5.4 Rank of a Matrix

In this section we use the concept of dimension to clarify the definition of the rank of a matrix given in Section 1.2, and to study its properties. This requires that we deal with rows and columns in the same way. While it has been our custom to write the n-tuples in $\mathbb{R}^n$ as columns, in this section we will frequently write them as rows. Subspaces, independence, spanning, and dimension are defined for rows using matrix operations, just as for columns. If A is an $m \times n$ matrix, we define:

Definition 5.10 Column and Row Space of a Matrix

The ***column space****,* $\text{col}\, A$*, of* A *is the subspace of* $\mathbb{R}^m$ *spanned by the columns of* A*.*
The ***row space****,* $\text{row}\, A$*, of* A *is the subspace of* $\mathbb{R}^n$ *spanned by the rows of* A*.*

Much of what we do in this section involves these subspaces. We begin with:

Lemma 5.4.1

Let A *and* B *denote* $m \times n$ *matrices.*

1. *If* $A \to B$ *by elementary row operations, then* $\text{row}\, A = \text{row}\, B$*.*
2. *If* $A \to B$ *by elementary column operations, then* $\text{col}\, A = \text{col}\, B$*.*

Proof. We prove (1); the proof of (2) is analogous. It is enough to do it in the case when $A \to B$ by a single row operation. Let $R_1, R_2, \dots, R_m$ denote the rows of A. The row operation $A \to B$ either interchanges two rows, multiplies a row by a nonzero constant, or adds a multiple of a row to a different row. We leave the first two cases to the reader. In the last case, suppose that a times row p is added to row q where $p < q$. Then the rows of B are $R_1, \dots, R_p, \dots, R_q + aR_p, \dots, R_m$, and Theorem 5.1.1 shows that

$$\text{span}\,\{R_1, \dots, R_p, \dots, R_q, \dots, R_m\} = \text{span}\,\{R_1, \dots, R_p, \dots, R_q + aR_p, \dots, R_m\}$$

That is, $\text{row}\, A = \text{row}\, B$. □

If A is any matrix, we can carry $A \to R$ by elementary row operations where R is a row-echelon matrix. Hence $\text{row}\, A = \text{row}\, R$ by Lemma 5.4.1; so the first part of the following result is of interest.

Lemma 5.4.2

If R *is a row-echelon matrix, then*

1. *The nonzero rows of* R *are a basis of* $\text{row}\, R$*.*
2. *The columns of* R *containing leading ones are a basis of* $\text{col}\, R$*.*

Proof. The rows of R are independent by Example 5.2.6, and they span $\text{row}\, R$ by definition. This proves (1).

Let $\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}$ denote the columns of R containing leading 1s. Then $\{\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}\}$ is independent because the leading 1s are in different rows (and have zeros below and to the left of them). Let U denote the subspace of all columns in $\mathbb{R}^m$ in which the last $m-r$ entries are zero. Then $\dim U = r$ (it is just $\mathbb{R}^r$ with extra zeros). Hence the independent set $\{\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}\}$ is a basis of U by Theorem 5.2.7. Since each $\mathbf{c}_{j_i}$ is in $\operatorname{col} R$, it follows that $\operatorname{col} R = U$, proving (2). □

With Lemma 5.4.2 we can fill a gap in the definition of the rank of a matrix given in Chapter 1. Let A be any matrix and suppose A is carried to some row-echelon matrix R by row operations. Note that R is not unique. In Section 1.2 we defined the **rank** of A, denoted $\operatorname{rank} A$, to be the number of leading 1s in R, that is the number of nonzero rows of R. The fact that this number does not depend on the choice of R was not proved in Section 1.2. However part 1 of Lemma 5.4.2 shows that

$$\operatorname{rank} A = \dim(\operatorname{row} A)$$

and hence that $\operatorname{rank} A$ is independent of R.

Lemma 5.4.2 can be used to find bases of subspaces of $\mathbb{R}^n$ (written as rows). Here is an example.

