5. Vector Space $\mathbb{R}^n$

5.1 Subspaces and Spanning

In Section 2.2 we introduced the set $\mathbb{R}^n$ of all n-tuples (called *vectors*), and began our investigation of the matrix transformations $\mathbb{R}^n \to \mathbb{R}^m$ given by matrix multiplication by an $m \times n$ matrix. Particular attention was paid to the euclidean plane $\mathbb{R}^2$ where certain simple geometric transformations were seen to be matrix transformations. Then in Section 2.6 we introduced linear transformations, showed that they are all matrix transformations, and found the matrices of rotations and reflections in $\mathbb{R}^2$. We returned to this in Section 4.4 where we showed that projections, reflections, and rotations of $\mathbb{R}^2$ and $\mathbb{R}^3$ were all linear, and where we related areas and volumes to determinants.

In this chapter we investigate $\mathbb{R}^n$ in full generality, and introduce some of the most important concepts and methods in linear algebra. The n-tuples in $\mathbb{R}^n$ will continue to be denoted $\mathbf{x}$, $\mathbf{y}$, and so on, and will be written as rows or columns depending on the context.

Subspaces of $\mathbb{R}^n$

Definition 5.1 Subspace of $\mathbb{R}^n$

A set[1] U *of vectors in* $\mathbb{R}^n$ *is called a* **subspace** *of* $\mathbb{R}^n$ *if it satisfies the following properties:*

S1. The zero vector $\mathbf{0} \in U$.

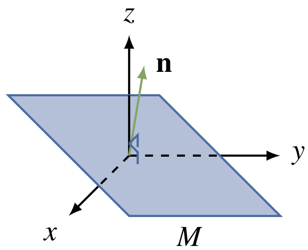
S2. If $\mathbf{x} \in U$ *and* $\mathbf{y} \in U$, *then* $\mathbf{x} + \mathbf{y} \in U$.

S3. If $\mathbf{x} \in U$, *then* $a\mathbf{x} \in U$ *for every real number* a.

We say that the subset U is **closed under addition** if S2 holds, and that U is **closed under scalar multiplication** if S3 holds.

Clearly $\mathbb{R}^n$ is a subspace of itself, and this chapter is about these subspaces and their properties. The set $U = \{\mathbf{0}\}$, consisting of only the zero vector, is also a subspace because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for each a in $\mathbb{R}$; it is called the **zero subspace**. Any subspace of $\mathbb{R}^n$ other than $\{\mathbf{0}\}$ or $\mathbb{R}^n$ is called a **proper** subspace.

[1] We use the language of sets. Informally, a **set** X is a collection of objects, called the **elements** of the set. The fact that x is an element of X is denoted $x \in X$. Two sets X and Y are called equal (written $X = Y$) if they have the same elements. If every element of X is in the set Y, we say that X is a **subset** of Y, and write $X \subseteq Y$. Hence $X \subseteq Y$ and $Y \subseteq X$ both hold if and only if $X = Y$.



We saw in Section 4.2 that every plane M through the origin in $\mathbb{R}^3$ has equation $ax+by+cz=0$ where a, b, and c are not all zero. Here $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal for the plane and

$$M = \{\mathbf{v} \text{ in } \mathbb{R}^3 \mid \mathbf{n}\cdot\mathbf{v} = 0\}$$

where $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{n}\cdot\mathbf{v}$ denotes the dot product introduced in Section 2.2 (see the diagram).[2] Then M is a subspace of $\mathbb{R}^3$. Indeed we show that M satisfies S1, S2, and S3 as follows:

S1. $\mathbf{0} \in M$ *because* $\mathbf{n}\cdot\mathbf{0} = 0$*;*

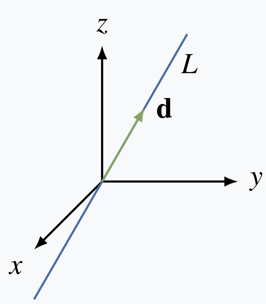
S2. If $\mathbf{v} \in M$ *and* $\mathbf{v}_1 \in M$ *, then* $\mathbf{n}\cdot(\mathbf{v}+\mathbf{v}_1) = \mathbf{n}\cdot\mathbf{v}+\mathbf{n}\cdot\mathbf{v}_1 = 0+0=0$ *, so* $\mathbf{v}+\mathbf{v}_1 \in M$*;*

S3. If $\mathbf{v} \in M$ *, then* $\mathbf{n}\cdot(a\mathbf{v}) = a(\mathbf{n}\cdot\mathbf{v}) = a(0) = 0$ *, so* $a\mathbf{v} \in M$*.*

This proves the first part of

Example 5.1.1

Planes and lines through the origin in $\mathbb{R}^3$ are all subspaces of $\mathbb{R}^3$.



