4.2 Projections and Planes

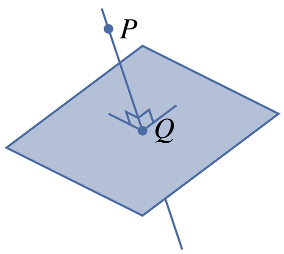


Figure 4.2.1

Any student of geometry soon realizes that the notion of perpendicular lines is fundamental. As an illustration, suppose a point P and a plane are given and it is desired to find the point Q that lies in the plane and is closest to P, as shown in Figure 4.2.1. Clearly, what is required is to find the line through P that is perpendicular to the plane and then to obtain Q as the point of intersection of this line with the plane. Finding the line *perpendicular* to the plane requires a way to determine when two vectors are perpendicular. This can be done using the idea of the dot product of two vectors.

The Dot Product and Angles

Definition 4.4 Dot Product in $\mathbb{R}^3$

Given vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *their* ***dot product*** $\mathbf{v} \cdot \mathbf{w}$ *is a number defined*

$$\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2 + z_1z_2 = \mathbf{v}^T\mathbf{w}$$

Because $\mathbf{v} \cdot \mathbf{w}$ is a number, it is sometimes called the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$.[11]

Example 4.2.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 + (-1) \cdot 4 + 3 \cdot (-1) = -5$.

The next theorem lists several basic properties of the dot product.

Theorem 4.2.1

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

1. $\mathbf{v} \cdot \mathbf{w}$ *is a real number.*

2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

3. $\mathbf{v} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{v}$.

4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

[11] Similarly, if $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, then $\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2$.

5. $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{v}) = \mathbf{v} \cdot (k\mathbf{w})$ *for all scalars* k.

6. $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$

Proof. (1), (2), and (3) are easily verified, and (4) comes from Theorem 4.1.1. The rest are properties of matrix arithmetic (because $\mathbf{w} \cdot \mathbf{v} = \mathbf{v}^T \mathbf{w}$), and are left to the reader. □

The properties in Theorem 4.2.1 enable us to do calculations like

$$3\mathbf{u} \cdot (2\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}) = 6(\mathbf{u} \cdot \mathbf{v}) - 9(\mathbf{u} \cdot \mathbf{w}) + 12(\mathbf{u} \cdot \mathbf{z})$$

and such computations will be used without comment below. Here is an example.

Example 4.2.2

Verify that $\|\mathbf{v} - 3\mathbf{w}\|^2 = 1$ when $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 1$, and $\mathbf{v} \cdot \mathbf{w} = 2$.

Solution. We apply Theorem 4.2.1 several times:

$$\begin{aligned}
\|\mathbf{v} - 3\mathbf{w}\|^2 &= (\mathbf{v} - 3\mathbf{w}) \cdot (\mathbf{v} - 3\mathbf{w}) \\
&= \mathbf{v} \cdot (\mathbf{v} - 3\mathbf{w}) - 3\mathbf{w} \cdot (\mathbf{v} - 3\mathbf{w}) \\
&= \mathbf{v} \cdot \mathbf{v} - 3(\mathbf{v} \cdot \mathbf{w}) - 3(\mathbf{w} \cdot \mathbf{v}) + 9(\mathbf{w} \cdot \mathbf{w}) \\
&= \|\mathbf{v}\|^2 - 6(\mathbf{v} \cdot \mathbf{w}) + 9\|\mathbf{w}\|^2 \\
&= 4 - 12 + 9 = 1
\end{aligned}$$

There is an intrinsic description of the dot product of two nonzero vectors in $\mathbb{R}^3$. To understand it we require the following result from trigonometry.

Law of Cosines

If a triangle has sides a, b, *and* c, *and if* θ *is the interior angle opposite* c *then*

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

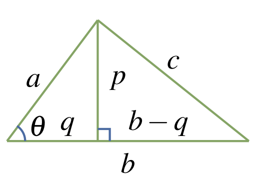


Figure 4.2.2

Proof. We prove it when is θ acute, that is $0 \leq \theta < \frac{\pi}{2}$; the obtuse case is similar. In Figure 4.2.2 we have $p = a\sin\theta$ and $q = a\cos\theta$. Hence Pythagoras' theorem gives

$$\begin{aligned}
c^2 = p^2 + (b - q)^2 &= a^2 \sin^2\theta + (b - a\cos\theta)^2 \\
&= a^2(\sin^2\theta + \cos^2\theta) + b^2 - 2ab\cos\theta
\end{aligned}$$

The law of cosines follows because $\sin^2\theta + \cos^2\theta = 1$ for any angle θ. □

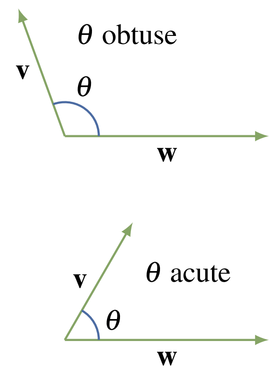


Figure 4.2.3

Note that the law of cosines reduces to Pythagoras' theorem if θ is a right angle (because $\cos\frac{\pi}{2} = 0$).