Example 5.4.1

Find a basis of $U = \operatorname{span}\{(1, 1, 2, 3), (2, 4, 1, 0), (1, 5, -4, -9)\}$.

Solution. U is the row space of $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 1 & 0 \\ 1 & 5 & -4 & -9 \end{bmatrix}$. This matrix has row-echelon form $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -\frac{3}{2} & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so $\{(1, 1, 2, 3), (0, 1, -\frac{3}{2}, -3)\}$ is basis of U by Lemma 5.4.2.
Note that $\{(1, 1, 2, 3), (0, 2, -3, -6)\}$ is another basis that avoids fractions.

Lemmas 5.4.1 and 5.4.2 are enough to prove the following fundamental theorem.

Theorem 5.4.1: Rank Theorem

Let A denote any $m \times n$ matrix of rank r. Then

$$\dim(\operatorname{col} A) = \dim(\operatorname{row} A) = r$$

Moreover, if A is carried to a row-echelon matrix R by row operations, then

1. *The r nonzero rows of R are a basis of $\operatorname{row} A$.*
2. *If the leading 1s lie in columns $j_1, j_2, \dots, j_r$ of R, then columns $j_1, j_2, \dots, j_r$ of A are a basis of $\operatorname{col} A$.*

Proof. We have $\operatorname{row} A = \operatorname{row} R$ by Lemma 5.4.1, so (1) follows from Lemma 5.4.2. Moreover, $R = UA$ for some invertible matrix U by Theorem 2.5.1. Now write $A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{bmatrix}$ where $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A. Then

$$R = UA = U\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} U\mathbf{c}_1 & U\mathbf{c}_2 & \cdots & U\mathbf{c}_n \end{bmatrix}$$

Thus, in the notation of (2), the set $B = \{U\mathbf{c}_{j_1}, U\mathbf{c}_{j_2}, \dots, U\mathbf{c}_{j_r}\}$ is a basis of col R by Lemma 5.4.2. So, to prove (2) and the fact that $\dim(\text{col } A) = r$, it is enough to show that $D = \{\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}\}$ is a basis of col A. First, D is linearly independent because U is invertible (verify), so we show that, for each j, column $\mathbf{c}_j$ is a linear combination of the $\mathbf{c}_{j_i}$. But $U\mathbf{c}_j$ is column j of R, and so is a linear combination of the $U\mathbf{c}_{j_i}$, say $U\mathbf{c}_j = a_1U\mathbf{c}_{j_1} + a_2U\mathbf{c}_{j_2} + \cdots + a_rU\mathbf{c}_{j_r}$ where each a_i is a real number.

Since U is invertible, it follows that $\mathbf{c}_j = a_1\mathbf{c}_{j_1} + a_2\mathbf{c}_{j_2} + \cdots + a_r\mathbf{c}_{j_r}$ and the proof is complete. □

Example 5.4.2

Compute the rank of $A = \left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{array}\right]$ and find bases for row A and col A.

Solution. The reduction of A to row-echelon form is as follows:

$$\left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{array}\right] \to \left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence rank $A = 2$, and $\left\{\left[\begin{array}{rrrr} 1 & 2 & 2 & -1 \end{array}\right], \left[\begin{array}{rrrr} 0 & 0 & 1 & -3 \end{array}\right]\right\}$ is a basis of row A by Lemma 5.4.2. Since the leading 1s are in columns 1 and 3 of the row-echelon matrix, Theorem 5.4.1 shows that columns 1 and 3 of A are a basis $\left\{\left[\begin{array}{r} 1 \\ 3 \\ 1 \end{array}\right], \left[\begin{array}{r} 2 \\ 5 \\ 1 \end{array}\right]\right\}$ of col A.

Theorem 5.4.1 has several important consequences. The first, Corollary 5.4.1 below, follows because the rows of A are independent (respectively span row A) if and only if their transposes are independent (respectively span col A).

Corollary 5.4.1

If A is any matrix, then rank A = rank (A^T).

