Exercise 6.1.14 Verify axioms A2—A5 and S2—S5 for the space $\mathbf{F}[a, b]$ of functions on $[a, b]$ (Example 6.1.7).

Exercise 6.1.15 Prove each of the following for vectors $\mathbf{u}$ and $\mathbf{v}$ and scalars a and b.

a. If $a\mathbf{v} = \mathbf{0}$, then $a = 0$ or $\mathbf{v} = \mathbf{0}$.

b. If $a\mathbf{v} = b\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, then $a = b$.

c. If $a\mathbf{v} = a\mathbf{w}$ and $a \neq 0$, then $\mathbf{v} = \mathbf{w}$.

Exercise 6.1.16 By calculating $(1+1)(\mathbf{v}+\mathbf{w})$ in two ways (using axioms S2 and S3), show that axiom A2 follows from the other axioms.

Exercise 6.1.17 Let V be a vector space, and define V^n to be the set of all n-tuples $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ of n vectors $\mathbf{v}_i$, each belonging to V. Define addition and scalar multiplication in V^n as follows:

$$\begin{aligned} &(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n) + (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) \\ &\quad = (\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2, \dots, \mathbf{u}_n + \mathbf{v}_n) \\ &a(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = (a\mathbf{v}_1, a\mathbf{v}_2, \dots, a\mathbf{v}_n) \end{aligned}$$

Show that V^n is a vector space.

Exercise 6.1.18 Let V^n be the vector space of n-tuples from the preceding exercise, written as columns. If A is an $m \times n$ matrix, and X is in V^n, define AX in V^m by matrix multiplication. More precisely, if

$$A = [a_{ij}] \text{ and } X = \left[\begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array}\right], \text{ let } AX = \left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{array}\right]$$

where $\mathbf{u}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + \cdots + a_{in}\mathbf{v}_n$ for each i.
Prove that:

a. $B(AX) = (BA)X$

b. $(A + A_1)X = AX + A_1X$

c. $A(X + X_1) = AX + AX_1$

d. $(kA)X = k(AX) = A(kX)$ if k is any number

e. $IX = X$ if I is the $n \times n$ identity matrix

f. Let E be an elementary matrix obtained by performing a row operation on the rows of I_n (see Section 2.5). Show that EX is the column resulting from performing that same row operation on the vectors (call them rows) of X. [*Hint*: Lemma 2.5.1.]

6.2 Subspaces and Spanning Sets

Chapter 5 is essentially about the subspaces of $\mathbb{R}^n$. We now extend this notion.

Definition 6.2 Subspaces of a Vector Space

If V is a vector space, a nonempty subset $U \subseteq V$ is called a ***subspace*** *of V if U is itself a vector space using the addition and scalar multiplication of V.*

Subspaces of $\mathbb{R}^n$ (as defined in Section 5.1) are subspaces in the present sense by Example 6.1.3. Moreover, the defining properties for a subspace of $\mathbb{R}^n$ actually *characterize* subspaces in general.

Theorem 6.2.1: Subspace Test

A subset U of a vector space is a subspace of V if and only if it satisfies the following three conditions:

1. *$\mathbf{0}$ lies in U where $\mathbf{0}$ is the zero vector of V.*
2. *If $\mathbf{u}_1$ and $\mathbf{u}_2$ are in U, then $\mathbf{u}_1 + \mathbf{u}_2$ is also in U.*
3. *If $\mathbf{u}$ is in U, then $a\mathbf{u}$ is also in U for each scalar a.*

Proof. If U is a subspace of V, then (2) and (3) hold by axioms A1 and S1 respectively, applied to the vector space U. Since U is nonempty (it is a vector space), choose $\mathbf{u}$ in U. Then (1) holds because $\mathbf{0} = 0\mathbf{u}$ is in U by (3) and Theorem 6.1.3.