Solution. We dealt with planes above. If L is a line through the origin with direction vector $\mathbf{d}$, then $L = \{t\mathbf{d} \mid t \in \mathbb{R}\}$ (see the diagram). We leave it as an exercise to verify that L satisfies S1, S2, and S3.

Example 5.1.1 shows that lines through the origin in $\mathbb{R}^2$ are subspaces; in fact, they are the *only* proper subspaces of $\mathbb{R}^2$ (Exercise 5.1.24). Indeed, we shall see in Example 5.2.14 that lines and planes through the origin in $\mathbb{R}^3$ are the only proper subspaces of $\mathbb{R}^3$. Thus the geometry of lines and planes through the origin is captured by the subspace concept. (Note that *every* line or plane is just a translation of one of these.)

Subspaces can also be used to describe important features of an $m \times n$ matrix A. The **null space** of A, denoted $\operatorname{null} A$, and the **image space** of A, denoted $\operatorname{im} A$, are defined by

$$\operatorname{null} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \quad \text{and} \quad \operatorname{im} A = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$$

In the language of Chapter 2, $\operatorname{null} A$ consists of all solutions $\mathbf{x}$ in $\mathbb{R}^n$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, and $\operatorname{im} A$ is the set of all vectors $\mathbf{y}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{y}$ *has* a solution $\mathbf{x}$. Note that $\mathbf{x}$ is in $\operatorname{null} A$ if it

[2] We are using set notation here. In general $\{q \mid p\}$ means the set of all objects q with property p.

satisfies the *condition* $A\mathbf{x} = \mathbf{0}$, while $\operatorname{im} A$ consists of vectors of the *form* $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. These two ways to describe subsets occur frequently.

Example 5.1.2

If A is an $m \times n$ matrix, then:

1. $\operatorname{null} A$ is a subspace of $\mathbb{R}^n$.
2. $\operatorname{im} A$ is a subspace of $\mathbb{R}^m$.

Solution.

1. The zero vector $\mathbf{0} \in \mathbb{R}^n$ lies in $\operatorname{null} A$ because $A\mathbf{0} = \mathbf{0}$.[3] If $\mathbf{x}$ and $\mathbf{x}_1$ are in $\operatorname{null} A$, then $\mathbf{x} + \mathbf{x}_1$ and $a\mathbf{x}$ are in $\operatorname{null} A$ because they satisfy the required condition:

$$A(\mathbf{x}+\mathbf{x}_1) = A\mathbf{x} + A\mathbf{x}_1 = \mathbf{0}+\mathbf{0} = \mathbf{0} \quad \text{and} \quad A(a\mathbf{x}) = a(A\mathbf{x}) = a\mathbf{0} = \mathbf{0}$$

   Hence $\operatorname{null} A$ satisfies S1, S2, and S3, and so is a subspace of $\mathbb{R}^n$.

2. The zero vector $\mathbf{0} \in \mathbb{R}^m$ lies in $\operatorname{im} A$ because $\mathbf{0} = A\mathbf{0}$. Suppose that $\mathbf{y}$ and $\mathbf{y}_1$ are in $\operatorname{im} A$, say $\mathbf{y} = A\mathbf{x}$ and $\mathbf{y}_1 = A\mathbf{x}_1$ where $\mathbf{x}$ and $\mathbf{x}_1$ are in $\mathbb{R}^n$. Then

$$\mathbf{y}+\mathbf{y}_1 = A\mathbf{x} + A\mathbf{x}_1 = A(\mathbf{x}+\mathbf{x}_1) \quad \text{and} \quad a\mathbf{y} = a(A\mathbf{x}) = A(a\mathbf{x})$$

   show that $\mathbf{y}+\mathbf{y}_1$ and $a\mathbf{y}$ are both in $\operatorname{im} A$ (they have the required form). Hence $\operatorname{im} A$ is a subspace of $\mathbb{R}^m$.

