c. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

$\mathbf{b} = \begin{bmatrix} 4 \\ -6 \\ 4 \\ 5 \end{bmatrix}$

Exercise 2.7.4 Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = LU$ is impossible where L is lower triangular and U is upper triangular.

Exercise 2.7.5 Show that we can accomplish any row interchange by using only row operations of other types.

Exercise 2.7.6

a. Let L and L_1 be invertible lower triangular matrices, and let U and U_1 be invertible upper triangular matrices. Show that $LU = L_1U_1$ if and only if there exists an invertible diagonal matrix D such that $L_1 = LD$ and $U_1 = D^{-1}U$. [*Hint*: Scrutinize $L^{-1}L_1 = UU_1^{-1}$.]

b. Use part (a) to prove Theorem 2.7.3 in the case that A is invertible.

Exercise 2.7.7 Prove Lemma 2.7.1(1). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.8 Prove Lemma 2.7.1(2). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.9 A triangular matrix is called **unit triangular** if it is square and every main diagonal element is a 1.

a. If A can be carried by the gaussian algorithm to row-echelon form using no row interchanges, show that $A = LU$ where L is unit lower triangular and U is upper triangular.

b. Show that the factorization in (a.) is unique.

Exercise 2.7.10 Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ be columns of lengths $m, m-1, \dots, m-r+1$. If $\mathbf{k}_j$ denotes column j of I_m, show that $L^{(m)}[\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r] = L^{(m)}[\mathbf{c}_1]L^{(m)}[\mathbf{k}_1, \mathbf{c}_2]L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_3]\cdots L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{r-1}, \mathbf{c}_r]$. The notation is as in the proof of Theorem 2.7.2. [*Hint*: Use induction on m and block multiplication.]

Exercise 2.7.11 Prove Lemma 2.7.2. [*Hint*: $P_k^{-1} = P_k$. Write $P_k = \begin{bmatrix} I_k & 0 \\ 0 & P_0 \end{bmatrix}$ in block form where P_0 is an $(m-k) \times (m-k)$ permutation matrix.]

2.8 An Application to Input-Output Economic Models[16]

In 1973 Wassily Leontief was awarded the Nobel prize in economics for his work on mathematical models.[17] Roughly speaking, an economic system in this model consists of several industries, each of which produces a product and each of which uses some of the production of the other industries. The following example is typical.

[16]The applications in this section and the next are independent and may be taken in any order.

[17]See W. W. Leontief, "The world economy of the year 2000," *Scientific American,* Sept. 1980.

Example 2.8.1

A primitive society has three basic needs: food, shelter, and clothing. There are thus three industries in the society—the farming, housing, and garment industries—that produce these commodities. Each of these industries consumes a certain proportion of the total output of each commodity according to the following table.

		OUTPUT		
		Farming	**Housing**	**Garment**
	Farming	0.4	0.2	0.3
CONSUMPTION	**Housing**	0.2	0.6	0.4
	Garment	0.4	0.2	0.3

Find the annual prices that each industry must charge for its income to equal its expenditures.

Solution. Let p_1, p_2, and p_3 be the prices charged per year by the farming, housing, and garment industries, respectively, for their total output. To see how these prices are determined, consider the farming industry. It receives p_1 for its production in any year. But it *consumes* products from all these industries in the following amounts (from row 1 of the table): 40% of the food, 20% of the housing, and 30% of the clothing. Hence, the expenditures of the farming industry are $0.4p_1 + 0.2p_2 + 0.3p_3$, so

$$0.4p_1 + 0.2p_2 + 0.3p_3 = p_1$$

A similar analysis of the other two industries leads to the following system of equations.

$$\begin{aligned} 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_1 \\ 0.2p_1 + 0.6p_2 + 0.4p_3 &= p_2 \\ 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_3 \end{aligned}$$

This has the matrix form $E\mathbf{p} = \mathbf{p}$, where

$$E = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.6 & 0.4 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

The equations can be written as the homogeneous system

$$(I - E)\mathbf{p} = \mathbf{0}$$

where I is the 3×3 identity matrix, and the solutions are

$$\mathbf{p} = \begin{bmatrix} 2t \\ 3t \\ 2t \end{bmatrix}$$

where t is a parameter. Thus, the pricing must be such that the total output of the farming industry has the same value as the total output of the garment industry, whereas the total value of the housing industry must be $\frac{3}{2}$ as much.