If A is an $m \times n$ matrix, we have col $A \subseteq \mathbb{R}^m$ and row $A \subseteq \mathbb{R}^n$. Hence Theorem 5.2.8 shows that $\dim(\text{col } A) \leq \dim(\mathbb{R}^m) = m$ and $\dim(\text{row } A) \leq \dim(\mathbb{R}^n) = n$. Thus Theorem 5.4.1 gives:

Corollary 5.4.2

If A is an $m \times n$ matrix, then rank $A \leq m$ *and* rank $A \leq n$.

Corollary 5.4.3

rank A = rank (UA) = rank (AV) *whenever U and V are invertible.*

Proof. Lemma 5.4.1 gives $\text{rank}\, A = \text{rank}\,(UA)$. Using this and Corollary 5.4.1 we get

$$\text{rank}\,(AV) = \text{rank}\,(AV)^T = \text{rank}\,(V^T A^T) = \text{rank}\,(A^T) = \text{rank}\, A$$

The next corollary requires a preliminary lemma. □

Lemma 5.4.3

Let A, U, and V be matrices of sizes $m \times n$, $p \times m$, and $n \times q$ respectively.

1. $\text{col}\,(AV) \subseteq \text{col}\, A$, *with equality if* $VV' = I_n$ *for some* V'.
2. $\text{row}\,(UA) \subseteq \text{row}\, A$, *with equality if* $U'U = I_m$ *for some* U'.

Proof. For (1), write $V = \begin{bmatrix} \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q \end{bmatrix}$ where $\mathbf{v}_j$ is column j of V. Then we have $AV = \begin{bmatrix} A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_q \end{bmatrix}$, and each $A\mathbf{v}_j$ is in $\text{col}\, A$ by Definition 2.4. It follows that $\text{col}\,(AV) \subseteq \text{col}\, A$. If $VV' = I_n$, we obtain $\text{col}\, A = \text{col}\, [(AV)V'] \subseteq \text{col}\,(AV)$ in the same way. This proves (1).

As to (2), we have $\text{col}\, \left[(UA)^T\right] = \text{col}\,(A^T U^T) \subseteq \text{col}\,(A^T)$ by (1), from which $\text{row}\,(UA) \subseteq \text{row}\, A$. If $U'U = I_m$, this is equality as in the proof of (1). □

Corollary 5.4.4

If A is $m \times n$ and B is $n \times m$, then $\text{rank}\, AB \leq \text{rank}\, A$ *and* $\text{rank}\, AB \leq \text{rank}\, B$.

Proof. By Lemma 5.4.3, $\text{col}\,(AB) \subseteq \text{col}\, A$ and $\text{row}\,(BA) \subseteq \text{row}\, A$, so Theorem 5.4.1 applies. □

In Section 5.1 we discussed two other subspaces associated with an $m \times n$ matrix A: the null space $\text{null}\,(A)$ and the image space $\text{im}\,(A)$

$$\text{null}\,(A) = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \text{ and } \text{im}\,(A) = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Using rank, there are simple ways to find bases of these spaces. If A has rank r, we have $\text{im}\,(A) = \text{col}\,(A)$ by Example 5.1.8, so $\dim\,[\text{im}\,(A)] = \dim\,[\text{col}\,(A)] = r$. Hence Theorem 5.4.1 provides a method of finding a basis of $\text{im}\,(A)$. This is recorded as part (2) of the following theorem.

Theorem 5.4.2

Let A denote an $m \times n$ matrix of rank r. Then

1. *The $n - r$ basic solutions to the system $A\mathbf{x} = \mathbf{0}$ provided by the gaussian algorithm are a basis of* $\text{null}\,(A)$, *so* $\dim\,[\text{null}\,(A)] = n - r$.
2. *Theorem 5.4.1 provides a basis of* $\text{im}\,(A) = \text{col}\,(A)$, *and* $\dim\,[\text{im}\,(A)] = r$.