Conversely, if (1), (2), and (3) hold, then axioms A1 and S1 hold because of (2) and (3), and axioms A2, A3, S2, S3, S4, and S5 hold in U because they hold in V. Axiom A4 holds because the zero vector $\mathbf{0}$ of V is actually in U by (1), and so serves as the zero of U. Finally, given $\mathbf{u}$ in U, then its negative $-\mathbf{u}$ in V is again in U by (3) because $-\mathbf{u} = (-1)\mathbf{u}$ (again using Theorem 6.1.3). Hence $-\mathbf{u}$ serves as the negative of $\mathbf{u}$ in U. □

Note that the proof of Theorem 6.2.1 shows that if U is a subspace of V, then U and V share the same zero vector, and that the negative of a vector in the space U is the same as its negative in V.

Example 6.2.1

If V is any vector space, show that $\{\mathbf{0}\}$ and V are subspaces of V.

Solution. $U = V$ clearly satisfies the conditions of the subspace test. As to $U = \{\mathbf{0}\}$, it satisfies the conditions because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for all a in $\mathbb{R}$.

The vector space $\{\mathbf{0}\}$ is called the **zero subspace** of V.

Example 6.2.2

Let $\mathbf{v}$ be a vector in a vector space V. Show that the set

$$\mathbb{R}\mathbf{v} = \{a\mathbf{v} \mid a \text{ in } \mathbb{R}\}$$

of all scalar multiples of $\mathbf{v}$ is a subspace of V.

Solution. Because $\mathbf{0} = 0\mathbf{v}$, it is clear that $\mathbf{0}$ lies in $\mathbb{R}\mathbf{v}$. Given two vectors $a\mathbf{v}$ and $a_1\mathbf{v}$ in $\mathbb{R}\mathbf{v}$, their sum $a\mathbf{v} + a_1\mathbf{v} = (a + a_1)\mathbf{v}$ is also a scalar multiple of $\mathbf{v}$ and so lies in $\mathbb{R}\mathbf{v}$. Hence $\mathbb{R}\mathbf{v}$ is closed under addition. Finally, given $a\mathbf{v}$, $r(a\mathbf{v}) = (ra)\mathbf{v}$ lies in $\mathbb{R}\mathbf{v}$ for all $r \in \mathbb{R}$, so $\mathbb{R}\mathbf{v}$ is closed under scalar multiplication. Hence the subspace test applies.

In particular, given $\mathbf{d} \neq \mathbf{0}$ in $\mathbb{R}^3$, $\mathbb{R}\mathbf{d}$ is the line through the origin with direction vector $\mathbf{d}$.

The space $\mathbb{R}\mathbf{v}$ in Example 6.2.2 is described by giving the *form* of each vector in $\mathbb{R}\mathbf{v}$. The next example describes a subset U of the space $\mathbf{M}_{nn}$ by giving a *condition* that each matrix of U must satisfy.

Example 6.2.3

Let A be a fixed matrix in $\mathbf{M}_{nn}$. Show that $U = \{X \text{ in } \mathbf{M}_{nn} \mid AX = XA\}$ is a subspace of $\mathbf{M}_{nn}$.

Solution. If 0 is the $n \times n$ zero matrix, then $A0 = 0A$, so 0 satisfies the condition for membership in U. Next suppose that X and X_1 lie in U so that $AX = XA$ and $AX_1 = X_1A$. Then

$$\begin{aligned} A(X + X_1) &= AX + AX_1 = XA + X_1A + (X + X_1)A \\ A(aX) &= a(AX) = a(XA) = (aX)A \end{aligned}$$

for all a in $\mathbb{R}$, so both $X + X_1$ and aX lie in U. Hence U is a subspace of $\mathbf{M}_{nn}$.

