 & 1 & 2 & -5 & 0 \\ 3 & -9 & -1 & 0 & 7 & 0 \\ -3 & 9 & 2 & 6 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the general solution is $x_1 = 3r - 2s - 2t$, $x_2 = r$, $x_3 = -6s + t$, $x_4 = s$, and $x_5 = t$ where r, s, and t are parameters. In matrix form this is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence basic solutions are

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Exercises for 1.3

Exercise 1.3.1 Consider the following statements about a system of linear equations with augmented matrix A. In each case either prove the statement or give an example for which it is false.

a. If the system is homogeneous, every solution is trivial.

b. If the system has a nontrivial solution, it cannot be homogeneous.

c. If there exists a trivial solution, the system is homogeneous.

d. If the system is consistent, it must be homogeneous.

Now assume that the system is homogeneous.

e. If there exists a nontrivial solution, there is no trivial solution.

f. If there exists a solution, there are infinitely many solutions.

g. If there exist nontrivial solutions, the row-echelon form of A has a row of zeros.

h. If the row-echelon form of A has a row of zeros, there exist nontrivial solutions.

i. If a row operation is applied to the system, the new system is also homogeneous.

Exercise 1.3.2 In each of the following, find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

a. $$\begin{aligned} x-2y+\ z&=0\\ x+ay-3z&=0\\ -x+6y-5z&=0 \end{aligned}$$

b. $$\begin{aligned} x+2y+\ z&=0\\ x+3y+6z&=0\\ 2x+3y+az&=0 \end{aligned}$$

c. $$\begin{aligned} x+\ y-\ z&=0\\ ay-\ z&=0\\ x+\ y+az&=0 \end{aligned}$$

d. $$\begin{aligned} ax+y+\ z&=0\\ x+y-\ z&=0\\ x+y+az&=0 \end{aligned}$$

Exercise 1.3.3 Let $\mathbf{x}=\left[\begin{array}{r} 2\\ 1\\ -1 \end{array}\right]$, $\mathbf{y}=\left[\begin{array}{r} 1\\ 0\\ 1 \end{array}\right]$, and $\mathbf{z}=\left[\begin{array}{r} 1\\ 1\\ -2 \end{array}\right]$. In each case, either write $\mathbf{v}$ as a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, or show that it is not such a linear combination.

a. $\mathbf{v}=\left[\begin{array}{r} 0\\ 1\\ -3 \end{array}\right]$

b. $\mathbf{v}=\left[\begin{array}{r} 4\\ 3\\ -4 \end{array}\right]$

c. $\mathbf{v}=\left[\begin{array}{r} 3\\ 1\\ 0 \end{array}\right]$

d. $\mathbf{v}=\left[\begin{array}{r} 3\\ 0\\ 3 \end{array}\right]$

Exercise 1.3.4 In each case, either express $\mathbf{y}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, or show that it is not such a linear combination. Here:

$$\mathbf{a}_1=\left[\begin{array}{r} -1\\ 3\\ 0\\ 1 \end{array}\right],\ \mathbf{a}_2=\left[\begin{array}{r} 3\\ 1\\ 2\\ 0 \end{array}\right],\ \text{and } \mathbf{a}_3=\left[\begin{array}{r} 1\\ 1\\ 1\\ 1 \end{array}\right]$$

a. $\mathbf{y}=\left[\begin{array}{r} 1\\ 2\\ 4\\ 0 \end{array}\right]$

b. $\mathbf{y}=\left[\begin{array}{r} -1\\ 9\\ 2\\ 6 \end{array}\right]$

Exercise 1.3.5 For each of the following homogeneous systems, find a set of basic solutions and express the general solution as a linear combination of these basic solutions.

a. $$\begin{aligned} x_1+2x_2-\ x_3+2x_4+x_5&=0\\ x_1+2x_2+2x_3\qquad\ \ +x_5&=0\\ 2x_1+4x_2-2x_3+3x_4+x_5&=0 \end{aligned}$$

b. $$\begin{aligned} x_1+2x_2-\ x_3+x_4+\ x_5&=0\\ -x_1-2x_2+2x_3\qquad\ +\ x_5&=0\\ -x_1-2x_2+3x_3+x_4+3x_5&=0 \end{aligned}$$

c. $$\begin{aligned} x_1+\ x_2-\ x_3+2x_4+\ x_5&=0\\ x_1+2x_2-\ x_3+\ x_4+\ x_5&=0\\ 2x_1+3x_2-\ x_3+2x_4+\ x_5&=0\\ 4x_1+5x_2-2x_3+5x_4+2x_5&=0 \end{aligned}$$

d. $$\begin{aligned} x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 &= 0 \\ 2x_1 + 2x_2 - 4x_3 - 4x_4 + x_5 &= 0 \\ x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 0 \\ -2x_1 - 4x_2 + 8x_3 + 10x_4 + x_5 &= 0 \end{aligned}$$

Exercise 1.3.6

a. Does Theorem 1.3.1 imply that the system $\begin{cases} -z + 3y = 0 \\ 2x - 6y = 0 \end{cases}$ has nontrivial solutions? Explain.

b. Show that the converse to Theorem 1.3.1 is not true. That is, show that the existence of nontrivial solutions does *not* imply that there are more variables than equations.

Exercise 1.3.7 In each case determine how many solutions (and how many parameters) are possible for a homogeneous system of four linear equations in six variables with augmented matrix A. Assume that A has nonzero entries. Give all possibilities.

a. Rank $A = 2$.

b. Rank $A = 1$.

c. A has a row of zeros.

d. The row-echelon form of A has a row of zeros.

Exercise 1.3.8 The graph of an equation $ax + by + cz = 0$ is a plane through the origin (provided that not all of a, b, and c are zero). Use Theorem 1.3.1 to show that two planes through the origin have a point in common other than the origin $(0, 0, 0)$.

Exercise 1.3.9

a. Show that there is a line through any pair of points in the plane. [*Hint*: Every line has equation $ax + by + c = 0$, where a, b, and c are not all zero.]

b. Generalize and show that there is a plane $ax + by + cz + d = 0$ through any three points in space.

Exercise 1.3.10 The graph of

$$a(x^2 + y^2) + bx + cy + d = 0$$

is a circle if $a \neq 0$. Show that there is a circle through any three points in the plane that are not all on a line.

Exercise 1.3.11 Consider a homogeneous system of linear equations in n variables, and suppose that the augmented matrix has rank r. Show that the system has nontrivial solutions if and only if $n > r$.

Exercise 1.3.12 If a consistent (possibly nonhomogeneous) system of linear equations has more variables than equations, prove that it has more than one solution.

1.4 An Application to Network Flow

There are many types of problems that concern a network of conductors along which some sort of flow is observed. Examples of these include an irrigation network and a network of streets or freeways. There are often points in the system at which a net flow either enters or leaves the system. The basic principle behind the analysis of such systems is that the total flow into the system must equal the total flow out. In fact, we apply this principle at every junction in the system.

Junction Rule

At each of the junctions in the network, the total flow into that junction must equal the total flow out.

This requirement gives a linear equation relating the flows in conductors emanating from the junction.