Proof. It remains to prove (1). We already know (Theorem 2.2.1) that $\text{null}\,(A)$ is spanned by the $n - r$ basic solutions of $A\mathbf{x} = \mathbf{0}$. Hence using Theorem 5.2.7, it suffices to show that $\dim\,[\text{null}\,(A)] = n - r$. So let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis of $\text{null}\,(A)$, and extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n\}$ of $\mathbb{R}^n$ (by

Theorem 5.2.6). It is enough to show that $\{A\mathbf{x}_{k+1}, \dots, A\mathbf{x}_n\}$ is a basis of $\text{im}(A)$; then $n-k=r$ by the above and so $k=n-r$ as required.

Spanning. Choose $A\mathbf{x}$ in $\text{im}(A)$, $\mathbf{x}$ in $\mathbb{R}^n$, and write $\mathbf{x}=a_1\mathbf{x}_1+\cdots+a_k\mathbf{x}_k+a_{k+1}\mathbf{x}_{k+1}+\cdots+a_n\mathbf{x}_n$ where the a_i are in $\mathbb{R}$. Then $A\mathbf{x}=a_{k+1}A\mathbf{x}_{k+1}+\cdots+a_nA\mathbf{x}_n$ because $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}\subseteq \text{null}(A)$.

Independence. Let $t_{k+1}A\mathbf{x}_{k+1}+\cdots+t_nA\mathbf{x}_n=\mathbf{0}$, t_i in $\mathbb{R}$. Then $t_{k+1}\mathbf{x}_{k+1}+\cdots+t_n\mathbf{x}_n$ is in $\text{null}\,A$, so $t_{k+1}\mathbf{x}_{k+1}+\cdots+t_n\mathbf{x}_n=t_1\mathbf{x}_1+\cdots+t_k\mathbf{x}_k$ for some $t_1, \dots, t_k$ in $\mathbb{R}$. But then the independence of the $\mathbf{x}_i$ shows that $t_i=0$ for every i. □

Example 5.4.3

If $A=\left[\begin{array}{rrrr} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{array}\right]$, find bases of $\text{null}(A)$ and $\text{im}(A)$, and so find their dimensions.

Solution. If $\mathbf{x}$ is in $\text{null}(A)$, then $A\mathbf{x}=\mathbf{0}$, so $\mathbf{x}$ is given by solving the system $A\mathbf{x}=\mathbf{0}$. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $r=\text{rank}(A)=2$. Here, $\text{im}(A)=\text{col}(A)$ has basis $\left\{\left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right], \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]\right\}$ by Theorem 5.4.1 because the leading 1s are in columns 1 and 3. In particular, $\dim[\text{im}(A)]=2=r$ as in Theorem 5.4.2.

Turning to $\text{null}(A)$, we use gaussian elimination. The leading variables are x_1 and x_3, so the nonleading variables become parameters: $x_2=s$ and $x_4=t$. It follows from the reduced matrix that $x_1=2s+t$ and $x_3=-2t$, so the general solution is

$$\mathbf{x}=\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right]=\left[\begin{array}{c} 2s+t \\ s \\ -2t \\ t \end{array}\right]=s\mathbf{x}_1+t\mathbf{x}_2 \text{ where } \mathbf{x}_1=\left[\begin{array}{r} 2 \\ 1 \\ 0 \\ 0 \end{array}\right], \text{ and } \mathbf{x}_2=\left[\begin{array}{r} 1 \\ 0 \\ -2 \\ 1 \end{array}\right].$$

Hence $\text{null}(A)$. But $\mathbf{x}_1$ and $\mathbf{x}_2$ *are* solutions (basic), so

$$\text{null}(A)=\text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$$

However Theorem 5.4.2 asserts that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a *basis* of $\text{null}(A)$. (In fact it is easy to verify directly that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is independent in this case.) In particular, $\dim[\text{null}(A)]=2=n-r$, as Theorem 5.4.2 asserts.