Suppose $p(x)$ is a polynomial and a is a number. Then the number $p(a)$ obtained by replacing x by a in the expression for $p(x)$ is called the **evaluation** of $p(x)$ at a. For example, if $p(x) = 5 - 6x + 2x^2$, then the evaluation of $p(x)$ at $a = 2$ is $p(2) = 5 - 12 + 8 = 1$. If $p(a) = 0$, the number a is called a **root** of $p(x)$.

Example 6.2.4

Consider the set U of all polynomials in $\mathbf{P}$ that have 3 as a root:

$$U = \{p(x) \in \mathbf{P} \mid p(3) = 0\}$$

Show that U is a subspace of $\mathbf{P}$.

Solution. Clearly, the zero polynomial lies in U. Now let $p(x)$ and $q(x)$ lie in U so $p(3) = 0$ and $q(3) = 0$. We have $(p+q)(x) = p(x) + q(x)$ for all x, so $(p+q)(3) = p(3) + q(3) = 0 + 0 = 0$, and U is closed under addition. The verification that U is closed under scalar multiplication is similar.

Recall that the space $\mathbf{P}_n$ consists of all polynomials of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, and so is closed under the addition and scalar multiplication in $\mathbf{P}$. Moreover, the zero polynomial is included in $\mathbf{P}_n$. Thus the subspace test gives Example 6.2.5.

Example 6.2.5

$\mathbf{P}_n$ is a subspace of $\mathbf{P}$ for each $n \geq 0$.

The next example involves the notion of the derivative f' of a function f. (If the reader is not familiar with calculus, this example may be omitted.) A function f defined on the interval $[a, b]$ is called **differentiable** if the derivative $f'(r)$ exists at every r in $[a, b]$.

Example 6.2.6

Show that the subset $\mathbf{D}[a, b]$ of all **differentiable functions** on $[a, b]$ is a subspace of the vector space $\mathbf{F}[a, b]$ of all functions on $[a, b]$.

Solution. The derivative of any constant function is the constant function 0; in particular, 0 itself is differentiable and so lies in $\mathbf{D}[a, b]$. If f and g both lie in $\mathbf{D}[a, b]$ (so that f' and g' exist), then it is a theorem of calculus that $f+g$ and rf are both differentiable for any $r \in \mathbb{R}$. In fact, $(f+g)' = f' + g'$ and $(rf)' = rf'$, so both lie in $\mathbf{D}[a, b]$. This shows that $\mathbf{D}[a, b]$ is a subspace of $\mathbf{F}[a, b]$.

Linear Combinations and Spanning Sets

Definition 6.3 Linear Combinations and Spanning

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a set of vectors in a vector space V. As in $\mathbb{R}^n$, a vector $\mathbf{v}$ is called a ***linear combination*** *of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ if it can be expressed in the form*

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where $a_1, a_2, \dots, a_n$ are scalars, called the ***coefficients*** *of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. The set of all linear combinations of these vectors is called their* ***span****, and is denoted by*

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \mid a_i \textit{ in } \mathbb{R}\}$$

If it happens that $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, these vectors are called a **spanning set** for V. For example, the span of two vectors $\mathbf{v}$ and $\mathbf{w}$ is the set

$$\operatorname{span}\{\mathbf{v}, \mathbf{w}\} = \{s\mathbf{v} + t\mathbf{w} \mid s \text{ and } t \text{ in } \mathbb{R}\}$$

of all sums of scalar multiples of these vectors.

Example 6.2.7

Consider the vectors $p_1 = 1 + x + 4x^2$ and $p_2 = 1 + 5x + x^2$ in $\mathbf{P}_2$. Determine whether p_1 and p_2 lie in $\operatorname{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Solution. For p_1, we want to determine if s and t exist such that

$$p_1 = s(1 + 2x - x^2) + t(3 + 5x + 2x^2)$$

Equating coefficients of powers of x (where $x^0 = 1$) gives

$$1 = s + 3t, \quad 1 = 2s + 5t, \quad \text{and} \quad 4 = -s + 2t$$

These equations have the solution $s = -2$ and $t = 1$, so p_1 is indeed in $\operatorname{span}\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Turning to $p_2 = 1+5x+x^2$, we are looking for s and t such that

$$p_2 = s(1+2x-x^2)+t(3+5x+2x^2)$$

Again equating coefficients of powers of x gives equations $1 = s+3t$, $5 = 2s+5t$, and $1 = -s+2t$. But in this case there is no solution, so p_2 is *not* in $\operatorname{span}\{1+2x-x^2,\ 3+5x+2x^2\}$.