There are other important subspaces associated with a matrix A that clarify basic properties of A. If A is an $n \times n$ matrix and λ is any number, let

$$E_\lambda(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

A vector $\mathbf{x}$ is in $E_\lambda(A)$ if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$, so Example 5.1.2 gives:

Example 5.1.3

$E_\lambda(A) = \operatorname{null}(\lambda I - A)$ is a subspace of $\mathbb{R}^n$ for each $n \times n$ matrix A and number λ.

$E_\lambda(A)$ is called the **eigenspace** of A corresponding to λ. The reason for the name is that, in the terminology of Section 3.3, λ is an **eigenvalue** of A if $E_\lambda(A) \neq \{\mathbf{0}\}$. In this case the nonzero vectors in $E_\lambda(A)$ are called the **eigenvectors** of A corresponding to λ.

The reader should not get the impression that *every* subset of $\mathbb{R}^n$ is a subspace. For example:

$$U_1 = \left\{ \left[\begin{array}{c} x \\ y \end{array} \right] \middle|\, x \geq 0 \right\} \text{ satisfies S1 and S2, but not S3;}$$

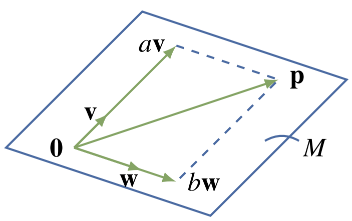
[3] We are using $\mathbf{0}$ to represent the zero vector in both $\mathbb{R}^m$ and $\mathbb{R}^n$. This abuse of notation is common and causes no confusion once everybody knows what is going on.

$$U_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x^2 = y^2 \right\} \text{ satisfies S1 and S3, but not S2;}$$

Hence neither U_1 nor U_2 is a subspace of $\mathbb{R}^2$. (However, see Exercise 5.1.20.)

Spanning Sets

Let $\mathbf{v}$ and $\mathbf{w}$ be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. The plane M through the origin containing these vectors is described in Section 4.2 by saying that $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is a *normal* for M, and that M consists of all vectors $\mathbf{p}$ such that $\mathbf{n} \cdot \mathbf{p} = 0$.[4] While this is a very useful way to look at planes, there is another approach that is at least as useful in $\mathbb{R}^3$ and, more importantly, works for all subspaces of $\mathbb{R}^n$ for any $n \geq 1$.



The idea is as follows: Observe that, by the diagram, a vector $\mathbf{p}$ is in M if and only if it has the form

$$\mathbf{p} = a\mathbf{v} + b\mathbf{w}$$

for certain real numbers a and b (we say that $\mathbf{p}$ is a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$). Hence we can describe M as

$$M = \{a\mathbf{x} + b\mathbf{w} \mid a,\ b \in \mathbb{R}\}.^5$$

and we say that $\{\mathbf{v},\ \mathbf{w}\}$ is a *spanning set* for M. It is this notion of a spanning set that provides a way to describe all subspaces of $\mathbb{R}^n$.

As in Section 1.3, given vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ in $\mathbb{R}^n$, a vector of the form

$$t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \quad \text{where the } t_i \text{ are scalars}$$

is called a **linear combination** of the $\mathbf{x}_i$, and t_i is called the **coefficient** of $\mathbf{x}_i$ in the linear combination.

Definition 5.2 Linear Combinations and Span in $\mathbb{R}^n$

The set of all such linear combinations is called the ***span*** *of the* $\mathbf{x}_i$ *and is denoted*

$$\operatorname{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\} = \{t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \mid t_i \text{ in } \mathbb{R}\}$$

If $V = \operatorname{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$, *we say that* V *is* ***spanned*** *by the vectors* $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$, *and that the vectors* $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ ***span*** *the space* V.

Here are two examples:

$$\operatorname{span}\{\mathbf{x}\} = \{t\mathbf{x} \mid t \in \mathbb{R}\}$$

which we write as $\operatorname{span}\{\mathbf{x}\} = \mathbb{R}\mathbf{x}$ for simplicity.

$$\operatorname{span}\{\mathbf{x},\ \mathbf{y}\} = \{r\mathbf{x} + s\mathbf{y} \mid r,\ s \in \mathbb{R}\}$$

[4] The vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is nonzero because $\mathbf{v}$ and $\mathbf{w}$ are not parallel.