Let A be an $m\times n$ matrix. Corollary 5.4.2 of Theorem 5.4.1 asserts that $\text{rank}\,A\le m$ and $\text{rank}\,A\le n$, and it is natural to ask when these extreme cases arise. If $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A, Theorem 5.2.2 shows that $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ spans $\mathbb{R}^m$ if and only if the system $A\mathbf{x}=\mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$, and that $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ is independent if and only if $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, implies $\mathbf{x} = \mathbf{0}$. The next two useful theorems improve on both these results, and relate them to when the rank of A is n or m.

Theorem 5.4.3

The following are equivalent for an $m \times n$ matrix A:

1. rank $A = n$.
2. *The rows of A span $\mathbb{R}^n$.*
3. *The columns of A are linearly independent in $\mathbb{R}^m$.*
4. *The $n \times n$ matrix A^TA is invertible.*
5. *$CA = I_n$ for some $n \times m$ matrix C.*
6. *If $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, then $\mathbf{x} = \mathbf{0}$.*

Proof. (1) $\Rightarrow$ (2). We have row $A \subseteq \mathbb{R}^n$, and $\dim(\text{row } A) = n$ by (1), so row $A = \mathbb{R}^n$ by Theorem 5.2.8. This is (2).

(2) $\Rightarrow$ (3). By (2), row $A = \mathbb{R}^n$, so rank $A = n$. This means $\dim(\text{col } A) = n$. Since the n columns of A span col A, they are independent by Theorem 5.2.7.

(3) $\Rightarrow$ (4). If $(A^TA)\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, we show that $\mathbf{x} = \mathbf{0}$ (Theorem 2.4.5). We have

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T \mathbf{0} = \mathbf{0}$$

Hence $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ by (3) and Theorem 5.2.2.

(4) $\Rightarrow$ (5). Given (4), take $C = (A^TA)^{-1}A^T$.

(5) $\Rightarrow$ (6). If $A\mathbf{x} = \mathbf{0}$, then left multiplication by C (from (5)) gives $\mathbf{x} = \mathbf{0}$.

(6) $\Rightarrow$ (1). Given (6), the columns of A are independent by Theorem 5.2.2. Hence $\dim(\text{col } A) = n$, and (1) follows. □

Theorem 5.4.4

The following are equivalent for an $m \times n$ matrix A:

1. rank $A = m$.
2. *The columns of A span $\mathbb{R}^m$.*
3. *The rows of A are linearly independent in $\mathbb{R}^n$.*
4. *The $m \times m$ matrix AA^T is invertible.*
5. *$AC = I_m$ for some $n \times m$ matrix C.*
6. *The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.*

Proof. (1) $\Rightarrow$ (2). By (1), $\dim(\text{col } A = m$, so $\text{col } A = \mathbb{R}^m$ by Theorem 5.2.8.

(2) $\Rightarrow$ (3). By (2), $\text{col } A = \mathbb{R}^m$, so $\text{rank } A = m$. This means $\dim(\text{row } A) = m$. Since the m rows of A span row A, they are independent by Theorem 5.2.7.

(3) $\Rightarrow$ (4). We have $\text{rank } A = m$ by (3), so the $n \times m$ matrix A^T has rank m. Hence applying Theorem 5.4.3 to A^T in place of A shows that $(A^T)^T A^T$ is invertible, proving (4).

(4) $\Rightarrow$ (5). Given (4), take $C = A^T(AA_T)^{-1}$ in (5).

(5) $\Rightarrow$ (6). Comparing columns in $AC = I_m$ gives $A\mathbf{c}_j = \mathbf{e}_j$ for each j, where $\mathbf{c}_j$ and $\mathbf{e}_j$ denote column j of C and I_m respectively. Given $\mathbf{b}$ in $\mathbb{R}^m$, write $\mathbf{b} = \sum_{j=1}^m r_j\mathbf{e}_j$, r_j in $\mathbb{R}$. Then $A\mathbf{x} = \mathbf{b}$ holds with $\mathbf{x} = \sum_{j=1}^m r_j\mathbf{c}_j$ as the reader can verify.