Now let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors positioned with a common tail as in Figure 4.2.3. Then they determine a unique angle θ in the range

$$0 \leq \theta \leq \pi$$

This angle θ will be called the **angle between** $\mathbf{v}$ and $\mathbf{w}$. Figure 4.2.3 illustrates when θ is acute (less than $\frac{\pi}{2}$) and obtuse (greater than $\frac{\pi}{2}$). Clearly $\mathbf{v}$ and $\mathbf{w}$ are parallel if θ is either 0 or π. Note that we do not define the angle between $\mathbf{v}$ and $\mathbf{w}$ if one of these vectors is $\mathbf{0}$.

The next result gives an easy way to compute the angle between two nonzero vectors using the dot product.

Theorem 4.2.2

Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

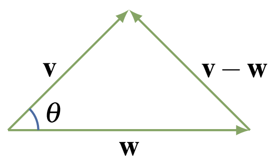


Figure 4.2.4

Proof. We calculate $\|\mathbf{v}-\mathbf{w}\|^2$ in two ways. First apply the law of cosines to the triangle in Figure 4.2.4 to obtain:

$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

On the other hand, we use Theorem 4.2.1:

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}-\mathbf{w}) \cdot (\mathbf{v}-\mathbf{w}) \\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} \\ &= \|\mathbf{v}\|^2 - 2(\mathbf{v}\cdot\mathbf{w}) + \|\mathbf{w}\|^2\end{aligned}$$

Comparing these we see that $-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = -2(\mathbf{v}\cdot\mathbf{w})$, and the result follows. □

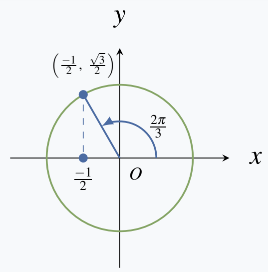
If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, Theorem 4.2.2 gives an intrinsic description of $\mathbf{v}\cdot\mathbf{w}$ because $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and the angle θ between $\mathbf{v}$ and $\mathbf{w}$ do not depend on the choice of coordinate system. Moreover, since $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ are nonzero ($\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors), it gives a formula for the cosine of the angle θ:

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \tag{4.1}$$

Since $0 \leq \theta \leq \pi$, this can be used to find θ.

Example 4.2.3

Compute the angle between $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.



Solution. Compute $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-2+1-2}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}$. Now recall that $\cos\theta$ and $\sin\theta$ are defined so that $(\cos\theta, \sin\theta)$ is the point on the unit circle determined by the angle θ (drawn counterclockwise, starting from the positive x axis). In the present case, we know that $\cos\theta = -\frac{1}{2}$ and that $0 \leq \theta \leq \pi$. Because $\cos\frac{\pi}{3} = \frac{1}{2}$, it follows that $\theta = \frac{2\pi}{3}$ (see the diagram).

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, equation (4.1) shows that $\cos\theta$ has the same sign as $\mathbf{v}\cdot\mathbf{w}$, so

$$\begin{array}{lll} \mathbf{v}\cdot\mathbf{w} > 0 & \text{if and only if} & \theta \text{ is acute } (0 \leq \theta < \frac{\pi}{2}) \\ \mathbf{v}\cdot\mathbf{w} < 0 & \text{if and only if} & \theta \text{ is obtuse } (\frac{\pi}{2} < \theta \leq 0) \\ \mathbf{v}\cdot\mathbf{w} = 0 & \text{if and only if} & \theta = \frac{\pi}{2} \end{array}$$

In this last case, the (nonzero) vectors are perpendicular. The following terminology is used in linear algebra:

Definition 4.5 Orthogonal Vectors in $\mathbb{R}^3$

*Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **orthogonal** if $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or the angle between them is $\frac{\pi}{2}$.*

Since $\mathbf{v}\cdot\mathbf{w} = 0$ if either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, we have the following theorem:

Theorem 4.2.3

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

Example 4.2.4

Show that the points $P(3, -1, 1)$, $Q(4, 1, 4)$, and $R(6, 0, 4)$ are the vertices of a right triangle.

Solution. The vectors along the sides of the triangle are

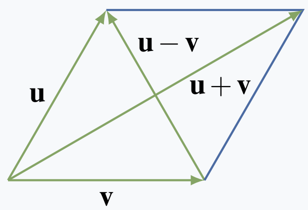
$$\overrightarrow{PQ} = \left[\begin{array}{r} 1 \\ 2 \\ 3 \end{array}\right], \quad \overrightarrow{PR} = \left[\begin{array}{r} 3 \\ 1 \\ 3 \end{array}\right], \quad \text{and } \overrightarrow{QR} = \left[\begin{array}{r} 2 \\ -1 \\ 0 \end{array}\right]$$

Evidently $\overrightarrow{PQ}\cdot\overrightarrow{QR} = 2 - 2 + 0 = 0$, so $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are orthogonal vectors. This means sides PQ and QR are perpendicular—that is, the angle at Q is a right angle.