[5] In particular, this implies that any vector $\mathbf{p}$ orthogonal to $\mathbf{v} \times \mathbf{w}$ must be a linear combination $\mathbf{p} = a\mathbf{v} + b\mathbf{w}$ of $\mathbf{v}$ and $\mathbf{w}$ for some a and b. Can you prove this directly?

In particular, the above discussion shows that, if $\mathbf{v}$ and $\mathbf{w}$ are two nonzero, nonparallel vectors in $\mathbb{R}^3$, then

$$M = \text{span}\,\{\mathbf{v},\ \mathbf{w}\}$$

is the plane in $\mathbb{R}^3$ containing $\mathbf{v}$ and $\mathbf{w}$. Moreover, if $\mathbf{d}$ is any nonzero vector in $\mathbb{R}^3$ (or $\mathbb{R}^2$), then

$$L = \text{span}\,\{\mathbf{v}\} = \{t\mathbf{d} \mid t \in \mathbb{R}\} = \mathbb{R}\mathbf{d}$$

is the line with direction vector $\mathbf{d}$. Hence lines and planes can both be described in terms of spanning sets.

Example 5.1.4

Let $\mathbf{x} = (2,\ -1,\ 2,\ 1)$ and $\mathbf{y} = (3,\ 4,\ -1,\ 1)$ in $\mathbb{R}^4$. Determine whether $\mathbf{p} = (0,\ -11,\ 8,\ 1)$ or $\mathbf{q} = (2,\ 3,\ 1,\ 2)$ are in $U = \text{span}\,\{\mathbf{x},\ \mathbf{y}\}$.

Solution. The vector $\mathbf{p}$ is in U if and only if $\mathbf{p} = s\mathbf{x} + t\mathbf{y}$ for scalars s and t. Equating components gives equations

$$2s + 3t = 0, \quad -s + 4t = -11, \quad 2s - t = 8, \quad \text{and} \quad s + t = 1$$

This linear system has solution $s = 3$ and $t = -2$, so $\mathbf{p}$ is in U. On the other hand, asking that $\mathbf{q} = s\mathbf{x} + t\mathbf{y}$ leads to equations

$$2s + 3t = 2, \quad -s + 4t = 3, \quad 2s - t = 1, \quad \text{and} \quad s + t = 2$$

and this system has *no* solution. So $\mathbf{q}$ does *not* lie in U.

Theorem 5.1.1: Span Theorem

Let $U = \text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ *in* $\mathbb{R}^n$. *Then:*

1. U *is a subspace of* $\mathbb{R}^n$ *containing each* $\mathbf{x}_i$.
2. *If* W *is a subspace of* $\mathbb{R}^n$ *and each* $\mathbf{x}_i \in W$, *then* $U \subseteq W$.

Proof.

1. The zero vector $\mathbf{0}$ is in U because $\mathbf{0} = 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k$ is a linear combination of the $\mathbf{x}_i$. If $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ and $\mathbf{y} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$ are in U, then $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ are in U because

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (t_1 + s_1)\mathbf{x}_1 + (t_2 + s_2)\mathbf{x}_2 + \cdots + (t_k + s_k)\mathbf{x}_k, \quad \text{and} \\ a\mathbf{x} &= (at_1)\mathbf{x}_1 + (at_2)\mathbf{x}_2 + \cdots + (at_k)\mathbf{x}_k \end{aligned}$$

   Finally each $\mathbf{x}_i$ is in U (for example, $\mathbf{x}_2 = 0\mathbf{x}_1 + 1\mathbf{x}_2 + \cdots + 0\mathbf{x}_k$) so S1, S2, and S3 are satisfied for U, proving (1).

2. Let $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ where the t_i are scalars and each $\mathbf{x}_i \in W$. Then each $t_i\mathbf{x}_i \in W$ because W satisfies S3. But then $\mathbf{x} \in W$ because W satisfies S2 (verify). This proves (2). □

Condition (2) in Theorem 5.1.1 can be expressed by saying that $\text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is the *smallest* subspace of $\mathbb{R}^n$ that contains each $\mathbf{x}_i$. This is useful for showing that two subspaces U and W are equal, since this amounts to showing that both $U \subseteq W$ and $W \subseteq U$. Here is an example of how it is used.