We saw in Example 5.1.6 that $\mathbb{R}^m = \operatorname{span}\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_m\}$ where the vectors $\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_m$ are the columns of the $m \times m$ identity matrix. Of course $\mathbb{R}^m = \mathbf{M}_{m1}$ is the set of all $m \times 1$ matrices, and there is an analogous spanning set for each space $\mathbf{M}_{mn}$. For example, each 2×2 matrix has the form

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = a\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + b\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] + c\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right] + d\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$$

so

$$\mathbf{M}_{22} = \operatorname{span}\left\{\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right],\ \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right],\ \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right],\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]\right\}$$

Similarly, we obtain

Example 6.2.8

$\mathbf{M}_{mn}$ is the span of the set of all $m \times n$ matrices with exactly one entry equal to 1, and all other entries zero.

The fact that every polynomial in $\mathbf{P}_n$ has the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where each a_i is in $\mathbb{R}$ shows that

Example 6.2.9

$\mathbf{P}_n = \operatorname{span}\{1,\ x,\ x^2,\ \dots,\ x^n\}$.

In Example 6.2.2 we saw that $\operatorname{span}\{\mathbf{v}\} = \{a\mathbf{v} \mid a \text{ in } \mathbb{R}\} = \mathbb{R}\mathbf{v}$ is a subspace for any vector $\mathbf{v}$ in a vector space V. More generally, the span of *any* set of vectors is a subspace. In fact, the proof of Theorem 5.1.1 goes through to prove:

Theorem 6.2.2

Let $U = \operatorname{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ in a vector space V. Then:

1. *U is a subspace of V containing each of $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$.*
2. *U is the "smallest" subspace containing these vectors in the sense that any subspace that contains each of $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$ must contain U.*

Here is how condition 2 in Theorem 6.2.2 is used. Given vectors $\mathbf{v}_1,\ \dots,\ \mathbf{v}_k$ in a vector space V and a subspace $U \subseteq V$, then:

$$\operatorname{span}\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\} \subseteq U \Leftrightarrow \text{ each } \mathbf{v}_i \in U$$

The following examples illustrate this.

Example 6.2.10

Show that $\mathbf{P}_3 = \text{span}\{x^2 + x^3,\ x,\ 2x^2 + 1,\ 3\}$.

Solution. Write $U = \text{span}\{x^2 + x^3,\ x,\ 2x^2 + 1,\ 3\}$. Then $U \subseteq \mathbf{P}_3$, and we use the fact that $\mathbf{P}_3 = \text{span}\{1,\ x,\ x^2,\ x^3\}$ to show that $\mathbf{P}_3 \subseteq U$. In fact, x and $1 = \frac{1}{3} \cdot 3$ clearly lie in U. But then successively,

$$x^2 = \frac{1}{2}[(2x^2 + 1) - 1] \quad \text{and} \quad x^3 = (x^2 + x^3) - x^2$$

also lie in U. Hence $\mathbf{P}_3 \subseteq U$ by Theorem 6.2.2.