Example 4.2.5 demonstrates how the dot product can be used to verify geometrical theorems involving perpendicular lines.

Example 4.2.5

A parallelogram with sides of equal length is called a **rhombus**. Show that the diagonals of a rhombus are perpendicular.



Solution. Let $\mathbf{u}$ and $\mathbf{v}$ denote vectors along two adjacent sides of a rhombus, as shown in the diagram. Then the diagonals are $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$, and we compute

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v})-\mathbf{v}\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2-\|\mathbf{v}\|^2\\ &= 0\end{aligned}$$

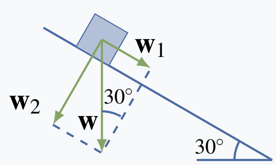
because $\|\mathbf{u}\| = \|\mathbf{v}\|$ (it is a rhombus). Hence $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are orthogonal.

Projections

In applications of vectors, it is frequently useful to write a vector as the sum of two orthogonal vectors. Here is an example.

Example 4.2.6

Suppose a ten-kilogram block is placed on a flat surface inclined 30° to the horizontal as in the diagram. Neglecting friction, how much force is required to keep the block from sliding down the surface?



Solution. Let $\mathbf{w}$ denote the weight (force due to gravity) exerted on the block. Then $\|\mathbf{w}\| = 10$ kilograms and the direction of $\mathbf{w}$ is vertically down as in the diagram. The idea is to write $\mathbf{w}$ as a sum $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is parallel to the inclined surface and $\mathbf{w}_2$ is perpendicular to the surface. Since there is no friction, the force required is $-\mathbf{w}_1$ because the force $\mathbf{w}_2$ has no effect parallel to the surface. As the angle between $\mathbf{w}$ and $\mathbf{w}_2$ is 30° in the diagram, we have $\frac{\|\mathbf{w}_1\|}{\|\mathbf{w}\|} = \sin 30^\circ = \frac{1}{2}$. Hence $\|\mathbf{w}_1\| = \frac{1}{2}\|\mathbf{w}\| = \frac{1}{2}10 = 5$. Thus the required force has a magnitude of 5 kilograms weight directed up the surface.

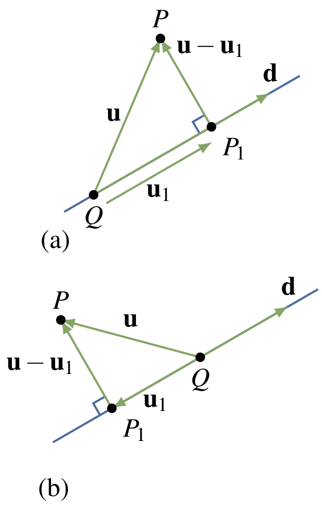


Figure 4.2.5

If a nonzero vector $\mathbf{d}$ is specified, the key idea in Example 4.2.6 is to be able to write an arbitrary vector $\mathbf{u}$ as a sum of two vectors,

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ is orthogonal to $\mathbf{d}$. Suppose that $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ emanate from a common tail Q (see Figure 4.2.5). Let P be the tip of $\mathbf{u}$, and let P_1 denote the foot of the perpendicular from P to the line through Q parallel to $\mathbf{d}$.

Then $\mathbf{u}_1 = \overrightarrow{QP}_1$ has the required properties:

1. $\mathbf{u}_1$ *is parallel to* $\mathbf{d}$.
2. $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ *is orthogonal to* $\mathbf{d}$.
3. $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

Definition 4.6 Projection in $\mathbb{R}^3$

*The vector $\mathbf{u}_1 = \overrightarrow{QP}_1$ in Figure 4.2.5 is called **the projection** of $\mathbf{u}$ on $\mathbf{d}$. It is denoted*

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$$

In Figure 4.2.5(a) the vector $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ has the same direction as $\mathbf{d}$; however, $\mathbf{u}_1$ and $\mathbf{d}$ have opposite directions if the angle between $\mathbf{u}$ and $\mathbf{d}$ is greater than $\frac{\pi}{2}$ (Figure 4.2.5(b)). Note that the projection $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ is zero if and only if $\mathbf{u}$ and $\mathbf{d}$ are orthogonal.

Calculating the projection of $\mathbf{u}$ on $\mathbf{d} \neq \mathbf{0}$ is remarkably easy.

Theorem 4.2.4

Let $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ be vectors.

