b. Show that $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

[*Hint*: When is $\cos\theta = \pm 1$?]

c. Show that $|x_1x_2 + y_1y_2 + z_1z_2| \leq \sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}$ holds for all numbers x_1, x_2, y_1, y_2, z_1, and z_2.

d. Show that $|xy+yz+zx| \leq x^2+y^2+z^2$ for all x, y, and z.

e. Show that $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ holds for all x, y, and z.

Exercise 4.2.45 Prove that the **triangle inequality** $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: Consider the triangle with $\mathbf{u}$ and $\mathbf{v}$ as two sides.]

4.3 More on the Cross Product

The cross product $\mathbf{v}\times\mathbf{w}$ of two $\mathbb{R}^3$-vectors $\mathbf{v} = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]$ and $\mathbf{w} = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$ was defined in Section 4.2 where we observed that it can be best remembered using a determinant:

$$\mathbf{v}\times\mathbf{w} = \det\left[\begin{array}{ccc} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{array}\right] = \left|\begin{array}{cc} y_1 & y_2 \\ z_1 & z_2 \end{array}\right|\mathbf{i} - \left|\begin{array}{cc} x_1 & x_2 \\ z_1 & z_2 \end{array}\right|\mathbf{j} + \left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right|\mathbf{k} \tag{4.3}$$

Here $\mathbf{i} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$, $\mathbf{j} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$, and $\mathbf{k} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$ are the coordinate vectors, and the determinant is expanded along the first column. We observed (but did not prove) in Theorem 4.2.5 that $\mathbf{v}\times\mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. This follows easily from the next result.

Theorem 4.3.1

If $\mathbf{u} = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array}\right]$, $\mathbf{v} = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]$, *and* $\mathbf{w} = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$, *then* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \det\left[\begin{array}{ccc} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{array}\right]$.

Proof. Recall that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is computed by multiplying corresponding components of $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$ and then adding. Using equation (4.3), the result is:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = x_0\left(\left|\begin{array}{cc} y_1 & y_2 \\ z_1 & z_2 \end{array}\right|\right) + y_0\left(-\left|\begin{array}{cc} x_1 & x_2 \\ z_1 & z_2 \end{array}\right|\right) + z_0\left(\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right|\right) = \det\left[\begin{array}{ccc} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{array}\right]$$

where the last determinant is expanded along column 1. □

The result in Theorem 4.3.1 can be succinctly stated as follows: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$, then

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \det\left[\begin{array}{ccc} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{array}\right]$$

where $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ denotes the matrix with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns. Now it is clear that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ because the determinant of a matrix is zero if two columns are identical.

Because of (4.3) and Theorem 4.3.1, several of the following properties of the cross product follow from properties of determinants (they can also be verified directly).

Theorem 4.3.2

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ denote arbitrary vectors in $\mathbb{R}^3$.

1. $\mathbf{u} \times \mathbf{v}$ *is a vector.*
2. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$.
3. $\mathbf{u} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{u}$.
4. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
5. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.
6. $(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$ *for any scalar* k.
7. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.
8. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$.

Proof. (1) is clear; (2) follows from Theorem 4.3.1; and (3) and (4) follow because the determinant of a matrix is zero if one column is zero or if two columns are identical. If two columns are interchanged, the determinant changes sign, and this proves (5). The proofs of (6), (7), and (8) are left as Exercise 4.3.15. □

We now come to a fundamental relationship between the dot and cross products.

Theorem 4.3.3: Lagrange Identity[12]

If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in $\mathbb{R}^3$, then

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof. Given $\mathbf{u}$ and $\mathbf{v}$, introduce a coordinate system and write $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ in component form. Then all the terms in the identity can be computed in terms of the components. The detailed proof is left as Exercise 4.3.14. □

An expression for the magnitude of the vector $\mathbf{u} \times \mathbf{v}$ can be easily obtained from the Lagrange identity. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, substituting $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$ into the Lagrange identity gives

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2\theta = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2\theta$$

[12]Joseph Louis Lagrange (1736–1813) was born in Italy and spent his early years in Turin. At the age of 19 he solved a famous problem by inventing an entirely new method, known today as the calculus of variations, and went on to become one of the greatest mathematicians of all time. His work brought a new level of rigour to analysis and his *Mécanique Analytique* is a masterpiece in which he introduced methods still in use. In 1766 he was appointed to the Berlin Academy by Frederik the Great who asserted that the "greatest mathematician in Europe" should be at the court of the "greatest king in Europe." After the death of Frederick, Lagrange went to Paris at the invitation of Louis XVI. He remained there throughout the revolution and was made a count by Napoleon.

using the fact that $1-\cos^2\theta = \sin^2\theta$. But $\sin\theta$ is nonnegative on the range $0 \leq \theta \leq \pi$, so taking the positive square root of both sides gives

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

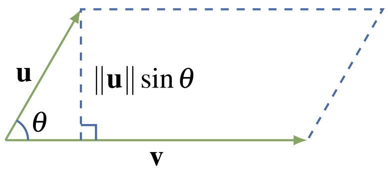


Figure 4.3.1

This expression for $\|\mathbf{u}\times\mathbf{v}\|$ makes no reference to a coordinate system and, moreover, it has a nice geometrical interpretation. The parallelogram determined by the vectors $\mathbf{u}$ and $\mathbf{v}$ has base length $\|\mathbf{v}\|$ and altitude $\|\mathbf{u}\|\sin\theta$ (see Figure 4.3.1). Hence the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is

$$(\|\mathbf{u}\|\sin\theta)\|\mathbf{v}\| = \|\mathbf{u}\times\mathbf{v}\|$$

This proves the first part of Theorem 4.3.4.

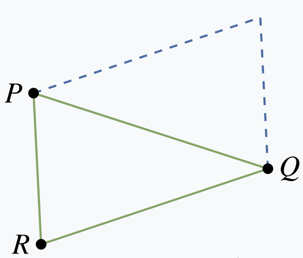
Theorem 4.3.4

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

1. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta =$ *the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.*
2. *$\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.*

Proof of (2). By (1), $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if the area of the parallelogram is zero. By Figure 4.3.1 the area vanishes if and only if $\mathbf{u}$ and $\mathbf{v}$ have the same or opposite direction—that is, if and only if they are parallel. □

Example 4.3.1



Find the area of the triangle with vertices $P(2,\ 1,\ 0)$, $Q(3,\ -1,\ 1)$, and $R(1,\ 0,\ 1)$.

Solution. We have $\overrightarrow{RP} = \left[\begin{array}{r} 1 \\ 1 \\ -1 \end{array}\right]$ and $\overrightarrow{RQ} = \left[\begin{array}{r} 2 \\ -1 \\ 0 \end{array}\right]$. The area of the triangle is half the area of the parallelogram (see the diagram), and so equals $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\|$. We have

$$\overrightarrow{RP}\times\overrightarrow{RQ} = \det\left[\begin{array}{crr} \mathbf{i} & 1 & 2 \\ \mathbf{j} & 1 & -1 \\ \mathbf{k} & -1 & 0 \end{array}\right] = \left[\begin{array}{r} -1 \\ -2 \\ -3 \end{array}\right]$$

so the area of the triangle is $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\| = \frac{1}{2}\sqrt{1+4+9} = \frac{1}{2}\sqrt{14}$.

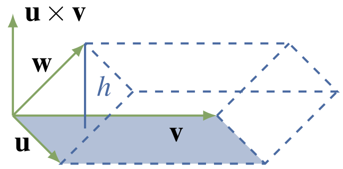


Figure 4.3.2

If three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given, they determine a "squashed" rectangular solid called a **parallelepiped** (Figure 4.3.2), and it is often useful to be able to find the volume of such a solid. The base of the solid is the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, so it has area $A = \|\mathbf{u} \times \mathbf{v}\|$ by Theorem 4.3.4. The height of the solid is the length h of the projection of $\mathbf{w}$ on $\mathbf{u} \times \mathbf{v}$. Hence

$$h = \left| \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})}{\|\mathbf{u} \times \mathbf{v}\|^2} \right| \|\mathbf{u} \times \mathbf{v}\| = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{A}$$

Thus the volume of the parallelepiped is $hA = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$. This proves

Theorem 4.3.5

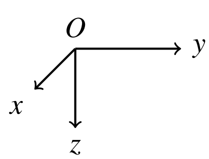
The volume of the parallelepiped determined by three vectors $\mathbf{w}$, $\mathbf{u}$, *and* $\mathbf{v}$ *(Figure 4.3.2) is given by* $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$.