Example 6.2.11

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in a vector space V. Show that

$$\text{span}\{\mathbf{u},\ \mathbf{v}\} = \text{span}\{\mathbf{u} + 2\mathbf{v},\ \mathbf{u} - \mathbf{v}\}$$

Solution. We have $\text{span}\{\mathbf{u} + 2\mathbf{v},\ \mathbf{u} - \mathbf{v}\} \subseteq \text{span}\{\mathbf{u},\ \mathbf{v}\}$ by Theorem 6.2.2 because both $\mathbf{u} + 2\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ lie in $\text{span}\{\mathbf{u},\ \mathbf{v}\}$. On the other hand,

$$\mathbf{u} = \frac{1}{3}(\mathbf{u} + 2\mathbf{v}) + \frac{2}{3}(\mathbf{u} - \mathbf{v}) \quad \text{and} \quad \mathbf{v} = \frac{1}{3}(\mathbf{u} + 2\mathbf{v}) - \frac{1}{3}(\mathbf{u} - \mathbf{v})$$

so $\text{span}\{\mathbf{u},\ \mathbf{v}\} \subseteq \text{span}\{\mathbf{u} + 2\mathbf{v},\ \mathbf{u} - \mathbf{v}\}$, again by Theorem 6.2.2.

Exercises for 6.2

Exercise 6.2.1 Which of the following are subspaces of $\mathbf{P}_3$? Support your answer.

a. $U = \{f(x) \mid f(x) \in \mathbf{P}_3,\ f(2) = 1\}$

b. $U = \{xg(x) \mid g(x) \in \mathbf{P}_2\}$

c. $U = \{xg(x) \mid g(x) \in \mathbf{P}_3\}$

d. $U = \{xg(x) + (1 - x)h(x) \mid g(x) \text{ and } h(x) \in \mathbf{P}_2\}$

e. $U =$ The set of all polynomials in $\mathbf{P}_3$ with constant term 0

f. $U = \{f(x) \mid f(x) \in \mathbf{P}_3,\ \deg f(x) = 3\}$

Exercise 6.2.2 Which of the following are subspaces of $\mathbf{M}_{22}$? Support your answer.

a. $U = \left\{ \left[\begin{array}{cc} a & b \\ 0 & c \end{array} \right] \middle|\ a,\ b, \text{ and } c \text{ in } \mathbb{R} \right\}$

b. $U = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \middle|\ a + b = c + d;\ a,\ b,\ c,\ d \text{ in } \mathbb{R} \right\}$

c. $U = \{A \mid A \in \mathbf{M}_{22},\ A = A^T\}$

d. $U = \{A \mid A \in \mathbf{M}_{22},\ AB = 0\}$, B a fixed 2×2 matrix

e. $U = \{A \mid A \in \mathbf{M}_{22},\ A^2 = A\}$

f. $U = \{A \mid A \in \mathbf{M}_{22},\ A \text{ is not invertible}\}$

g. $U = \{A \mid A \in \mathbf{M}_{22},\ BAC = CAB\}$, B and C fixed 2×2 matrices

Exercise 6.2.3 Which of the following are subspaces of $\mathbf{F}[0,\ 1]$? Support your answer.

a. $U = \{f \mid f(0) = 0\}$

b. $U = \{f \mid f(0) = 1\}$

c. $U = \{f \mid f(0) = f(1)\}$

d. $U = \{f \mid f(x) \geq 0$ for all x in $[0,\ 1]\}$

e. $U = \{f \mid f(x) = f(y)$ for all x and y in $[0,\ 1]\}$

f. $U = \{f \mid f(x+y) = f(x) + f(y)$ for all x and y in $[0,\ 1]\}$

g. $U = \{f \mid f$ is integrable and $\int_0^1 f(x)dx = 0\}$

Exercise 6.2.4 Let A be an $m \times n$ matrix. For which columns $\mathbf{b}$ in $\mathbb{R}^m$ is $U = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n,\ A\mathbf{x} = \mathbf{b}\}$ a subspace of $\mathbb{R}^n$? Support your answer.

Exercise 6.2.5 Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ (written as a column), and define $U = \{A\mathbf{x} \mid A \in \mathbf{M}_{mn}\}$.

a. Show that U is a subspace of $\mathbb{R}^m$.

b. Show that $U = \mathbb{R}^m$ if $\mathbf{x} \neq \mathbf{0}$.