1. *The projection of $\mathbf{u}$ on $\mathbf{d}$ is given by $\text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$.*
2. *The vector $\mathbf{u} - \text{proj}_{\mathbf{d}}\, \mathbf{u}$ is orthogonal to $\mathbf{d}$.*

Proof. The vector $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ is parallel to $\mathbf{d}$ and so has the form $\mathbf{u}_1 = t\mathbf{d}$ for some scalar t. The requirement that $\mathbf{u} - \mathbf{u}_1$ and $\mathbf{d}$ are orthogonal determines t. In fact, it means that $(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{d} = 0$ by Theorem 4.2.3. If $\mathbf{u}_1 = t\mathbf{d}$ is substituted here, the condition is

$$0 = (\mathbf{u} - t\mathbf{d}) \cdot \mathbf{d} = \mathbf{u} \cdot \mathbf{d} - t(\mathbf{d} \cdot \mathbf{d}) = \mathbf{u} \cdot \mathbf{d} - t\|\mathbf{d}\|^2$$

It follows that $t = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$, where the assumption that $\mathbf{d} \neq \mathbf{0}$ guarantees that $\|\mathbf{d}\|^2 \neq 0$. □

Example 4.2.7

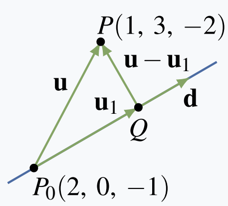
Find the projection of $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ on $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and express $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{d}$.

Solution. The projection $\mathbf{u}_1$ of $\mathbf{u}$ on $\mathbf{d}$ is

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{2+3+3}{1^2+(-1)^2+3^2}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{8}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Hence $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \frac{1}{11}\begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix}$, and this is orthogonal to $\mathbf{d}$ by Theorem 4.2.4 (alternatively, observe that $\mathbf{d}\cdot\mathbf{u}_2 = 0$). Since $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, we are done.

Example 4.2.8



Find the shortest distance (see diagram) from the point $P(1, 3, -2)$ to the line through $P_0(2, 0, -1)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Also find the point Q that lies on the line and is closest to P.

Solution. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ denote the vector from P_0 to P, and let $\mathbf{u}_1$ denote the projection of $\mathbf{u}$ on $\mathbf{d}$. Thus

$$\mathbf{u}_1 = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{-1-3+0}{1^2+(-1)^2+0^2}\mathbf{d} = -2\mathbf{d} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

by Theorem 4.2.4. We see geometrically that the point Q on the line is closest to P, so the distance is

$$\|\overrightarrow{QP}\| = \|\mathbf{u} - \mathbf{u}_1\| = \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{3}$$

To find the coordinates of Q, let $\mathbf{p}_0$ and $\mathbf{q}$ denote the vectors of P_0 and Q, respectively. Then $\mathbf{p}_0 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \mathbf{p}_0 + \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Hence $Q(0, 2, -1)$ is the required point. It can be checked that the distance from Q to P is $\sqrt{3}$, as expected.

Planes

It is evident geometrically that among all planes that are perpendicular to a given straight line there is exactly one containing any given point. This fact can be used to give a very simple description of a plane. To do this, it is necessary to introduce the following notion:

Definition 4.7 Normal Vector in a Plane

A nonzero vector $\mathbf{n}$ *is called a* ***normal*** *for a plane if it is orthogonal to every vector in the plane.*

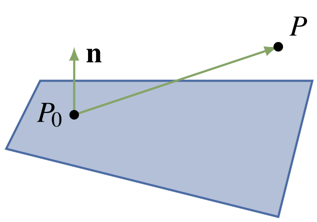


Figure 4.2.6

For example, the coordinate vector $\mathbf{k}$ is a normal for the x-y plane.

Given a point $P_0 = P_0(x_0, y_0, z_0)$ and a nonzero vector $\mathbf{n}$, there is a unique plane through P_0 with normal $\mathbf{n}$, shaded in Figure 4.2.6. A point $P = P(x, y, z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$—that is, if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. Because $\overrightarrow{P_0P} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$ this gives the following result:

Scalar Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ *with normal* $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *as a normal vector is given by*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In other words, a point $P(x, y, z)$ *is on this plane if and only if* x, y, *and* z *satisfy this equation.*

Example 4.2.9

Find an equation of the plane through $P_0(1, -1, 3)$ with $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ as normal.

Solution. Here the general scalar equation becomes

$$3(x - 1) - (y + 1) + 2(z - 3) = 0$$

This simplifies to $3x - y + 2z = 10$.

If we write $d = ax_0 + by_0 + cz_0$, the scalar equation shows that every plane with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a linear equation of the form

$$ax+by+cz=d \tag{4.2}$$

for some constant d. Conversely, the graph of this equation is a plane with $\mathbf{n}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a normal vector (assuming that a, b, and c are not all zero).

Example 4.2.10

Find an equation of the plane through $P_0(3, -1, 2)$ that is parallel to the plane with equation $2x-3y=6$.

Solution. The plane with equation $2x-3y=6$ has normal $\mathbf{n}=\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$. Because the two planes are parallel, $\mathbf{n}$ serves as a normal for the plane we seek, so the equation is $2x-3y=d$ for some d by Equation 4.2. Insisting that $P_0(3, -1, 2)$ lies on the plane determines d; that is, $d=2\cdot 3-3(-1)=9$. Hence, the equation is $2x-3y=9$.

Consider points $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$ with vectors $\mathbf{p}_0=\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{p}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Given a nonzero vector $\mathbf{n}$, the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal $\mathbf{n}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ takes the vector form:

Vector Equation of a Plane

The plane with normal $\mathbf{n}\neq\mathbf{0}$ through the point with vector $\mathbf{p}_0$ is given by

$$\mathbf{n}\cdot(\mathbf{p}-\mathbf{p}_0)=0$$

In other words, the point with vector $\mathbf{p}$ is on the plane if and only if $\mathbf{p}$ satisfies this condition.