Example 4.3.2

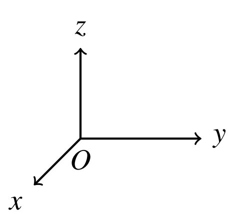
Find the volume of the parallelepiped determined by the vectors

$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Solution. By Theorem 4.3.1, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \det \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = -3$. Hence the volume is $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| = |-3| = 3$ by Theorem 4.3.5.



Left-hand system



Right-hand system

Figure 4.3.3

We can now give an intrinsic description of the cross product $\mathbf{u} \times \mathbf{v}$. Its magnitude $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$ is coordinate-free. If $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, its direction is very nearly determined by the fact that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and so points along the line normal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. It remains only to decide which of the two possible directions is correct.

Before this can be done, the basic issue of how coordinates are assigned must be clarified. When coordinate axes are chosen in space, the procedure is as follows: An origin is selected, two perpendicular lines (the x and y axes) are chosen through the origin, and a positive direction on each of these axes is selected quite arbitrarily. Then the line through the origin normal to this x-y plane is called the z axis, but there is a choice of which direction on this axis is the positive one. The two possibilities are shown in Figure 4.3.3, and it is a standard convention that cartesian coordinates are always **right-hand coordinate systems**. The reason for this terminology is that, in such a system, if the z axis is grasped in the right hand with the thumb pointing in the positive z direction, then the fingers curl around from the positive x axis to the positive y axis (through a right angle).

Suppose now that $\mathbf{u}$ and $\mathbf{v}$ are given and that θ is the angle between them (so $0 \leq \theta \leq \pi$). Then the direction of $\|\mathbf{u} \times \mathbf{v}\|$ is given by the right-hand rule.

Right-hand Rule

If the vector $\mathbf{u} \times \mathbf{v}$ is grasped in the right hand and the fingers curl around from $\mathbf{u}$ to $\mathbf{v}$ through the angle θ, the thumb points in the direction for $\mathbf{u} \times \mathbf{v}$.

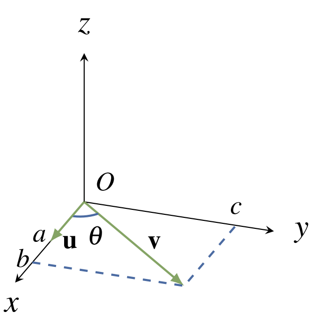


Figure 4.3.4

To indicate why this is true, introduce coordinates in $\mathbb{R}^3$ as follows: Let $\mathbf{u}$ and $\mathbf{v}$ have a common tail O, choose the origin at O, choose the x axis so that $\mathbf{u}$ points in the positive x direction, and then choose the y axis so that $\mathbf{v}$ is in the x-y plane and the positive y axis is on the same side of the x axis as $\mathbf{v}$. Then, in this system, $\mathbf{u}$ and $\mathbf{v}$ have component form $\mathbf{u} = \left[\begin{array}{c} a \\ 0 \\ 0 \end{array}\right]$ and $\mathbf{v} = \left[\begin{array}{c} b \\ c \\ 0 \end{array}\right]$ where $a > 0$ and $c > 0$. The situation is depicted in Figure 4.3.4. The right-hand rule asserts that $\mathbf{u} \times \mathbf{v}$ should point in the positive z direction. But our definition of $\mathbf{u} \times \mathbf{v}$ gives

$$\mathbf{u} \times \mathbf{v} = \det \left[\begin{array}{ccc} \mathbf{i} & a & b \\ \mathbf{j} & 0 & c \\ \mathbf{k} & 0 & 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ ac \end{array}\right] = (ac)\mathbf{k}$$

and $(ac)\mathbf{k}$ has the positive z direction because $ac > 0$.

Exercises for 4.3

Exercise 4.3.1 If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the coordinate vectors, verify that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Exercise 4.3.2 Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ need not equal $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ by calculating both when

$$\mathbf{u} = \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right], \mathbf{v} = \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right], \text{ and } \mathbf{w} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

Exercise 4.3.3 Find two unit vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ if:

a. $\mathbf{u} = \left[\begin{array}{c} 1 \\ 2 \\ 2 \end{array}\right], \mathbf{v} = \left[\begin{array}{r} 2 \\ -1 \\ 2 \end{array}\right]$

b. $\mathbf{u} = \left[\begin{array}{r} 1 \\ 2 \\ -1 \end{array}\right], \mathbf{v} = \left[\begin{array}{c} 3 \\ 1 \\ 2 \end{array}\right]$