Exercise 6.2.6 Write each of the following as a linear combination of $x+1$, x^2+x, and x^2+2.

a. x^2+3x+2

b. $2x^2-3x+1$

c. x^2+1

d. x

Exercise 6.2.7 Determine whether $\mathbf{v}$ lies in $\text{span}\,\{\mathbf{u},\ \mathbf{w}\}$ in each case.

a. $\mathbf{v} = 3x^2-2x-1$; $\mathbf{u} = x^2+1$, $\mathbf{w} = x+2$

b. $\mathbf{v} = x$; $\mathbf{u} = x^2+1$, $\mathbf{w} = x+2$

c. $\mathbf{v} = \left[\begin{array}{rr} 1 & 3 \\ -1 & 1 \end{array}\right]$; $\mathbf{u} = \left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right]$, $\mathbf{w} = \left[\begin{array}{rr} 2 & 1 \\ 1 & 0 \end{array}\right]$

d. $\mathbf{v} = \left[\begin{array}{rr} 1 & -4 \\ 5 & 3 \end{array}\right]$; $\mathbf{u} = \left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right]$, $\mathbf{w} = \left[\begin{array}{rr} 2 & 1 \\ 1 & 0 \end{array}\right]$

Exercise 6.2.8 Which of the following functions lie in $\text{span}\,\{\cos^2 x,\ \sin^2 x\}$? (Work in $\mathbf{F}[0,\ \pi]$.)

a. $\cos 2x$

b. 1

c. x^2

d. $1+x^2$

Exercise 6.2.9

a. Show that $\mathbb{R}^3$ is spanned by $\{(1,\ 0,\ 1),\ (1,\ 1,\ 0),\ (0,\ 1,\ 1)\}$.

b. Show that $\mathbf{P}_2$ is spanned by $\{1+2x^2,\ 3x,\ 1+x\}$.

c. Show that $\mathbf{M}_{22}$ is spanned by $\left\{\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]\right\}$.

Exercise 6.2.10 If X and Y are two sets of vectors in a vector space V, and if $X \subseteq Y$, show that $\text{span}\,X \subseteq \text{span}\,Y$.

Exercise 6.2.11 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ denote vectors in a vector space V. Show that:

a. $\text{span}\,\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\} = \text{span}\,\{\mathbf{u}+\mathbf{v},\ \mathbf{u}+\mathbf{w},\ \mathbf{v}+\mathbf{w}\}$

b. $\text{span}\,\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\} = \text{span}\,\{\mathbf{u}-\mathbf{v},\ \mathbf{u}+\mathbf{w},\ \mathbf{w}\}$

Exercise 6.2.12 Show that

$$\text{span}\,\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n,\ \mathbf{0}\} = \text{span}\,\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$$

holds for any set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$.

Exercise 6.2.13 If X and Y are nonempty subsets of a vector space V such that $\text{span}\,X = \text{span}\,Y = V$, must there be a vector common to both X and Y? Justify your answer.

Exercise 6.2.14 Is it possible that $\{(1,\ 2,\ 0),\ (1,\ 1,\ 1)\}$ can span the subspace $U = \{(a,\ b,\ 0) \mid a$ and b in $\mathbb{R}\}$?

Exercise 6.2.15 Describe $\text{span}\,\{\mathbf{0}\}$.

Exercise 6.2.16 Let $\mathbf{v}$ denote any vector in a vector space V. Show that $\text{span}\,\{\mathbf{v}\} = \text{span}\,\{a\mathbf{v}\}$ for any $a \neq 0$.

Exercise 6.2.17 Determine all subspaces of $\mathbb{R}\mathbf{v}$ where $\mathbf{v} \neq \mathbf{0}$ in some vector space V.