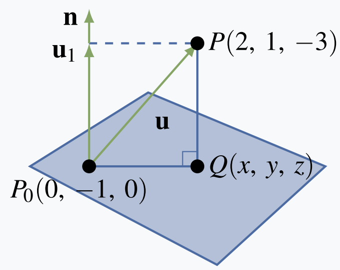
Moreover, Equation 4.2 translates as follows:

Every plane with normal $\mathbf{n}$ has vector equation $\mathbf{n}\cdot\mathbf{p}=d$ for some number d.

This is useful in the second solution of Example 4.2.11.

Example 4.2.11

Find the shortest distance from the point $P(2, 1, -3)$ to the plane with equation $3x-y+4z=1$. Also find the point Q on this plane closest to P.



Solution 1. The plane in question has normal $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$.

Choose any point P_0 on the plane—say $P_0(0, -1, 0)$—and let $Q(x, y, z)$ be the point on the plane closest to P (see the diagram). The vector from P_0 to P is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. Now erect $\mathbf{n}$ with its tail at P_0. Then $\overrightarrow{QP} = \mathbf{u}_1$ and $\mathbf{u}_1$ is the projection of $\mathbf{u}$ on $\mathbf{n}$:

$$\mathbf{u}_1 = \frac{\mathbf{n}\cdot\mathbf{u}}{\|\mathbf{n}\|^2}\mathbf{n} = \frac{-8}{26}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \frac{-4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

Hence the distance is $\|\overrightarrow{QP}\| = \|\mathbf{u}_1\| = \frac{4\sqrt{26}}{13}$. To calculate the point Q, let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p}_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ be the vectors of Q and P_0. Then

$$\mathbf{q} = \mathbf{p}_0 + \mathbf{u} - \mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{38}{13} \\ \frac{9}{13} \\ \frac{-23}{13} \end{bmatrix}$$

This gives the coordinates of $Q(\frac{38}{13}, \frac{9}{13}, \frac{-23}{13})$.

Solution 2. Let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ be the vectors of Q and P. Then Q is on the line through P with direction vector $\mathbf{n}$, so $\mathbf{q} = \mathbf{p} + t\mathbf{n}$ for some scalar t. In addition, Q lies on the plane, so $\mathbf{n}\cdot\mathbf{q} = 1$. This determines t:

$$1 = \mathbf{n}\cdot\mathbf{q} = \mathbf{n}\cdot(\mathbf{p} + t\mathbf{n}) = \mathbf{n}\cdot\mathbf{p} + t\|\mathbf{n}\|^2 = -7 + t(26)$$

This gives $t = \frac{8}{26} = \frac{4}{13}$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{q} = \mathbf{p} + t\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \frac{1}{13}\begin{bmatrix} 38 \\ 9 \\ -23 \end{bmatrix}$$

as before. This determines Q (in the diagram), and the reader can verify that the required distance is $\|\overrightarrow{QP}\| = \frac{4}{13}\sqrt{26}$, as before.

The Cross Product

If P, Q, and R are three distinct points in $\mathbb{R}^3$ that are not all on some line, it is clear geometrically that there is a unique plane containing all three. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ both lie in this plane, so finding a normal amounts to finding a nonzero vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The cross product provides a systematic way to do this.

Definition 4.8 Cross Product

Given vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *define the* ***cross product*** $\mathbf{v}_1 \times \mathbf{v}_2$ *by*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1z_2 - z_1y_2 \\ -(x_1z_2 - z_1x_2) \\ x_1y_2 - y_1x_2 \end{bmatrix}$$

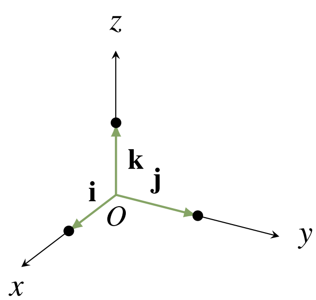


Figure 4.2.7

(Because it is a vector, $\mathbf{v}_1 \times \mathbf{v}_2$ is often called the **vector product**.) There is an easy way to remember this definition using the **coordinate vectors**:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

They are vectors of length 1 pointing along the positive x, y, and z axes, respectively, as in Figure 4.2.7. The reason for the name is that any vector can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

With this, the cross product can be described as follows:

Determinant Form of the Cross Product

If $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *are two vectors, then*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \boldsymbol{i} & x_1 & x_2 \\ \boldsymbol{j} & y_1 & y_2 \\ \boldsymbol{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \boldsymbol{k}$$

where the determinant is expanded along the first column.

Example 4.2.12

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, then

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & -1 & 3 \\ \mathbf{k} & 4 & 7 \end{bmatrix} &= \begin{vmatrix} -1 & 3 \\ 4 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{k} \\ &= -19\mathbf{i} - 10\mathbf{j} + 7\mathbf{k} \\ &= \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix} \end{aligned}$$

Observe that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ in Example 4.2.12. This holds in general as can be verified directly by computing $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})$, and is recorded as the first part of the following theorem. It will follow from a more general result which, together with the second part, will be proved in Section 4.3 where a more detailed study of the cross product will be undertaken.