(6) $\Rightarrow$ (1). Given (6), the columns of A span $\mathbb{R}^m$ by Theorem 5.2.2. Thus $\text{col } A = \mathbb{R}^m$ and (1) follows. □

Example 5.4.4

Show that $\begin{bmatrix} 3 & x+y+z \\ x+y+z & x^2+y^2+z^2 \end{bmatrix}$ is invertible if x, y, and z are not all equal.

Solution. The given matrix has the form A^TA where $A = \begin{bmatrix} 1 & x \\ 1 & y \\ 1 & z \end{bmatrix}$ has independent columns because x, y, and z are not all equal (verify). Hence Theorem 5.4.3 applies.

Theorem 5.4.3 and Theorem 5.4.4 relate several important properties of an $m \times n$ matrix A to the invertibility of the square, symmetric matrices A^TA and AA^T. In fact, even if the columns of A are not independent or do not span $\mathbb{R}^m$, the matrices A^TA and AA^T are both symmetric and, as such, have real eigenvalues as we shall see. We return to this in Chapter 7.

Exercises for 5.4

Exercise 5.4.1 In each case find bases for the row and column spaces of A and determine the rank of A.

a. $\begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ -3 & -6 & 3 & -2 \end{bmatrix}$

Exercise 5.4.2 In each case find a basis of the subspace U.

a. $U = \text{span}\{(1, -1, 0, 3), (2, 1, 5, 1), (4, -2, 5, 7)\}$

b. $U = \text{span}\{(1, -1, 2, 5, 1), (3, 1, 4, 2, 7), (1, 1, 0, 0, 0), (5, 1, 6, 7, 8)\}$

c. $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

d. $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -10 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \right\}$

Exercise 5.4.3

a. Can a 3×4 matrix have independent columns? Independent rows? Explain.

b. If A is 4×3 and rank $A = 2$, can A have independent columns? Independent rows? Explain.

c. If A is an $m \times n$ matrix and rank $A = m$, show that $m \leq n$.

d. Can a nonsquare matrix have its rows independent and its columns independent? Explain.

e. Can the null space of a 3×6 matrix have dimension 2? Explain.

f. Suppose that A is 5×4 and null $(A) = \mathbb{R}\mathbf{x}$ for some column $\mathbf{x} \neq \mathbf{0}$. Can dim (im A) $= 2$?

Exercise 5.4.4 If A is $m \times n$ show that

$$\text{col}\,(A) = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Exercise 5.4.5 If A is $m \times n$ and B is $n \times m$, show that $AB = 0$ if and only if col $B \subseteq$ null A.

Exercise 5.4.6 Show that the rank does not change when an elementary row or column operation is performed on a matrix.

Exercise 5.4.7 In each case find a basis of the null space of A. Then compute rank A and verify (1) of Theorem 5.4.2.

a. $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & 5 & 5 & 2 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -4 & -2 \end{bmatrix}$

Exercise 5.4.8 Let $A = \mathbf{cr}$ where $\mathbf{c} \neq \mathbf{0}$ is a column in $\mathbb{R}^m$ and $\mathbf{r} \neq \mathbf{0}$ is a row in $\mathbb{R}^n$.

a. Show that col $A = \text{span}\,\{\mathbf{c}\}$ and row $A = \text{span}\,\{\mathbf{r}\}$.

b. Find dim (null A).

c. Show that null A = null $\mathbf{r}$.

Exercise 5.4.9 Let A be $m \times n$ with columns $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$.

a. If $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ is independent, show null $A = \{\mathbf{0}\}$.

b. If null $A = \{\mathbf{0}\}$, show that $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ is independent.

Exercise 5.4.10 Let A be an $n \times n$ matrix.

a. Show that $A^2 = 0$ if and only if col $A \subseteq$ null A.

b. Conclude that if $A^2 = 0$, then rank $A \leq \frac{n}{2}$.

c. Find a matrix A for which col A = null A.