Exercise 4.3.4 Find the area of the triangle with the following vertices.

a. $A(3, -1, 2)$, $B(1, 1, 0)$, and $C(1, 2, -1)$

b. $A(3, 0, 1)$, $B(5, 1, 0)$, and $C(7, 2, -1)$

c. $A(1, 1, -1)$, $B(2, 0, 1)$, and $C(1, -1, 3)$

d. $A(3, -1, 1)$, $B(4, 1, 0)$, and $C(2, -3, 0)$

Exercise 4.3.5 Find the volume of the parallelepiped determined by $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ when:

a. $\mathbf{w} = \left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$, $\mathbf{v} = \left[\begin{array}{r} 1 \\ 0 \\ 2 \end{array}\right]$, and $\mathbf{u} = \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right]$

b. $\mathbf{w} = \left[\begin{array}{r} 1 \\ 0 \\ 3 \end{array}\right]$, $\mathbf{v} = \left[\begin{array}{r} 2 \\ 1 \\ -3 \end{array}\right]$, and $\mathbf{u} = \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right]$

Exercise 4.3.6 Let P_0 be a point with vector $\mathbf{p}_0$, and let $ax + by + cz = d$ be the equation of a plane with normal $\mathbf{n} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right]$.

a. Show that the point on the plane closest to P_0 has vector $\mathbf{p}$ given by

$$\mathbf{p} = \mathbf{p}_0 + \tfrac{d-(\mathbf{p}_0\cdot\mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}.$$

[*Hint*: $\mathbf{p} = \mathbf{p}_0 + t\mathbf{n}$ for some t, and $\mathbf{p}\cdot\mathbf{n} = \mathbf{d}$.]

b. Show that the shortest distance from P_0 to the plane is $\frac{|d-(\mathbf{p}_0\cdot\mathbf{n})|}{\|\mathbf{n}\|}$.

c. Let P_0' denote the reflection of P_0 in the plane—that is, the point on the opposite side of the plane such that the line through P_0 and P_0' is perpendicular to the plane.

Show that $\mathbf{p}_0 + 2\frac{d-(\mathbf{p}_0\cdot\mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}$ is the vector of P_0'.

Exercise 4.3.7 Simplify $(a\mathbf{u} + b\mathbf{v}) \times (c\mathbf{u} + d\mathbf{v})$.

Exercise 4.3.8 Show that the shortest distance from a point P to the line through P_0 with direction vector $\mathbf{d}$ is $\frac{\|\overrightarrow{P_0P}\times\mathbf{d}\|}{\|\mathbf{d}\|}$.

Exercise 4.3.9 Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero, nonorthogonal vectors. If θ is the angle between them, show that $\tan\theta = \frac{\|\mathbf{u}\times\mathbf{v}\|}{\mathbf{u}\cdot\mathbf{v}}$.

Exercise 4.3.10 Show that points A, B, and C are all on one line if and only if $\overrightarrow{AB} \times \overrightarrow{AC} = 0$

Exercise 4.3.11 Show that points A, B, C, and D are all on one plane if and only if $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$

Exercise 4.3.12 Use Theorem 4.3.5 to confirm that, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular, the (rectangular) parallelepiped they determine has volume $\|\mathbf{u}\|\|\mathbf{v}\|\|\mathbf{w}\|$.

Exercise 4.3.13 Show that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ is $\|\mathbf{u}\times\mathbf{v}\|^2$.

Exercise 4.3.14 Complete the proof of Theorem 4.3.3.

Exercise 4.3.15 Prove the following properties in Theorem 4.3.2.

a. Property 6

b. Property 7

c. Property 8

Exercise 4.3.16

a. Show that $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\times(\mathbf{w}\times\mathbf{u})$ holds for all vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$.

b. Show that $\mathbf{v} - \mathbf{w}$ and $(\mathbf{u}\times\mathbf{v}) + (\mathbf{v}\times\mathbf{w}) + (\mathbf{w}\times\mathbf{u})$ are orthogonal.

Exercise 4.3.17 Show $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\times\mathbf{v})\mathbf{w}$. [*Hint*: First do it for $\mathbf{u} = \mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; then write $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and use Theorem 4.3.2.]