Example 5.1.5

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, show that $\text{span}\,\{\mathbf{x},\ \mathbf{y}\} = \text{span}\,\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$.

Solution. Since both $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are in $\text{span}\,\{\mathbf{x},\ \mathbf{y}\}$, Theorem 5.1.1 gives

$$\text{span}\,\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\} \subseteq \text{span}\,\{\mathbf{x},\ \mathbf{y}\}$$

But $\mathbf{x} = \frac{1}{2}(\mathbf{x}+\mathbf{y}) + \frac{1}{2}(\mathbf{x}-\mathbf{y})$ and $\mathbf{y} = \frac{1}{2}(\mathbf{x}+\mathbf{y}) - \frac{1}{2}(\mathbf{x}-\mathbf{y})$ are both in $\text{span}\,\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$, so

$$\text{span}\,\{\mathbf{x},\ \mathbf{y}\} \subseteq \text{span}\,\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$$

again by Theorem 5.1.1. Thus $\text{span}\,\{\mathbf{x},\ \mathbf{y}\} = \text{span}\,\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$, as desired.

It turns out that many important subspaces are best described by giving a spanning set. Here are three examples, beginning with an important spanning set for $\mathbb{R}^n$ itself. Column j of the $n \times n$ identity matrix I_n is denoted $\mathbf{e}_j$ and called the jth **coordinate vector** in $\mathbb{R}^n$, and the set $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ is called the **standard basis** of $\mathbb{R}^n$. If $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ is any vector in $\mathbb{R}^n$, then $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$, as the reader can verify. This proves:

Example 5.1.6

$\mathbb{R}^n = \text{span}\,\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ where $\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n$ are the columns of I_n.

If A is an $m \times n$ matrix A, the next two examples show that it is a routine matter to find spanning sets for $\text{null}\,A$ and $\text{im}\,A$.

Example 5.1.7

Given an $m \times n$ matrix A, let $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ denote the basic solutions to the system $A\mathbf{x} = \mathbf{0}$ given by the gaussian algorithm. Then

$$\text{null}\,A = \text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$$

Solution. If $\mathbf{x} \in \text{null}\,A$, then $A\mathbf{x} = \mathbf{0}$ so Theorem 1.3.2 shows that $\mathbf{x}$ is a linear combination of the basic solutions; that is, $\text{null}\,A \subseteq \text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$. On the other hand, if $\mathbf{x}$ is in $\text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$, then $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ for scalars t_i, so

$$A\mathbf{x} = t_1A\mathbf{x}_1 + t_2A\mathbf{x}_2 + \cdots + t_kA\mathbf{x}_k = t_1\mathbf{0} + t_2\mathbf{0} + \cdots + t_k\mathbf{0} = \mathbf{0}$$

This shows that $\mathbf{x} \in \text{null}\,A$, and hence that $\text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\} \subseteq \text{null}\,A$. Thus we have equality.

Example 5.1.8

Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ denote the columns of the $m \times n$ matrix A. Then

$$\text{im}\, A = \text{span}\,\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$$

Solution. If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, observe that

$$\left[\begin{array}{cccc} A\mathbf{e}_1 & A\mathbf{e}_2 & \cdots & A\mathbf{e}_n \end{array}\right] = A\left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = AI_n = A = \left[\begin{array}{ccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots \mathbf{c}_n \end{array}\right].$$

Hence $\mathbf{c}_i = A\mathbf{e}_i$ is in $\text{im}\, A$ for each i, so $\text{span}\,\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \subseteq \text{im}\, A$.

Conversely, let $\mathbf{y}$ be in $\text{im}\, A$, say $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$, then Definition 2.5 gives

$$\mathbf{y} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n \text{ is in } \text{span}\,\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$$

This shows that $\text{im}\, A \subseteq \text{span}\,\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$, and the result follows.

Exercises for 5.1

We often write vectors in $\mathbb{R}^n$ as rows.

Exercise 5.1.1 In each case determine whether U is a subspace of $\mathbb{R}^3$. Support your answer.

a. $U = \{(1, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.

b. $U = \{(0, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.

c. $U = \{(r, s, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}, -r + 3s + 2t = 0\}$.

d. $U = \{(r, 3s, r-2) \mid r \text{ and } s \text{ in } \mathbb{R}\}$.

e. $U = \{(r, 0, s) \mid r^2 + s^2 = 0, r \text{ and } s \text{ in } \mathbb{R}\}$.

f. $U = \{(2r, -s^2, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}\}$.