Exercise 5.4.11 Let B be $m \times n$ and let AB be $k \times n$. If rank B = rank (AB), show that null B = null (AB). [*Hint*: Theorem 5.4.1.]

Exercise 5.4.12 Give a careful argument why rank (A^T) = rank A.

Exercise 5.4.13 Let A be an $m \times n$ matrix with columns $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$. If rank $A = n$, show that $\{A^T\mathbf{c}_1, A^T\mathbf{c}_2, \dots, A^T\mathbf{c}_n\}$ is a basis of $\mathbb{R}^n$.

Exercise 5.4.14 If A is $m \times n$ and $\mathbf{b}$ is $m \times 1$, show that $\mathbf{b}$ lies in the column space of A if and only if rank $[A\ \mathbf{b}]$ = rank A.

Exercise 5.4.15

a. Show that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if rank A = rank $[A\ \mathbf{b}]$. [*Hint*: Exercises 5.4.12 and 5.4.14.]

b. If $A\mathbf{x} = \mathbf{b}$ has no solution, show that rank $[A\ \mathbf{b}] = 1 +$ rank A.

Exercise 5.4.16 Let X be a $k \times m$ matrix. If I is the $m \times m$ identity matrix, show that $I + X^TX$ is invertible.

[*Hint*: $I + X^TX = A^TA$ where $A = \begin{bmatrix} I \\ X \end{bmatrix}$ in block form.]

Exercise 5.4.17 If A is $m \times n$ of rank r, show that A can be factored as $A = PQ$ where P is $m \times r$ with r independent columns, and Q is $r \times n$ with r independent rows. [*Hint*: Let $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by Theorem 2.5.3, and write $U^{-1} = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ and $V^{-1} = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$ in block form, where U_1 and V_1 are $r \times r$.]

Exercise 5.4.18

a. Show that if A and B have independent columns, so does AB.

b. Show that if A and B have independent rows, so does AB.

Exercise 5.4.19 A matrix obtained from A by deleting rows and columns is called a **submatrix** of A. If A has an invertible $k \times k$ submatrix, show that rank $A \geq k$. [*Hint*: Show that row and column operations carry $A \to \begin{bmatrix} I_k & P \\ 0 & Q \end{bmatrix}$ in block form.] *Remark*: It can be shown that rank A is the largest integer r such that A has an invertible $r \times r$ submatrix.

5.5 Similarity and Diagonalization

In Section 3.3 we studied diagonalization of a square matrix A, and found important applications (for example to linear dynamical systems). We can now utilize the concepts of subspace, basis, and dimension to clarify the diagonalization process, reveal some new results, and prove some theorems which could not be demonstrated in Section 3.3.

Before proceeding, we introduce a notion that simplifies the discussion of diagonalization, and is used throughout the book.

Similar Matrices

Definition 5.11 Similar Matrices

*If A and B are $n \times n$ matrices, we say that A and B are **similar**, and write $A \sim B$, if $B = P^{-1}AP$ for some invertible matrix P.*

Note that $A \sim B$ if and only if $B = QAQ^{-1}$ where Q is invertible (write $P^{-1} = Q$). The language of similarity is used throughout linear algebra. For example, a matrix A is diagonalizable if and only if it is similar to a diagonal matrix.

If $A \sim B$, then necessarily $B \sim A$. To see why, suppose that $B = P^{-1}AP$. Then $A = PBP^{-1} = Q^{-1}BQ$ where $Q = P^{-1}$ is invertible. This proves the second of the following properties of similarity (the others are left as an exercise):

$$\begin{aligned} &1.\ A \sim A \text{ for all square matrices } A. \\ &2.\ \text{If } A \sim B, \text{ then } B \sim A. \\ &3.\ \text{If } A \sim B \text{ and } B \sim A, \text{ then } A \sim C. \end{aligned} \tag{5.2}$$

These properties are often expressed by saying that the similarity relation $\sim$ is an **equivalence relation** on the set of $n \times n$ matrices. Here is an example showing how these properties are used.