1.6 An Application to Chemical Reactions

When a chemical reaction takes place a number of molecules combine to produce new molecules. Hence, when hydrogen H_2 and oxygen O_2 molecules combine, the result is water H_2O. We express this as

$$H_2 + O_2 \rightarrow H_2O$$

Individual atoms are neither created nor destroyed, so the number of hydrogen and oxygen atoms going into the reaction must equal the number coming out (in the form of water). In this case the reaction is said to be *balanced*. Note that each hydrogen molecule H_2 consists of two atoms as does each oxygen molecule O_2, while a water molecule H_2O consists of two hydrogen atoms and one oxygen atom. In the above reaction, this requires that twice as many hydrogen molecules enter the reaction; we express this as follows:

$$2H_2 + O_2 \rightarrow 2H_2O$$

This is now balanced because there are 4 hydrogen atoms and 2 oxygen atoms on each side of the reaction.

Example 1.6.1

Balance the following reaction for burning octane C_8H_{18} in oxygen O_2:

$$C_8H_{18} + O_2 \rightarrow CO_2 + H_2O$$

where CO_2 represents carbon dioxide. We must find positive integers x, y, z, and w such that

$$xC_8H_{18} + yO_2 \rightarrow zCO_2 + wH_2O$$

Equating the number of carbon, hydrogen, and oxygen atoms on each side gives $8x = z$, $18x = 2w$ and $2y = 2z + w$, respectively. These can be written as a homogeneous linear system

$$\begin{array}{rcrcrcrcl} 8x & & & - & z & & & = & 0 \\ 18x & & & & & - & 2w & = & 0 \\ & & 2y & - & 2z & - & w & = & 0 \end{array}$$

which can be solved by gaussian elimination. In larger systems this is necessary but, in such a simple situation, it is easier to solve directly. Set $w = t$, so that $x = \frac{1}{9}t$, $z = \frac{8}{9}t$, $2y = \frac{16}{9}t + t = \frac{25}{9}t$. But x, y, z, and w must be positive integers, so the smallest value of t that eliminates fractions is 18. Hence, $x = 2$, $y = 25$, $z = 16$, and $w = 18$, and the balanced reaction is

$$2C_8H_{18} + 25O_2 \rightarrow 16CO_2 + 18H_2O$$

The reader can verify that this is indeed balanced.

It is worth noting that this problem introduces a new element into the theory of linear equations: the insistence that the solution must consist of positive integers.

Exercises for 1.6

In each case balance the chemical reaction.

Exercise 1.6.1 $CH_4 + O_2 \to CO_2 + H_2O$. This is the burning of methane CH_4.

Exercise 1.6.2 $NH_3 + CuO \to N_2 + Cu + H_2O$. Here NH_3 is ammonia, CuO is copper oxide, Cu is copper, and N_2 is nitrogen.

Exercise 1.6.3 $CO_2 + H_2O \to C_6H_{12}O_6 + O_2$. This is called the photosynthesis reaction—$C_6H_{12}O_6$ is glucose.

Exercise 1.6.4 $Pb(N_3)_2 + Cr(MnO_4)_2 \to Cr_2O_3 + MnO_2 + Pb_3O_4 + NO$.

Supplementary Exercises for Chapter 1

Exercise 1.1 We show in Chapter 4 that the graph of an equation $ax+by+cz=d$ is a plane in space when not all of a, b, and c are zero.

a. By examining the possible positions of planes in space, show that three equations in three variables can have zero, one, or infinitely many solutions.

b. Can two equations in three variables have a unique solution? Give reasons for your answer.

Exercise 1.2 Find all solutions to the following systems of linear equations.

a.
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + 5x_2 - 2x_3 + x_4 &= 1 \\ -3x_1 - 7x_2 + 7x_3 - 5x_4 &= 7 \\ x_1 + 3x_2 - 4x_3 + 3x_4 &= -5 \end{aligned}$$

b.
$$\begin{aligned} x_1 + 4x_2 - x_3 + x_4 &= 2 \\ 3x_1 + 2x_2 + x_3 + 2x_4 &= 5 \\ x_1 - 6x_2 + 3x_3 &= 1 \\ x_1 + 14x_2 - 5x_3 + 2x_4 &= 3 \end{aligned}$$

Exercise 1.3 In each case find (if possible) conditions on a, b, and c such that the system has zero, one, or infinitely many solutions.

a.
$$\begin{aligned} x + 2y - 4z &= 4 \\ 3x - y + 13z &= 2 \\ 4x + y + a^2z &= a+3 \end{aligned}$$

b.
$$\begin{aligned} x + y + 3z &= a \\ ax + y + 5z &= 4 \\ x + ay + 4z &= a \end{aligned}$$

Exercise 1.4 Show that any two rows of a matrix can be interchanged by elementary row transformations of the other two types.

Exercise 1.5 If $ad \neq bc$, show that $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ has reduced row-echelon form $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$.

Exercise 1.6 Find a, b, and c so that the system

$$\begin{aligned} x + ay + cz &= 0 \\ bx + cy - 3z &= 1 \\ ax + 2y + bz &= 5 \end{aligned}$$

has the solution $x = 3$, $y = -1$, $z = 2$.

Exercise 1.7 Solve the system

$$\begin{aligned} x + 2y + 2z &= -3 \\ 2x + y + z &= -4 \\ x - y + iz &= i \end{aligned}$$

where $i^2 = -1$. [See Appendix A.]

Exercise 1.8 Show that the *real* system

$$\left\{\begin{aligned} x + y + z &= 5 \\ 2x - y - z &= 1 \\ -3x + 2y + 2z &= 0 \end{aligned}\right.$$

has a *complex* solution: $x = 2$, $y = i$, $z = 3 - i$ where $i^2 = -1$. Explain. What happens when such a real system has a unique solution?