6.5 An Application to Polynomials

The vector space of all polynomials of degree at most n is denoted $\mathbf{P}_n$, and it was established in Section 6.3 that $\mathbf{P}_n$ has dimension $n+1$; in fact, $\{1, x, x^2, \dots, x^n\}$ is a basis. More generally, *any* $n+1$ polynomials of distinct degrees form a basis, by Theorem 6.4.4 (they are independent by Example 6.3.4). This proves

Theorem 6.5.1

Let $p_0(x), p_1(x), p_2(x), \dots, p_n(x)$ be polynomials in $\mathbf{P}_n$ of degrees $0, 1, 2, \dots, n$, respectively. Then $\{p_0(x), \dots, p_n(x)\}$ is a basis of $\mathbf{P}_n$.

An immediate consequence is that $\{1, (x-a), (x-a)^2, \dots, (x-a)^n\}$ is a basis of $\mathbf{P}_n$ for any number a. Hence we have the following:

Corollary 6.5.1

If a is any number, every polynomial $f(x)$ of degree at most n has an expansion in powers of $(x-a)$:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n \tag{6.2}$$

If $f(x)$ is evaluated at $x = a$, then equation (6.2) becomes

$$f(x) = a_0 + a_1(a-a) + \cdots + a_n(a-a)^n = a_0$$

Hence $a_0 = f(a)$, and equation (6.2) can be written $f(x) = f(a) + (x-a)g(x)$, where $g(x)$ is a polynomial of degree $n-1$ (this assumes that $n \geq 1$). If it happens that $f(a) = 0$, then it is clear that $f(x)$ has the form $f(x) = (x-a)g(x)$. Conversely, every such polynomial certainly satisfies $f(a) = 0$, and we obtain:

Corollary 6.5.2

Let $f(x)$ be a polynomial of degree $n \geq 1$ and let a be any number. Then:

Remainder Theorem

1. *$f(x) = f(a) + (x-a)g(x)$ for some polynomial $g(x)$ of degree $n-1$.*

Factor Theorem

2. *$f(a) = 0$ if and only if $f(x) = (x-a)g(x)$ for some polynomial $g(x)$.*

The polynomial $g(x)$ can be computed easily by using "long division" to divide $f(x)$ by $(x-a)$—see Appendix D.

All the coefficients in the expansion (6.2) of $f(x)$ in powers of $(x-a)$ can be determined in terms of the derivatives of $f(x)$.[6] These will be familiar to students of calculus. Let $f^{(n)}(x)$ denote the nth derivative

[6] The discussion of Taylor's theorem can be omitted with no loss of continuity.

of the polynomial $f(x)$, and write $f^{(0)}(x) = f(x)$. Then, if

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$

it is clear that $a_0 = f(a) = f^{(0)}(a)$. Differentiation gives

$$f^{(1)}(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1}$$

and substituting $x = a$ yields $a_1 = f^{(1)}(a)$. This continues to give $a_2 = \frac{f^{(2)}(a)}{2!}$, $a_3 = \frac{f^{(3)}(a)}{3!}$, $\dots$, $a_k = \frac{f^{(k)}(a)}{k!}$, where $k!$ is defined as $k! = k(k-1)\cdots 2 \cdot 1$. Hence we obtain the following:

Corollary 6.5.3: Taylor's Theorem

If $f(x)$ is a polynomial of degree n, then

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example 6.5.1

Expand $f(x) = 5x^3 + 10x + 2$ as a polynomial in powers of $x - 1$.

Solution. The derivatives are $f^{(1)}(x) = 15x^2 + 10$, $f^{(2)}(x) = 30x$, and $f^{(3)}(x) = 30$. Hence the Taylor expansion is

$$\begin{aligned} f(x) &= f(1) + \frac{f^{(1)}(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 \\ &= 17 + 25(x-1) + 15(x-1)^2 + 5(x-1)^3 \end{aligned}$$

Taylor's theorem is useful in that it provides a formula for the coefficients in the expansion. It is dealt with in calculus texts and will not be pursued here.

Theorem 6.5.1 produces bases of $\mathbf{P}_n$ consisting of polynomials of distinct degrees. A different criterion is involved in the next theorem.