In general, suppose an economy has n industries, each of which uses some (possibly none) of the production of every industry. We assume first that the economy is **closed** (that is, no product is exported or imported) and that all product is used. Given two industries i and j, let e_{ij} denote the proportion of the total annual output of industry j that is consumed by industry i. Then $E = \left[e_{ij}\right]$ is called the **input-output** matrix for the economy. Clearly,

$$0 \leq e_{ij} \leq 1 \quad \text{for all } i \text{ and } j \tag{2.12}$$

Moreover, all the output from industry j is used by *some* industry (the model is closed), so

$$e_{1j} + e_{2j} + \cdots + e_{ij} = 1 \quad \text{for each } j \tag{2.13}$$

This condition asserts that each column of E sums to 1. Matrices satisfying conditions (2.12) and (2.13) are called **stochastic matrices**.

As in Example 2.8.1, let p_i denote the price of the total annual production of industry i. Then p_i is the annual revenue of industry i. On the other hand, industry i spends $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$ annually for the product it uses ($e_{ij}p_j$ is the cost for product from industry j). The closed economic system is said to be in **equilibrium** if the annual expenditure equals the annual revenue for each industry—that is, if

$$e_{1j}p_1 + e_{2j}p_2 + \cdots + e_{ij}p_n = p_i \quad \text{for each } i = 1,\ 2,\ \dots,\ n$$

If we write $\mathbf{p} = \left[\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array}\right]$, these equations can be written as the matrix equation

$$E\mathbf{p} = \mathbf{p}$$

This is called the **equilibrium condition**, and the solutions $\mathbf{p}$ are called **equilibrium price structures**. The equilibrium condition can be written as

$$(I - E)\mathbf{p} = \mathbf{0}$$

which is a system of homogeneous equations for $\mathbf{p}$. Moreover, there is always a nontrivial solution $\mathbf{p}$. Indeed, the column sums of $I - E$ are all 0 (because E is stochastic), so the row-echelon form of $I - E$ has a row of zeros. In fact, more is true:

Theorem 2.8.1

Let E be any $n \times n$ stochastic matrix. Then there is a nonzero $n \times 1$ vector $\mathbf{p}$ with nonnegative entries such that $E\mathbf{p} = \mathbf{p}$. If all the entries of E are positive, the matrix $\mathbf{p}$ can be chosen with all entries positive.

Theorem 2.8.1 guarantees the existence of an equilibrium price structure for any closed input-output system of the type discussed here. The proof is beyond the scope of this book.[18]

[18] The interested reader is referred to P. Lancaster's *Theory of Matrices* (New York: Academic Press, 1969) or to E. Seneta's *Non-negative Matrices* (New York: Wiley, 1973).

Example 2.8.2

Find the equilibrium price structures for four industries if the input-output matrix is

$$E = \begin{bmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0 \\ 0.1 & 0.3 & 0.5 & 0.2 \\ 0 & 0.1 & 0.2 & 0.7 \end{bmatrix}$$

Find the prices if the total value of business is \$1000.

Solution. If $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ is the equilibrium price structure, then the equilibrium condition reads $E\mathbf{p} = \mathbf{p}$. When we write this as $(I - E)\mathbf{p} = \mathbf{0}$, the methods of Chapter 1 yield the following family of solutions:

$$\mathbf{p} = \begin{bmatrix} 44t \\ 39t \\ 51t \\ 47t \end{bmatrix}$$

where t is a parameter. If we insist that $p_1 + p_2 + p_3 + p_4 = 1000$, then $t = 5.525$. Hence

$$\mathbf{p} = \begin{bmatrix} 243.09 \\ 215.47 \\ 281.76 \\ 259.67 \end{bmatrix}$$

to five figures.

The Open Model

We now assume that there is a demand for products in the **open sector** of the economy, which is the part of the economy other than the producing industries (for example, consumers). Let d_i denote the total value of the demand for product i in the open sector. If p_i and e_{ij} are as before, the value of the annual demand for product i by the producing industries themselves is $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$, so the total annual revenue p_i of industry i breaks down as follows:

$$p_i = (e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n) + d_i \quad \text{for each } i = 1, 2, \dots, n$$

The column $\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$ is called the **demand matrix**, and this gives a matrix equation

$$\mathbf{p} = E\mathbf{p} + \mathbf{d}$$

or

$$(I-E)\mathbf{p}=\mathbf{d} \tag{2.14}$$

This is a system of linear equations for $\mathbf{p}$, and we ask for a solution $\mathbf{p}$ with every entry nonnegative. Note that every entry of E is between 0 and 1, but the column sums of E need not equal 1 as in the closed model.

Before proceeding, it is convenient to introduce a useful notation. If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are matrices of the same size, we write $A>B$ if $a_{ij}>b_{ij}$ for all i and j, and we write $A\geq B$ if $a_{ij}\geq b_{ij}$ for all i and j. Thus $P\geq 0$ means that every entry of P is nonnegative. Note that $A\geq 0$ and $B\geq 0$ implies that $AB\geq 0$.