Exercise 4.3.18 Prove the **Jacobi identity**:

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$$

[*Hint*: The preceding exercise.]

Exercise 4.3.19 Show that

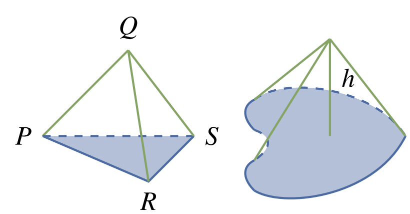
$$(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{z}) = \det\left[\begin{array}{cc} \mathbf{u}\cdot\mathbf{w} & \mathbf{u}\cdot\mathbf{z} \\ \mathbf{v}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{z} \end{array}\right]$$

[*Hint*: Exercises 4.3.16 and 4.3.17.]

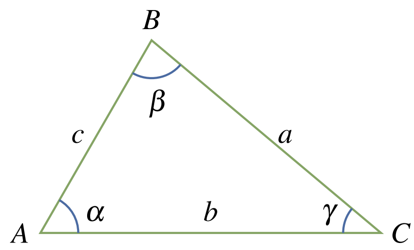
Exercise 4.3.20 Let P, Q, R, and S be four points, not all on one plane, as in the diagram. Show that the volume of the pyramid they determine is

$$\tfrac{1}{6}|\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})|.$$

[*Hint*: The volume of a cone with base area A and height h as in the diagram below right is $\frac{1}{3}Ah$.]



Exercise 4.3.21 Consider a triangle with vertices A, B, and C, as in the diagram below. Let α, β, and γ denote the angles at A, B, and C, respectively, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Write $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{CA}$.



a. Deduce that $\mathbf{u}+\mathbf{v}+\mathbf{w} = \mathbf{0}$.

b. Show that $\mathbf{u}\times\mathbf{v} = \mathbf{w}\times\mathbf{u} = \mathbf{v}\times\mathbf{w}$. [*Hint*: Compute $\mathbf{u}\times(\mathbf{u}+\mathbf{v}+\mathbf{w})$ and $\mathbf{v}\times(\mathbf{u}+\mathbf{v}+\mathbf{w})$.]

c. Deduce the **law of sines**:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Exercise 4.3.22 Show that the (shortest) distance between two planes $\mathbf{n}\cdot\mathbf{p} = d_1$ and $\mathbf{n}\cdot\mathbf{p} = d_2$ with $\mathbf{n}$ as normal is $\frac{|d_2-d_1|}{\|\mathbf{n}\|}$.

Exercise 4.3.23 Let A and B be points other than the origin, and let $\mathbf{a}$ and $\mathbf{b}$ be their vectors. If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, show that the plane through A, B, and the origin is given by

$$\left\{P(x,\ y,\ z) \mid \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = s\mathbf{a}+t\mathbf{b} \text{ for some } s \text{ and } t\right\}$$

Exercise 4.3.24 Let A be a 2×3 matrix of rank 2 with rows $\mathbf{r}_1$ and $\mathbf{r}_2$. Show that

$$P = \{XA \mid X = [xy]; x,\ y \text{ arbitrary}\}$$

is the plane through the origin with normal $\mathbf{r}_1\times\mathbf{r}_2$.

Exercise 4.3.25 Given the cube with vertices $P(x,\ y,\ z)$, where each of x, y, and z is either 0 or 2, consider the plane perpendicular to the diagonal through $P(0,\ 0,\ 0)$ and $P(2,\ 2,\ 2)$ and bisecting it.

a. Show that the plane meets six of the edges of the cube and bisects them.

b. Show that the six points in (a) are the vertices of a regular hexagon.

4.4 Linear Operators on $\mathbb{R}^3$

Recall that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ holds for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. In this case we showed (in Theorem 2.6.2) that there exists an $m\times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and we say that T is the **matrix transformation induced** by A.

Definition 4.9 Linear Operator on $\mathbb{R}^n$

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

*is called a **linear operator** on $\mathbb{R}^n$.*

In Section 2.6 we investigated three important linear operators on $\mathbb{R}^2$: rotations about the origin, reflections in a line through the origin, and projections on this line.

In this section we investigate the analogous operators on $\mathbb{R}^3$: Rotations about a line through the origin, reflections in a plane through the origin, and projections onto a plane or line through the origin in $\mathbb{R}^3$. In every case we show that the operator is linear, and we find the matrices of all the reflections and projections.