Theorem 6.5.2

Let $f_0(x), f_1(x), \dots, f_n(x)$ be nonzero polynomials in $\mathbf{P}_n$. Assume that numbers $a_0, a_1, \dots, a_n$ exist such that

$$\begin{aligned} f_i(a_i) &\neq 0 \quad \text{for each } i \\ f_i(a_j) &= 0 \quad \text{if } i \neq j \end{aligned}$$

Then

1. *$\{f_0(x), \dots, f_n(x)\}$ is a basis of $\mathbf{P}_n$.*
2. *If $f(x)$ is any polynomial in $\mathbf{P}_n$, its expansion as a linear combination of these basis vectors is*

$$f(x) = \frac{f(a_0)}{f_0(a_0)}f_0(x) + \frac{f(a_1)}{f_1(a_1)}f_1(x) + \cdots + \frac{f(a_n)}{f_n(a_n)}f_n(x)$$

Proof.

1. It suffices (by Theorem 6.4.4) to show that $\{f_0(x), \dots, f_n(x)\}$ is linearly independent (because $\dim \mathbf{P}_n = n+1$). Suppose that

$$r_0 f_0(x) + r_1 f_1(x) + \cdots + r_n f_n(x) = 0, \ r_i \in \mathbb{R}$$

Because $f_i(a_0) = 0$ for all $i > 0$, taking $x = a_0$ gives $r_0 f_0(a_0) = 0$. But then $r_0 = 0$ because $f_0(a_0) \neq 0$. The proof that $r_i = 0$ for $i > 0$ is analogous.

2. By (1), $f(x) = r_0 f_0(x) + \cdots + r_n f_n(x)$ for *some* numbers r_i. Once again, evaluating at a_0 gives $f(a_0) = r_0 f_0(a_0)$, so $r_0 = f(a_0)/f_0(a_0)$. Similarly, $r_i = f(a_i)/f_i(a_i)$ for each i. □

Example 6.5.2

Show that $\{x^2 - x, x^2 - 2x, x^2 - 3x + 2\}$ is a basis of $\mathbf{P}_2$.

Solution. Write $f_0(x) = x^2 - x = x(x-1)$, $f_1(x) = x^2 - 2x = x(x-2)$, and $f_2(x) = x^2 - 3x + 2 = (x-1)(x-2)$. Then the conditions of Theorem 6.5.2 are satisfied with $a_0 = 2$, $a_1 = 1$, and $a_2 = 0$.

We investigate one natural choice of the polynomials $f_i(x)$ in Theorem 6.5.2. To illustrate, let a_0, a_1, and a_2 be distinct numbers and write

$$f_0(x) = \frac{(x-a_1)(x-a_2)}{(a_0-a_1)(a_0-a_2)} \qquad f_1(x) = \frac{(x-a_0)(x-a_2)}{(a_1-a_0)(a_1-a_2)} \qquad f_2(x) = \frac{(x-a_0)(x-a_1)}{(a_2-a_0)(a_2-a_1)}$$

Then $f_0(a_0) = f_1(a_1) = f_2(a_2) = 1$, and $f_i(a_j) = 0$ for $i \neq j$. Hence Theorem 6.5.2 applies, and because $f_i(a_i) = 1$ for each i, the formula for expanding any polynomial is simplified.

In fact, this can be generalized with no extra effort. If $a_0, a_1, \dots, a_n$ are distinct numbers, define the **Lagrange polynomials** $\delta_0(x), \delta_1(x), \dots, \delta_n(x)$ relative to these numbers as follows:

$$\delta_k(x) = \frac{\prod_{i \neq k}(x-a_i)}{\prod_{i \neq k}(a_k-a_i)} \qquad k = 0, 1, 2, \dots, n$$

Here the numerator is the product of all the terms $(x-a_0), (x-a_1), \dots, (x-a_n)$ with $(x-a_k)$ omitted, and a similar remark applies to the denominator. If $n = 2$, these are just the polynomials in the preceding paragraph. For another example, if $n = 3$, the polynomial $\delta_1(x)$ takes the form

$$\delta_1(x) = \frac{(x-a_0)(x-a_2)(x-a_3)}{(a_1-a_0)(a_1-a_2)(a_1-a_3)}$$

In the general case, it is clear that $\delta_i(a_i) = 1$ for each i and that $\delta_i(a_j) = 0$ if $i \neq j$. Hence Theorem 6.5.2 specializes as Theorem 6.5.3.