Now, given a demand matrix $\mathbf{d}\geq\mathbf{0}$, we look for a production matrix $\mathbf{p}\geq\mathbf{0}$ satisfying equation (2.14). This certainly exists if $I-E$ is invertible and $(I-E)^{-1}\geq 0$. On the other hand, the fact that $\mathbf{d}\geq\mathbf{0}$ means any solution $\mathbf{p}$ to equation (2.14) satisfies $\mathbf{p}\geq E\mathbf{p}$. Hence, the following theorem is not too surprising.

Theorem 2.8.2

Let $E\geq 0$ be a square matrix. Then $I-E$ is invertible and $(I-E)^{-1}\geq 0$ if and only if there exists a column $\mathbf{p}>\mathbf{0}$ such that $\mathbf{p}>E\mathbf{p}$.

Heuristic Proof.

If $(I-E)^{-1}\geq 0$, the existence of $\mathbf{p}>\mathbf{0}$ with $\mathbf{p}>E\mathbf{p}$ is left as Exercise 2.8.11. Conversely, suppose such a column $\mathbf{p}$ exists. Observe that

$$(I-E)(I+E+E^2+\cdots+E^{k-1})=I-E^k$$

holds for all $k\geq 2$. If we can show that every entry of E^k approaches 0 as k becomes large then, intuitively, the infinite matrix sum

$$U=I+E+E^2+\cdots$$

exists and $(I-E)U=I$. Since $U\geq 0$, this does it. To show that E^k approaches 0, it suffices to show that $EP<\mu P$ for some number μ with $0<\mu<1$ (then $E^kP<\mu^kP$ for all $k\geq 1$ by induction). The existence of μ is left as Exercise 2.8.12. □

The condition $\mathbf{p}>E\mathbf{p}$ in Theorem 2.8.2 has a simple economic interpretation. If $\mathbf{p}$ is a production matrix, entry i of $E\mathbf{p}$ is the total value of all product used by industry i in a year. Hence, the condition $\mathbf{p}>E\mathbf{p}$ means that, for each i, the value of product produced by industry i exceeds the value of the product it uses. In other words, each industry runs at a profit.

Example 2.8.3

If $E=\left[\begin{array}{ccc}0.6 & 0.2 & 0.3\\ 0.1 & 0.4 & 0.2\\ 0.2 & 0.5 & 0.1\end{array}\right]$, show that $I-E$ is invertible and $(I-E)^{-1}\geq 0$.

Solution. Use $\mathbf{p}=(3,\ 2,\ 2)^T$ in Theorem 2.8.2.

If $\mathbf{p}_0=(1,\ 1,\ 1)^T$, the entries of $E\mathbf{p}_0$ are the row sums of E. Hence $\mathbf{p}_0>E\mathbf{p}_0$ holds if the row sums of E are all less than 1. This proves the first of the following useful facts (the second is Exercise 2.8.10).

Corollary 2.8.1

Let $E \geq 0$ be a square matrix. In each case, $I - E$ is invertible and $(I-E)^{-1} \geq 0$:

1. *All row sums of E are less than 1.*
2. *All column sums of E are less than 1.*

Exercises for 2.8

Exercise 2.8.1 Find the possible equilibrium price structures when the input-output matrices are:

a. $\left[\begin{array}{rrr} 0.1 & 0.2 & 0.3 \\ 0.6 & 0.2 & 0.3 \\ 0.3 & 0.6 & 0.4 \end{array}\right]$ b. $\left[\begin{array}{rrr} 0.5 & 0 & 0.5 \\ 0.1 & 0.9 & 0.2 \\ 0.4 & 0.1 & 0.3 \end{array}\right]$

c. $\left[\begin{array}{rrrr} 0.3 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 & 0 \\ 0.3 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.6 & 0.7 \end{array}\right]$

d. $\left[\begin{array}{rrrr} 0.5 & 0 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0.1 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.1 & 0.1 & 0.6 \end{array}\right]$

Exercise 2.8.2 Three industries A, B, and C are such that all the output of A is used by B, all the output of B is used by C, and all the output of C is used by A. Find the possible equilibrium price structures.

Exercise 2.8.3 Find the possible equilibrium price structures for three industries where the input-output matrix is $\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$. Discuss why there are two parameters here.

Exercise 2.8.4 Prove Theorem 2.8.1 for a 2×2 stochastic matrix E by first writing it in the form $E = \left[\begin{array}{cc} a & b \\ 1-a & 1-b \end{array}\right]$, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

Exercise 2.8.5 If E is an $n \times n$ stochastic matrix and $\mathbf{c}$ is an $n \times 1$ matrix, show that the sum of the entries of $\mathbf{c}$ equals the sum of the entries of the $n \times 1$ matrix $E\mathbf{c}$.