Exercise 6.2.18 Suppose $V = \text{span}\,\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$. If $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where the a_i are in $\mathbb{R}$ and $a_1 \neq 0$, show that $V = \text{span}\,\{\mathbf{u},\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$.

Exercise 6.2.19 If $\mathbf{M}_{nn} = \text{span}\,\{A_1,\ A_2,\ \dots,\ A_k\}$, show that $\mathbf{M}_{nn} = \text{span}\,\{A_1^T,\ A_2^T,\ \dots,\ A_k^T\}$.

Exercise 6.2.20 If $\mathbf{P}_n = \text{span}\{p_1(x),\ p_2(x),\ \dots,\ p_k(x)\}$ and a is in $\mathbb{R}$, show that $p_i(a) \neq 0$ for some i.

Exercise 6.2.21 Let U be a subspace of a vector space V.

a. If $a\mathbf{u}$ is in U where $a \neq 0$, show that $\mathbf{u}$ is in U.

b. If $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ are in U, show that $\mathbf{v}$ is in U.

Exercise 6.2.22 Let U be a nonempty subset of a vector space V. Show that U is a subspace of V if and only if $\mathbf{u}_1 + a\mathbf{u}_2$ lies in U for all $\mathbf{u}_1$ and $\mathbf{u}_2$ in U and all a in $\mathbb{R}$.

Exercise 6.2.23 Let $U = \{p(x) \text{ in } \mathbf{P} \mid p(3) = 0\}$ be the set in Example 6.2.4. Use the factor theorem (see Section 6.5) to show that U consists of multiples of $x-3$; that is, show that $U = \{(x-3)q(x) \mid q(x) \in \mathbf{P}\}$. Use this to show that U is a subspace of $\mathbf{P}$.

Exercise 6.2.24 Let $A_1,\ A_2,\ \dots,\ A_m$ denote $n \times n$ matrices. If $\mathbf{0} \neq \mathbf{y} \in \mathbb{R}^n$ and $A_1\mathbf{y} = A_2\mathbf{y} = \cdots = A_m\mathbf{y} = \mathbf{0}$, show that $\{A_1,\ A_2,\ \dots,\ A_m\}$ cannot span $\mathbf{M}_{nn}$.

Exercise 6.2.25 Let $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ and $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \dots,\ \mathbf{u}_n\}$ be sets of vectors in a vector space, and let

$$X = \left[\begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array}\right] \quad Y = \left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{array}\right]$$

as in Exercise 6.1.18.

a. Show that $\text{span}\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\} \subseteq \text{span}\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_n\}$ if and only if $AY = X$ for some $n \times n$ matrix A.

b. If $X = AY$ where A is invertible, show that $\text{span}\{\mathbf{v}_1,\ \dots,\ \mathbf{v}_n\} = \text{span}\{\mathbf{u}_1,\ \dots,\ \mathbf{u}_n\}$.

Exercise 6.2.26 If U and W are subspaces of a vector space V, let $U \cup W = \{\mathbf{v} \mid \mathbf{v} \text{ is in } U \text{ or } \mathbf{v} \text{ is in } W\}$. Show that $U \cup W$ is a subspace if and only if $U \subseteq W$ or $W \subseteq U$.

Exercise 6.2.27 Show that $\mathbf{P}$ cannot be spanned by a finite set of polynomials.

6.3 Linear Independence and Dimension

Definition 6.4 Linear Independence and Dependence

*As in $\mathbb{R}^n$, a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ in a vector space V is called **linearly independent** (or simply **independent**) if it satisfies the following condition:*

$$\textit{If} \quad s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n = \mathbf{0}, \qquad \textit{then} \quad s_1 = s_2 = \cdots = s_n = 0.$$

*A set of vectors that is not linearly independent is said to be **linearly dependent** (or simply **dependent**).*

The **trivial linear combination** of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$ is the one with every coefficient zero:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$$

This is obviously one way of expressing $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n$, and they are linearly independent when it is the *only* way.