Theorem 4.2.5

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

1. *$\mathbf{v} \times \mathbf{w}$ is a vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.*
2. *If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.*

It is interesting to contrast Theorem 4.2.5(2) with the assertion (in Theorem 4.2.3) that

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{if and only if } \mathbf{v} \text{ and } \mathbf{w} \text{ are orthogonal.}$$

Example 4.2.13

Find the equation of the plane through $P(1, 3, -2)$, $Q(1, 1, 5)$, and $R(2, -2, 3)$.

Solution. The vectors $\overrightarrow{PQ} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ lie in the plane, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \mathbf{i} & 0 & 1 \\ \mathbf{j} & -2 & -5 \\ \mathbf{k} & 7 & 5 \end{bmatrix} = 25\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$$

is a normal for the plane (being orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$). Hence the plane has equation

$$25x + 7y + 2z = d \quad \text{for some number } d.$$

Since $P(1, 3, -2)$ lies in the plane we have $25 \cdot 1 + 7 \cdot 3 + 2(-2) = d$. Hence $d = 42$ and the equation is $25x + 7y + 2z = 42$. Incidentally, the same equation is obtained (verify) if $\overrightarrow{QP}$ and $\overrightarrow{QR}$, or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$, are used as the vectors in the plane.

Example 4.2.14

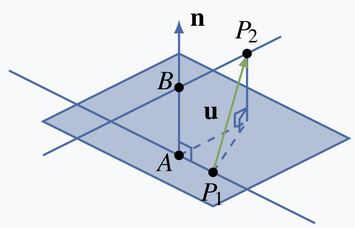
Find the shortest distance between the nonparallel lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Then find the points A and B on the lines that are closest together.

Solution. Direction vectors for the two lines are $\mathbf{d}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, so

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & 0 & 1 \\ \mathbf{k} & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$



is perpendicular to both lines. Consider the plane shaded in the diagram containing the first line with $\mathbf{n}$ as normal. This plane contains $P_1(1, 0, -1)$ and is parallel to the second line. Because $P_2(3, 1, 0)$ is on the second line, the distance in question is just the shortest distance between $P_2(3, 1, 0)$ and this plane. The vector $\mathbf{u}$ from P_1 to P_2 is $\mathbf{u} = \overrightarrow{P_1P_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and so, as in Example 4.2.11, the distance is the length of the projection of $\mathbf{u}$ on $\mathbf{n}$.

$$\text{distance} = \left\| \frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \right\| = \frac{|\mathbf{u} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

Note that it is necessary that $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$ be nonzero for this calculation to be possible. As is shown later (Theorem 4.3.4), this is guaranteed by the fact that $\mathbf{d}_1$ and $\mathbf{d}_2$ are *not* parallel.
The points A and B have coordinates $A(1 + 2t, 0, t - 1)$ and $B(3 + s, 1 + s, -s)$ for some s and t, so $\overrightarrow{AB} = \begin{bmatrix} 2 + s - 2t \\ 1 + s \\ 1 - s - t \end{bmatrix}$. This vector is orthogonal to both $\mathbf{d}_1$ and $\mathbf{d}_2$, and the conditions $\overrightarrow{AB} \cdot \mathbf{d}_1 = 0$ and $\overrightarrow{AB} \cdot \mathbf{d}_2 = 0$ give equations $5t - s = 5$ and $t - 3s = 2$. The solution is $s = \frac{-5}{14}$ and $t = \frac{13}{14}$, so the points are $A(\frac{40}{14}, 0, \frac{-1}{14})$ and $B(\frac{37}{14}, \frac{9}{14}, \frac{5}{14})$. We have $\|\overrightarrow{AB}\| = \frac{3\sqrt{14}}{14}$, as before.

Exercises for 4.2

Exercise 4.2.1 Compute $\mathbf{u} \cdot \mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \mathbf{u}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -7 \\ -5 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \mathbf{v} = \mathbf{0}$

Exercise 4.2.2 Find the angle between the following pairs of vectors.

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5\sqrt{2} \\ -7 \\ -1 \end{bmatrix}$

Exercise 4.2.3 Find all real numbers x such that:

a. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

b. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$.

Exercise 4.2.4 Find all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both:

a. $\mathbf{u}_1 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

c. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

d. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Exercise 4.2.5 Find two orthogonal vectors that are both orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Exercise 4.2.6 Consider the triangle with vertices $P(2, 0, -3)$, $Q(5, -2, 1)$, and $R(7, 5, 3)$.

a. Show that it is a right-angled triangle.

b. Find the lengths of the three sides and verify the Pythagorean theorem.

Exercise 4.2.7 Show that the triangle with vertices $A(4, -7, 9)$, $B(6, 4, 4)$, and $C(7, 10, -6)$ is not a right-angled triangle.