Exercise 2.8.6 Let $W = \left[\begin{array}{ccccc} 1 & 1 & 1 & \cdots & 1 \end{array}\right]$. Let E and F denote $n \times n$ matrices with nonnegative entries.

a. Show that E is a stochastic matrix if and only if $WE = W$.

b. Use part (a.) to deduce that, if E and F are both stochastic matrices, then EF is also stochastic.

Exercise 2.8.7 Find a 2×2 matrix E with entries between 0 and 1 such that:

a. $I - E$ has no inverse.

b. $I - E$ has an inverse but not all entries of $(I-E)^{-1}$ are nonnegative.

Exercise 2.8.8 If E is a 2×2 matrix with entries between 0 and 1, show that $I - E$ is invertible and $(I-E)^{-1} \geq 0$ if and only if $\text{tr } E < 1 + \det E$. Here, if $E = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, then $\text{tr } E = a + d$ and $\det E = ad - bc$.

Exercise 2.8.9 In each case show that $I - E$ is invertible and $(I-E)^{-1} \geq 0$.

a. $\left[\begin{array}{rrr} 0.6 & 0.5 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.4 \end{array}\right]$ b. $\left[\begin{array}{rrr} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.4 \end{array}\right]$

c. $\left[\begin{array}{rrr} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.1 \end{array}\right]$ d. $\left[\begin{array}{rrr} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.1 & 0.2 \\ 0.3 & 0.3 & 0.2 \end{array}\right]$

Exercise 2.8.10 Prove that (1) implies (2) in the Corollary to Theorem 2.8.2.

Exercise 2.8.11 If $(I-E)^{-1} \geq 0$, find $\mathbf{p} > 0$ such that $\mathbf{p} > E\mathbf{p}$.

Exercise 2.8.12 If $E\mathbf{p} < \mathbf{p}$ where $E \geq 0$ and $\mathbf{p} > 0$, find a number μ such that $E\mathbf{p} < \mu\mathbf{p}$ and $0 < \mu < 1$.

[*Hint*: If $E\mathbf{p} = (q_1, \dots, q_n)^T$ and $\mathbf{p} = (p_1, \dots, p_n)^T$, take any number μ where $\max \left\{\frac{q_1}{p_1}, \dots, \frac{q_n}{p_n}\right\} < \mu < 1$.]

2.9 An Application to Markov Chains

Many natural phenomena progress through various stages and can be in a variety of states at each stage. For example, the weather in a given city progresses day by day and, on any given day, may be sunny or rainy. Here the states are "sun" and "rain," and the weather progresses from one state to another in daily stages. Another example might be a football team: The stages of its evolution are the games it plays, and the possible states are "win," "draw," and "loss."

The general setup is as follows: A real conceptual "system" is run generating a sequence of outcomes. The system evolves through a series of "stages," and at any stage it can be in any one of a finite number of "states." At any given stage, the state to which it will go at the next stage depends on the past and present history of the system—that is, on the sequence of states it has occupied to date.

Definition 2.15 Markov Chain

*A **Markov chain** is such an evolving system wherein the state to which it will go next depends only on its present state and does not depend on the earlier history of the system.*[19]

Even in the case of a Markov chain, the state the system will occupy at any stage is determined only in terms of probabilities. In other words, chance plays a role. For example, if a football team wins a particular game, we do not know whether it will win, draw, or lose the next game. On the other hand, we may know that the team tends to persist in winning streaks; for example, if it wins one game it may win the next game $\frac{1}{2}$ of the time, lose $\frac{4}{10}$ of the time, and draw $\frac{1}{10}$ of the time. These fractions are called the **probabilities** of these various possibilities. Similarly, if the team loses, it may lose the next game with probability $\frac{1}{2}$ (that is, half the time), win with probability $\frac{1}{4}$, and draw with probability $\frac{1}{4}$. The probabilities of the various outcomes after a drawn game will also be known.

We shall treat probabilities informally here: *The probability that a given event will occur is the long-run proportion of the time that the event does indeed occur*. Hence, all probabilities are numbers between 0 and 1. A probability of 0 means the event is impossible and never occurs; events with probability 1 are certain to occur.

If a Markov chain is in a particular state, the probabilities that it goes to the various states at the next stage of its evolution are called the **transition probabilities** for the chain, and they are assumed to be known quantities. To motivate the general conditions that follow, consider the following simple example. Here the system is a man, the stages are his successive lunches, and the states are the two restaurants he chooses.

Example 2.9.1

A man always eats lunch at one of two restaurants, A and B. He never eats at A twice in a row. However, if he eats at B, he is three times as likely to eat at B next time as at A. Initially, he is equally likely to eat at either restaurant.

a. What is the probability that he eats at A on the third day after the initial one?

[19] The name honours Andrei Andreyevich Markov (1856–1922) who was a professor at the university in St. Petersburg, Russia.