Exercise 5.1.2 In each case determine if $\mathbf{x}$ lies in $U = \text{span}\,\{\mathbf{y}, \mathbf{z}\}$. If $\mathbf{x}$ is in U, write it as a linear combination of $\mathbf{y}$ and $\mathbf{z}$; if $\mathbf{x}$ is not in U, show why not.

a. $\mathbf{x} = (2, -1, 0, 1)$, $\mathbf{y} = (1, 0, 0, 1)$, and $\mathbf{z} = (0, 1, 0, 1)$.

b. $\mathbf{x} = (1, 2, 15, 11)$, $\mathbf{y} = (2, -1, 0, 2)$, and $\mathbf{z} = (1, -1, -3, 1)$.

c. $\mathbf{x} = (8, 3, -13, 20)$, $\mathbf{y} = (2, 1, -3, 5)$, and $\mathbf{z} = (-1, 0, 2, -3)$.

d. $\mathbf{x} = (2, 5, 8, 3)$, $\mathbf{y} = (2, -1, 0, 5)$, and $\mathbf{z} = (-1, 2, 2, -3)$.

Exercise 5.1.3 In each case determine if the given vectors span $\mathbb{R}^4$. Support your answer.

a. $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$.

b. $\{(1, 3, -5, 0), (-2, 1, 0, 0), (0, 2, 1, -1), (1, -4, 5, 0)\}$.

Exercise 5.1.4 Is it possible that $\{(1, 2, 0), (2, 0, 3)\}$ can span the subspace $U = \{(r, s, 0) \mid r \text{ and } s \text{ in } \mathbb{R}\}$? Defend your answer.

Exercise 5.1.5 Give a spanning set for the zero subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$.

Exercise 5.1.6 Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Defend your answer.

Exercise 5.1.7 If $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x}+t\mathbf{z}, \mathbf{y}, \mathbf{z}\}$ for every t in $\mathbb{R}$.

Exercise 5.1.8 If $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{z}+\mathbf{x}\}$.

Exercise 5.1.9 If $a \neq 0$ is a scalar, show that $\text{span}\{a\mathbf{x}\} = \text{span}\{\mathbf{x}\}$ for every vector $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 5.1.10 If $a_1, a_2, \dots, a_k$ are nonzero scalars, show that $\text{span}\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ for any vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

Exercise 5.1.11 If $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, determine all subspaces of $\text{span}\{\mathbf{x}\}$.

Exercise 5.1.12 Suppose that $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ where each $\mathbf{x}_i$ is in $\mathbb{R}^n$. If A is an $m \times n$ matrix and $A\mathbf{x}_i = \mathbf{0}$ for each i, show that $A\mathbf{y} = \mathbf{0}$ for every vector $\mathbf{y}$ in U.

Exercise 5.1.13 If A is an $m \times n$ matrix, show that, for each invertible $m \times m$ matrix U, $\text{null}(A) = \text{null}(UA)$.

Exercise 5.1.14 If A is an $m \times n$ matrix, show that, for each invertible $n \times n$ matrix V, $\text{im}(A) = \text{im}(AV)$.

Exercise 5.1.15 Let U be a subspace of $\mathbb{R}^n$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$.

a. If $a\mathbf{x}$ is in U where $a \neq 0$ is a number, show that $\mathbf{x}$ is in U.

b. If $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$ are in U where $\mathbf{y}$ is a vector in $\mathbb{R}^n$, show that $\mathbf{x}$ is in U.

Exercise 5.1.16 In each case either show that the statement is true or give an example showing that it is false.

a. If $U \neq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and $\mathbf{x}+\mathbf{y}$ is in U, then $\mathbf{x}$ and $\mathbf{y}$ are both in U.

b. If U is a subspace of $\mathbb{R}^n$ and $r\mathbf{x}$ is in U for all r in $\mathbb{R}$, then $\mathbf{x}$ is in U.

c. If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in U, then $-\mathbf{x}$ is also in U.

d. If $\mathbf{x}$ is in U and $U = \text{span}\{\mathbf{y}, \mathbf{z}\}$, then $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.

e. The empty set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

f. $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ is in span $\left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 2 \\ 0 \end{array}\right]\right\}$.