Exercise 4.2.8 Find the three internal angles of the triangle with vertices:

a. $A(3, 1, -2)$, $B(3, 0, -1)$, and $C(5, 2, -1)$

b. $A(3, 1, -2)$, $B(5, 2, -1)$, and $C(4, 3, -3)$

Exercise 4.2.9 Show that the line through $P_0(3, 1, 4)$ and $P_1(2, 1, 3)$ is perpendicular to the line through $P_2(1, -1, 2)$ and $P_3(0, 5, 3)$.

Exercise 4.2.10 In each case, compute the projection of $\mathbf{u}$ on $\mathbf{v}$.

a. $\mathbf{u} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

Exercise 4.2.11 In each case, write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ -1 \end{bmatrix}$

Exercise 4.2.12 Calculate the distance from the point P to the line in each case and find the point Q on the line closest to P.

a. $P(3, 2-1)$

line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

b. $P(1, -1, 3)$

line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

Exercise 4.2.13 Compute $\mathbf{u} \times \mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

Exercise 4.2.14 Find an equation of each of the following planes.

a. Passing through $A(2, 1, 3)$, $B(3, -1, 5)$, and $C(1, 2, -3)$.

b. Passing through $A(1, -1, 6)$, $B(0, 0, 1)$, and $C(4, 7, -11)$.

c. Passing through $P(2, -3, 5)$ and parallel to the plane with equation $3x - 2y - z = 0$.

d. Passing through $P(3, 0, -1)$ and parallel to the plane with equation $2x - y + z = 3$.

e. Containing $P(3, 0, -1)$ and the line
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

f. Containing $P(2, 1, 0)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

g. Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

h. Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

i. Each point of which is equidistant from $P(2, -1, 3)$ and $Q(1, 1, -1)$.

j. Each point of which is equidistant from $P(0, 1, -1)$ and $Q(2, -1, -3)$.

Exercise 4.2.15 In each case, find a vector equation of the line.

a. Passing through $P(3, -1, 4)$ and perpendicular to the plane $3x - 2y - z = 0$.

b. Passing through $P(2, -1, 3)$ and perpendicular to the plane $2x + y = 1$.

c. Passing through $P(0, 0, 0)$ and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

d. Passing through $P(1, 1, -1)$, and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

e. Passing through $P(2, 1, -1)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and perpendicular to that line.

f. Passing through $P(1, 1, 2)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and perpendicular to that line.

Exercise 4.2.16 In each case, find the shortest distance from the point P to the plane and find the point Q on the plane closest to P.

a. $P(2, 3, 0)$; plane with equation $5x + y + z = 1$.

b. $P(3, 1, -1)$; plane with equation $2x + y - z = 6$.

Exercise 4.2.17

a. Does the line through $P(1, 2, -3)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ lie in the plane $2x - y - z = 3$? Explain.

b. Does the plane through $P(4, 0, 5)$, $Q(2, 2, 1)$, and $R(1, -1, 2)$ pass through the origin? Explain.

Exercise 4.2.18 Show that every plane containing $P(1, 2, -1)$ and $Q(2, 0, 1)$ must also contain $R(-1, 6, -5)$.

Exercise 4.2.19 Find the equations of the line of intersection of the following planes.

a. $2x - 3y + 2z = 5$ and $x + 2y - z = 4$.

b. $3x + y - 2z = 1$ and $x + y + z = 5$.

Exercise 4.2.20 In each case, find all points of intersection of the given plane and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

a. $x - 3y + 2z = 4$

b. $2x - y - z = 5$

c. $3x - y + z = 8$

d. $-x - 4y - 3z = 6$

Exercise 4.2.21 Find the equation of *all* planes:

a. Perpendicular to the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

b. Perpendicular to the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

c. Containing the origin.

d. Containing $P(3, 2, -4)$.

e. Containing $P(1, 1, -1)$ and $Q(0, 1, 1)$.

f. Containing $P(2, -1, 1)$ and $Q(1, 0, 0)$.

g. Containing the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

h. Containing the line
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

Exercise 4.2.22 If a plane contains two distinct points P_1 and P_2, show that it contains every point on the line through P_1 and P_2.

Exercise 4.2.23 Find the shortest distance between the following pairs of parallel lines.

a. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

b. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Exercise 4.2.24 Find the shortest distance between the following pairs of nonparallel lines and find the points on the lines that are closest together.

a. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

b. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

c. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

d. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix};$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Exercise 4.2.25 Show that two lines in the plane with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. [*Hint*: Example 4.1.11.]

Exercise 4.2.26

a. Show that, of the four diagonals of a cube, no pair is perpendicular.

b. Show that each diagonal is perpendicular to the face diagonals it does not meet.

Exercise 4.2.27 Given a rectangular solid with sides of lengths 1, 1, and $\sqrt{2}$, find the angle between a diagonal and one of the longest sides.

Exercise 4.2.28 Consider a rectangular solid with sides of lengths a, b, and c. Show that it has two orthogonal diagonals if and only if the sum of two of a^2, b^2, and c^2 equals the third.

Exercise 4.2.29 Let A, B, and $C(2, -1, 1)$ be the vertices of a triangle where $\overrightarrow{AB}$ is parallel to $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\overrightarrow{AC}$ is parallel to $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and angle $C = 90^\circ$. Find the equation of the line through B and C.

Exercise 4.2.30 If the diagonals of a parallelogram have equal length, show that the parallelogram is a rectangle.

Exercise 4.2.31 Given $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in component form, show that the projections of $\mathbf{v}$ on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$, respectively.

Exercise 4.2.32

a. Can $\mathbf{u} \cdot \mathbf{v} = -7$ if $\|\mathbf{u}\| = 3$ and $\|\mathbf{v}\| = 2$? Defend your answer.

b. Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\|\mathbf{v}\| = 6$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

Exercise 4.2.33 Show $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

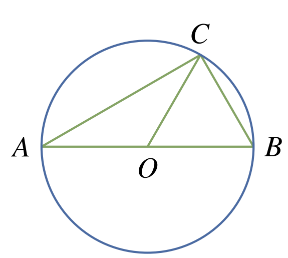
Exercise 4.2.34

a. Show $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

b. What does this say about parallelograms?

Exercise 4.2.35 Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus. [*Hint*: Example 4.2.5.]

Exercise 4.2.36 Let A and B be the end points of a diameter of a circle (see the diagram). If C is any point on the circle, show that AC and BC are perpendicular. [*Hint*: Express $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$ and $\overrightarrow{BC}$ in terms of $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OC}$, where O is the centre.]



Exercise 4.2.37 Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Exercise 4.2.38 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be pairwise orthogonal vectors.

a. Show that $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

b. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all the same length, show that they all make the same angle with $\mathbf{u}+\mathbf{v}+\mathbf{w}$.

Exercise 4.2.39

a. Show that $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is orthogonal to every vector along the line $ax + by + c = 0$.

b. Show that the shortest distance from $P_0(x_0, y_0)$ to the line is $\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

[*Hint*: If P_1 is on the line, project $\mathbf{u} = \overrightarrow{P_1P_0}$ on $\mathbf{n}$.]

Exercise 4.2.40 Assume $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors that are not parallel. Show that $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ is a nonzero vector that bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Exercise 4.2.41 Let α, β, and γ be the angles a vector $\mathbf{v} \neq \mathbf{0}$ makes with the positive x, y, and z axes, respectively. Then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector $\mathbf{v}$.

a. If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that $\cos\alpha = \frac{a}{\|\mathbf{v}\|}$, $\cos\beta = \frac{b}{\|\mathbf{v}\|}$, and $\cos\gamma = \frac{c}{\|\mathbf{v}\|}$.

b. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Exercise 4.2.42 Let $\mathbf{v} \neq \mathbf{0}$ be any nonzero vector and suppose that a vector $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p}$ is parallel to $\mathbf{v}$ and $\mathbf{q}$ is orthogonal to $\mathbf{v}$. Show that $\mathbf{p}$ must equal the projection of $\mathbf{u}$ on $\mathbf{v}$. [*Hint*: Argue as in the proof of Theorem 4.2.4.]

Exercise 4.2.43 Let $\mathbf{v} \neq \mathbf{0}$ be a nonzero vector and let $a \neq 0$ be a scalar. If $\mathbf{u}$ is any vector, show that the projection of $\mathbf{u}$ on $\mathbf{v}$ equals the projection of $\mathbf{u}$ on $a\mathbf{v}$.

Exercise 4.2.44

a. Show that the **Cauchy-Schwarz inequality** $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: $|\cos\theta| \leq 1$ for all angles θ.]

b. Show that $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

[*Hint*: When is $\cos\theta = \pm 1$?]

c. Show that $|x_1x_2 + y_1y_2 + z_1z_2| \leq \sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}$ holds for all numbers x_1, x_2, y_1, y_2, z_1, and z_2.

d. Show that $|xy + yz + zx| \leq x^2 + y^2 + z^2$ for all x, y, and z.

e. Show that $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ holds for all x, y, and z.

Exercise 4.2.45 Prove that the **triangle inequality** $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: Consider the triangle with $\mathbf{u}$ and $\mathbf{v}$ as two sides.]

4.3 More on the Cross Product

The cross product $\mathbf{v} \times \mathbf{w}$ of two $\mathbb{R}^3$-vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ was defined in Section 4.2 where we observed that it can be best remembered using a determinant:

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\mathbf{k} \tag{4.3}$$

Here $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are the coordinate vectors, and the determinant is expanded along the first column. We observed (but did not prove) in Theorem 4.2.5 that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. This follows easily from the next result.

Theorem 4.3.1

If $\mathbf{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *then* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$.

Proof. Recall that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is computed by multiplying corresponding components of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ and then adding. Using equation (4.3), the result is:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = x_0\left(\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}\right) + y_0\left(-\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}\right) + z_0\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$$

where the last determinant is expanded along column 1. □

The result in Theorem 4.3.1 can be succinctly stated as follows: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$$