Theorem 6.5.3: Lagrange Interpolation Expansion

Let $a_0, a_1, \dots, a_n$ be distinct numbers. The corresponding set

$$\{\delta_0(x), \delta_1(x), \dots, \delta_n(x)\}$$

of Lagrange polynomials is a basis of $\mathbf{P}_n$, and any polynomial $f(x)$ in $\mathbf{P}_n$ has the following unique expansion as a linear combination of these polynomials.

$$f(x) = f(a_0)\delta_0(x) + f(a_1)\delta_1(x) + \cdots + f(a_n)\delta_n(x)$$

Example 6.5.3

Find the Lagrange interpolation expansion for $f(x) = x^2 - 2x + 1$ relative to $a_0 = -1$, $a_1 = 0$, and $a_2 = 1$.

Solution. The Lagrange polynomials are

$$\begin{aligned}
\delta_0 &= \tfrac{(x-0)(x-1)}{(-1-0)(-1-1)} = \tfrac{1}{2}(x^2 - x) \\
\delta_1 &= \tfrac{(x+1)(x-1)}{(0+1)(0-1)} = -(x^2 - 1) \\
\delta_2 &= \tfrac{(x+1)(x-0)}{(1+1)(1-0)} = \tfrac{1}{2}(x^2 + x)
\end{aligned}$$

Because $f(-1) = 4$, $f(0) = 1$, and $f(1) = 0$, the expansion is

$$f(x) = 2(x^2 - x) - (x^2 - 1)$$

The Lagrange interpolation expansion gives an easy proof of the following important fact.

Theorem 6.5.4

Let $f(x)$ be a polynomial in $\mathbf{P}_n$, and let $a_0, a_1, \dots, a_n$ denote distinct numbers. If $f(a_i) = 0$ for all i, then $f(x)$ is the zero polynomial (that is, all coefficients are zero).

Proof. All the coefficients in the Lagrange expansion of $f(x)$ are zero. □

Exercises for 6.5

Exercise 6.5.1 If polynomials $f(x)$ and $g(x)$ satisfy $f(a) = g(a)$, show that $f(x) - g(x) = (x-a)h(x)$ for some polynomial $h(x)$.

Exercises 6.5.2, 6.5.3, 6.5.4, and 6.5.5 require polynomial differentiation.

Exercise 6.5.2 Expand each of the following as a polynomial in powers of $x - 1$.

a. $f(x) = x^3 - 2x^2 + x - 1$

b. $f(x) = x^3 + x + 1$

c. $f(x) = x^4$

d. $f(x) = x^3 - 3x^2 + 3x$

Exercise 6.5.3 Prove Taylor's theorem for polynomials.

Exercise 6.5.4 Use Taylor's theorem to derive the **binomial theorem**:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Here the **binomial coefficients** $\binom{n}{r}$ are defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n! = n(n-1)\cdots 2 \cdot 1$ if $n \geq 1$ and $0! = 1$.

Exercise 6.5.5 Let $f(x)$ be a polynomial of degree n. Show that, given any polynomial $g(x)$ in $\mathbf{P}_n$, there exist numbers $b_0, b_1, \dots, b_n$ such that

$$g(x) = b_0 f(x) + b_1 f^{(1)}(x) + \cdots + b_n f^{(n)}(x)$$

where $f^{(k)}(x)$ denotes the kth derivative of $f(x)$.

Exercise 6.5.6 Use Theorem 6.5.2 to show that the following are bases of $\mathbf{P}_2$.

a. $\{x^2 - 2x, x^2 + 2x, x^2 - 4\}$

b. $\{x^2 - 3x + 2, x^2 - 4x + 3, x^2 - 5x + 6\}$

Exercise 6.5.7 Find the Lagrange interpolation expansion of $f(x)$ relative to $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$ if:

a. $f(x) = x^2 + 1$

b. $f(x) = x^2 + x + 1$

Exercise 6.5.8 Let $a_0, a_1, \dots, a_n$ be distinct numbers. If $f(x)$ and $g(x)$ in $\mathbf{P}_n$ satisfy $f(a_i) = g(a_i)$ for all i, show that $f(x) = g(x)$. [*Hint*: See Theorem 6.5.4.]

Exercise 6.5.9 Let $a_0, a_1, \dots, a_n$ be distinct numbers. If $f(x) \in \mathbf{P}_{n+1}$ satisfies $f(a_i) = 0$ for each $i = 0, 1, \dots, n$, show that $f(x) = r(x-a_0)(x-a_1)\cdots(x-a_n)$ for some r in $\mathbb{R}$. [*Hint*: r is the coefficient of x^{n+1} in $f(x)$. Consider $f(x) - r(x-a_0)\cdots(x-a_n)$ and use Theorem 6.5.4.]

Exercise 6.5.10 Let a and b denote distinct numbers.

a. Show that $\{(x-a), (x-b)\}$ is a basis of $\mathbf{P}_1$.

b. Show that $\{(x-a)^2, (x-a)(x-b), (x-b)^2\}$ is a basis of $\mathbf{P}_2$.

c. Show that $\{(x-a)^n, (x-a)^{n-1}(x-b), \dots, (x-a)(x-b)^{n-1}, (x-b)^n\}$ is a basis of $\mathbf{P}_n$. [*Hint*: If a linear combination vanishes, evaluate at $x = a$ and $x = b$. Then reduce to the case $n-2$ by using the fact that if $p(x)q(x) = 0$ in $\mathbf{P}$, then either $p(x) = 0$ or $q(x) = 0$.]

Exercise 6.5.11 Let a and b be two distinct numbers. Assume that $n \geq 2$ and let

$$U_n = \{f(x) \text{ in } \mathbf{P}_n \mid f(a) = 0 = f(b)\}.$$

a. Show that

$$U_n = \{(x-a)(x-b)p(x) \mid p(x) \text{ in } \mathbf{P}_{n-2}\}$$

b. Show that $\dim U_n = n - 1$.

[*Hint*: If $p(x)q(x) = 0$ in $\mathbf{P}$, then either $p(x) = 0$, or $q(x) = 0$.]

c. Show $\{(x-a)^{n-1}(x-b), (x-a)^{n-2}(x-b)^2, \dots, (x-a)^2(x-b)^{n-2}, (x-a)(x-b)^{n-1}\}$ is a basis of U_n. [*Hint*: Exercise 6.5.10.]

6.6 An Application to Differential Equations

Call a function $f : \mathbb{R} \to \mathbb{R}$ **differentiable** if it can be differentiated as many times as we want. If f is a differentiable function, the nth derivative $f^{(n)}$ of f is the result of differentiating n times. Thus $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f^{(1)\prime}$, ... and, in general, $f^{(n+1)} = f^{(n)\prime}$ for each $n \geq 0$. For small values of n these are often written as $f, f', f'', f''', \dots$.

If a, b, and c are numbers, the differential equations

$$f'' + af' + bf = 0 \quad \text{or} \quad f''' + af'' + bf' + cf = 0$$

are said to be of **second-order** and **third-order**, respectively. In general, an equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + a_{n-2}f^{(n-2)} + \cdots + a_2 f^{(2)} + a_1 f^{(1)} + a_0 f^{(0)} = 0, \quad a_i \text{ in } \mathbb{R} \tag{6.3}$$

is called a **differential equation of order** n. In this section we investigate the set of solutions to (6.3) and, if n is 1 or 2, find explicit solutions. Of course an acquaintance with calculus is required.

Let f and g be solutions to (6.3). Then $f + g$ is also a solution because $(f + g)^{(k)} = f^{(k)} + g^{(k)}$ for all k, and af is a solution for any a in $\mathbb{R}$ because $(af)^{(k)} = af^{(k)}$. It follows that the set of solutions to (6.3) is a vector space, and we ask for the dimension of this space.

We have already dealt with the simplest case (see Theorem 3.5.1):

Theorem 6.6.1

The set of solutions of the first-order differential equation $f' + af = 0$ is a one-dimensional vector space and $\{e^{-ax}\}$ is a basis.

There is a far-reaching generalization of Theorem 6.6.1 that will be proved in Theorem 7.4.1.

Theorem 6.6.2

The set of solutions to the nth order equation (6.3) has dimension n.

Remark

Every differential equation of order n can be converted into a system of n linear first-order equations (see Exercises 3.5.6 and 3.5.7). In the case that the matrix of this system is diagonalizable, this approach provides a proof of Theorem 6.6.2. But if the matrix is not diagonalizable, Theorem 7.4.1 is required.

Theorem 6.6.1 suggests that we look for solutions to (6.3) of the form $e^{\lambda x}$ for some number λ. This is a good idea. If we write $f(x) = e^{\lambda x}$, it is easy to verify that $f^{(k)}(x) = \lambda^k e^{\lambda x}$ for each $k \geq 0$, so substituting f in (6.3) gives

$$(\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda^1 + a_0)e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$ for all x, this shows that $e^{\lambda x}$ is a solution of (6.3) if and only if λ is a root of the **characteristic polynomial** $c(x)$, defined to be

$$c(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$