Exercise 5.1.17

a. If A and B are $m \times n$ matrices, show that $U = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = B\mathbf{x}\}$ is a subspace of $\mathbb{R}^n$.

b. What if A is $m \times n$, B is $k \times n$, and $m \neq k$?

Exercise 5.1.18 Suppose that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are vectors in $\mathbb{R}^n$. If $\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ where $a_1 \neq 0$, show that $\text{span}\{\mathbf{x}_1\, \mathbf{x}_2, \dots, \mathbf{x}_k\} = \text{span}\{\mathbf{y}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$.

Exercise 5.1.19 If $U \neq \{\mathbf{0}\}$ is a subspace of $\mathbb{R}$, show that $U = \mathbb{R}$.

Exercise 5.1.20 Let U be a nonempty subset of $\mathbb{R}^n$. Show that U is a subspace if and only if S2 and S3 hold.

Exercise 5.1.21 If S and T are nonempty sets of vectors in $\mathbb{R}^n$, and if $S \subseteq T$, show that $\text{span}\{S\} \subseteq \text{span}\{T\}$.

Exercise 5.1.22 Let U and W be subspaces of $\mathbb{R}^n$. Define their **intersection** $U \cap W$ and their **sum** $U + W$ as follows:

$U \cap W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ belongs to both } U \text{ and } W\}$.

$U + W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is a sum of a vector in } U$ and a vector in $W\}$.

a. Show that $U \cap W$ is a subspace of $\mathbb{R}^n$.

b. Show that $U + W$ is a subspace of $\mathbb{R}^n$.

Exercise 5.1.23 Let P denote an invertible $n \times n$ matrix. If λ is a number, show that

$$E_\lambda(PAP^{-1}) = \{P\mathbf{x} \mid \mathbf{x} \text{ is in } E_\lambda(A)\}$$

for each $n \times n$ matrix A.

Exercise 5.1.24 Show that every proper subspace U of $\mathbb{R}^2$ is a line through the origin. [*Hint*: If $\mathbf{d}$ is a nonzero vector in U, let $L = \mathbb{R}\mathbf{d} = \{r\mathbf{d} \mid r \text{ in } \mathbb{R}\}$ denote the line with direction vector $\mathbf{d}$. If $\mathbf{u}$ is in U but not in L, argue geometrically that every vector $\mathbf{v}$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{d}$.]

5.2 Independence and Dimension

Some spanning sets are better than others. If $U = \text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is a subspace of $\mathbb{R}^n$, then every vector in U can be written as a linear combination of the $\mathbf{x}_i$ in at least one way. Our interest here is in spanning sets where each vector in U has a *exactly one* representation as a linear combination of these vectors.

Linear Independence

Given $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ in $\mathbb{R}^n$, suppose that two linear combinations are equal:

$$r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$$

We are looking for a condition on the set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ of vectors that guarantees that this representation is *unique*; that is, $r_i = s_i$ for each i. Taking all terms to the left side gives

$$(r_1 - s_1)\mathbf{x}_1 + (r_2 - s_2)\mathbf{x}_2 + \cdots + (r_k - s_k)\mathbf{x}_k = \mathbf{0}$$

so the required condition is that this equation forces all the coefficients $r_i - s_i$ to be zero.

Definition 5.3 Linear Independence in $\mathbb{R}^n$

With this in mind, we call a set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ of vectors ***linearly independent*** *(or simply* ***independent****) if it satisfies the following condition:*

$$\text{If } t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0} \text{ then } t_1 = t_2 = \cdots = t_k = 0$$

We record the result of the above discussion for reference.

Theorem 5.2.1

If $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is an independent set of vectors in $\mathbb{R}^n$, then every vector in span $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ *has a* ***unique*** *representation as a linear combination of the $\mathbf{x}_i$.*

It is useful to state the definition of independence in different language. Let us say that a linear combination **vanishes** if it equals the zero vector, and call a linear combination **trivial** if every coefficient is zero. Then the definition of independence can be compactly stated as follows:

> A set of vectors is independent if and only if the only linear combination that vanishes is the trivial one.

Hence we have a procedure for checking that a set of vectors is independent: