Exercise 7.3.27 Let $T : V \to W$ be a linear transformation, where V and W are finite dimensional.

a. Show that T is one-to-one if and only if there exists a linear transformation $S : W \to V$ with $ST = 1_V$. [*Hint*: If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of V and T is one-to-one, show that W has a basis $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n), \mathbf{f}_{n+1}, \dots, \mathbf{f}_{n+k}\}$ and use Theorem 7.1.2 and Theorem 7.1.3.]

b. Show that T is onto if and only if there exists a linear transformation $S : W \to V$ with $TS = 1_W$. [*Hint*: Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$. Use Theorem 7.2.5, Theorem 7.1.2 and Theorem 7.1.3.]

Exercise 7.3.28 Let S and T be linear transformations $V \to W$, where $\dim V = n$ and $\dim W = m$.

a. Show that $\ker S = \ker T$ if and only if $T = RS$ for some isomorphism $R : W \to W$. [*Hint*: Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker S = \ker T$. Use Theorem 7.2.5 to extend $\{S(\mathbf{e}_1), \dots, S(\mathbf{e}_r)\}$ and $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ to bases of W.]

b. Show that $\operatorname{im} S = \operatorname{im} T$ if and only if $T = SR$ for some isomorphism $R : V \to V$. [*Hint*: Show that $\dim(\ker S) = \dim(\ker T)$ and choose bases $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_r, \dots, \mathbf{f}_n\}$ of V where $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ are bases of $\ker S$ and $\ker T$, respectively. If $1 \le i \le r$, show that $S(\mathbf{e}_i) = T(\mathbf{g}_i)$ for some $\mathbf{g}_i$ in V, and prove that $\{\mathbf{g}_1, \dots, \mathbf{g}_r, \mathbf{f}_{r+1}, \dots, \mathbf{f}_n\}$ is a basis of V.]

Exercise 7.3.29 If $T : V \to V$ is a linear transformation where $\dim V = n$, show that $TST = T$ for some isomorphism $S : V \to V$. [*Hint*: Let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ be as in Theorem 7.2.5. Extend $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ to a basis of V, and use Theorem 7.3.1, Theorem 7.1.2 and Theorem 7.1.3.]

Exercise 7.3.30 Let A and B denote $m \times n$ matrices. In each case show that (1) and (2) are equivalent.

a. (1) A and B have the same null space. (2) $B = PA$ for some invertible $m \times m$ matrix P.

b. (1) A and B have the same range. (2) $B = AQ$ for some invertible $n \times n$ matrix Q.

[*Hint*: Use Exercise 7.3.28.]

7.4 A Theorem about Differential Equations

Differential equations are instrumental in solving a variety of problems throughout science, social science, and engineering. In this brief section, we will see that the set of solutions of a linear differential equation (with constant coefficients) is a vector space and we will calculate its dimension. The proof is pure linear algebra, although the applications are primarily in analysis. However, a key result (Lemma 7.4.3 below) can be applied much more widely.

We denote the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ by f', and f will be called **differentiable** if it can be differentiated any number of times. If f is a differentiable function, the nth derivative $f^{(n)}$ of f is the result of differentiating n times. Thus $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f^{(1)\prime}$, $\dots$, and in general $f^{(n+1)} = f^{(n)\prime}$ for each $n \ge 0$. For small values of n these are often written as $f, f', f'', f''', \dots$.

If a, b, and c are numbers, the differential equations

$$f'' - af' - bf = 0 \quad \text{or} \quad f''' - af'' - bf' - cf = 0$$

are said to be of **second order** and **third-order**, respectively. In general, an equation

$$f^{(n)} - a_{n-1}f^{(n-1)} - a_{n-2}f^{(n-2)} - \dots - a_2f^{(2)} - a_1f^{(1)} - a_0f^{(0)} = 0, \ a_i \text{ in } \mathbb{R} \tag{7.3}$$

is called a **differential equation of order** n. We want to describe all solutions of this equation. Of course a knowledge of calculus is required.

The set $\mathbf{F}$ of all functions $\mathbb{R} \to \mathbb{R}$ is a vector space with operations as described in Example 6.1.7. If f and g are differentiable, we have $(f+g)' = f' + g'$ and $(af)' = af'$ for all a in $\mathbb{R}$. With this it is a routine matter to verify that the following set is a subspace of $\mathbf{F}$:

$$\mathbf{D}_n = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable and is a solution to (7.3)}\}$$

Our sole objective in this section is to prove

Theorem 7.4.1

The space $\mathbf{D}_n$ has dimension n.

As will be clear later, the proof of Theorem 7.4.1 requires that we enlarge $\mathbf{D}_n$ somewhat and allow our differentiable functions to take values in the set $\mathbb{C}$ of complex numbers. To do this, we must clarify what it means for a function $f : \mathbb{R} \to \mathbb{C}$ to be differentiable. For each real number x write $f(x)$ in terms of its real and imaginary parts $f_r(x)$ and $f_i(x)$:

$$f(x) = f_r(x) + i f_i(x)$$

This produces new functions $f_r : \mathbb{R} \to \mathbb{R}$ and $f_i : \mathbb{R} \to \mathbb{R}$, called the **real** and **imaginary parts** of f, respectively. We say that f is **differentiable** if both f_r and f_i are differentiable (as real functions), and we define the **derivative** f' of f by

$$f' = f_r' + i f_i' \tag{7.4}$$

We refer to this frequently in what follows.[4]

With this, write $\mathbf{D}_\infty$ for the set of all differentiable complex valued functions $f : \mathbb{R} \to \mathbb{C}$. This is a *complex* vector space using pointwise addition (see Example 6.1.7), and the following scalar multiplication: For any w in $\mathbb{C}$ and f in $\mathbf{D}_\infty$, we define $wf : \mathbb{R} \to \mathbb{C}$ by $(wf)(x) = wf(x)$ for all x in $\mathbb{R}$. We will be working in $\mathbf{D}_\infty$ for the rest of this section. In particular, consider the following complex subspace of $\mathbf{D}_\infty$:

$$\mathbf{D}_n^* = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is a solution to (7.3)}\}$$

Clearly, $\mathbf{D}_n \subseteq \mathbf{D}_n^*$, and our interest in $\mathbf{D}_n^*$ comes from

Lemma 7.4.1

If $\dim_{\mathbb{C}}(\mathbf{D}_n^*) = n$, then $\dim_{\mathbb{R}}(\mathbf{D}_n) = n$.

Proof. Observe first that if $\dim_{\mathbb{C}}(\mathbf{D}_n^*) = n$, then $\dim_{\mathbb{R}}(\mathbf{D}_n^*) = 2n$. [In fact, if $\{g_1, \dots, g_n\}$ is a $\mathbb{C}$-basis of $\mathbf{D}_n^*$ then $\{g_1, \dots, g_n, ig_1, \dots, ig_n\}$ is a $\mathbb{R}$-basis of $\mathbf{D}_n^*$]. Now observe that the set $\mathbf{D}_n \times \mathbf{D}_n$ of all ordered pairs (f, g) with f and g in $\mathbf{D}_n$ is a real vector space with componentwise operations. Define

$$\theta : \mathbf{D}_n^* \to \mathbf{D}_n \times \mathbf{D}_n \quad \text{given by} \quad \theta(f) = (f_r, f_i) \text{ for } f \text{ in } \mathbf{D}_n^*$$

[4]Write $|w|$ for the absolute value of any complex number w. As for functions $\mathbb{R} \to \mathbb{R}$, we say that $\lim_{t\to 0} f(t) = w$ if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t) - w| <\in$ whenever $|t| < \delta$. (Note that t represents a real number here.) In particular, given a real number x, we define the *derivative* f' of a function $f : \mathbb{R} \to \mathbb{C}$ by $f'(x) = \lim_{t\to 0} \left\{\frac{1}{t}[f(x+t) - f(x)]\right\}$ and we say that f is *differentiable* if $f'(x)$ exists for all x in $\mathbb{R}$. Then we can *prove* that f is differentiable if and only if both f_r and f_i are differentiable, and that $f' = f_r' + i f_i'$ in this case.

One verifies that θ is onto and one-to-one, and it is $\mathbb{R}$-linear because $f \to f_r$ and $f \to f_i$ are both $\mathbb{R}$-linear. Hence $\mathbf{D}_n^* \cong \mathbf{D}_n \times \mathbf{D}_n$ as $\mathbb{R}$-spaces. Since $\dim_{\mathbb{R}}(\mathbf{D}_n^*)$ is finite, it follows that $\dim_{\mathbb{R}}(\mathbf{D}_n)$ is finite, and we have

$$2\dim_{\mathbb{R}}(\mathbf{D}_n) = \dim_{\mathbb{R}}(\mathbf{D}_n \times \mathbf{D}_n) = \dim_{\mathbb{R}}(\mathbf{D}_n^*) = 2n$$

Hence $\dim_{\mathbb{R}}(\mathbf{D}_n) = n$, as required. □

It follows that to prove Theorem 7.4.1 it suffices to show that $\dim_{\mathbb{C}}(\mathbf{D}_n^*) = n$.

There is one function that arises frequently in any discussion of differential equations. Given a complex number $w = a + ib$ (where a and b are real), we have $e^w = e^a(\cos b + i\sin b)$. The law of exponents, $e^w e^v = e^{w+v}$ for all w, v in $\mathbb{C}$ is easily verified using the formulas for $\sin(b+b_1)$ and $\cos(b+b_1)$. If x is a variable and $w = a+ib$ is a complex number, define the **exponential function** e^{wx} by

$$e^{wx} = e^{ax}(\cos bx + i\sin bx)$$

Hence e^{wx} is differentiable because its real and imaginary parts are differentiable for all x. Moreover, the following can be proved using (7.4):

$$(e^{wx})' = we^{wx}$$

In addition, (7.4) gives the **product rule** for differentiation:

$$\text{If } f \text{ and } g \text{ are in } \mathbf{D}_\infty, \text{ then } (fg)' = f'g + fg'$$

We omit the verifications.

To prove that $\dim_{\mathbb{C}}(\mathbf{D}_n^*) = n$, two preliminary results are required. Here is the first.

Lemma 7.4.2

Given f in $\mathbf{D}_\infty$ and w in $\mathbb{C}$, there exists g in $\mathbf{D}_\infty$ such that $g' - wg = f$.

Proof. Define $p(x) = f(x)e^{-wx}$. Then p is differentiable, whence p_r and p_i are both differentiable, hence continuous, and so both have antiderivatives, say $p_r = q_r'$ and $p_i = q_i'$. Then the function $q = q_r + iq_i$ is in $\mathbf{D}_\infty$, and $q' = p$ by (7.4). Finally define $g(x) = q(x)e^{wx}$. Then

$$g' = q'e^{wx} + qwe^{wx} = pe^{wx} + w(qe^{wx}) = f + wg$$

by the product rule, as required. □

The second preliminary result is important in its own right.

Lemma 7.4.3: Kernel Lemma

Let V be a vector space, and let S and T be linear operators $V \to V$. If S is onto and both $\ker(S)$ and $\ker(T)$ are finite dimensional, then $\ker(TS)$ is also finite dimensional and $\dim[\ker(TS)] = \dim[\ker(T)] + \dim[\ker(S)]$.

Proof. Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a basis of $\ker(T)$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of $\ker(S)$. Since S is onto, let $\mathbf{u}_i = S(\mathbf{w}_i)$ for some $\mathbf{w}_i$ in V. It suffices to show that

$$B = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

is a basis of $\ker(TS)$. Note $B \subseteq \ker(TS)$ because $TS(\mathbf{w}_i) = T(\mathbf{u}_i) = \mathbf{0}$ for each i and $TS(\mathbf{v}_j) = T(\mathbf{0}) = \mathbf{0}$ for each j.

Spanning. If $\mathbf{v}$ is in $\ker(TS)$, then $S(\mathbf{v})$ is in $\ker(T)$, say $S(\mathbf{v}) = \sum r_i \mathbf{u}_i = \sum r_i S(\mathbf{w}_i) = S(\sum r_i \mathbf{w}_i)$. It follows that $\mathbf{v} - \sum r_i \mathbf{w}_i$ is in $\ker(S) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, proving that $\mathbf{v}$ is in $\text{span}(B)$.

Independence. Let $\sum r_i \mathbf{w}_i + \sum t_j \mathbf{v}_j = \mathbf{0}$. Applying S, and noting that $S(\mathbf{v}_j) = \mathbf{0}$ for each j, yields $\mathbf{0} = \sum r_i S(\mathbf{w}_i) = \sum r_i \mathbf{u}_i$. Hence $r_i = 0$ for each i, and so $\sum t_j \mathbf{v}_j = \mathbf{0}$. This implies that each $t_j = 0$, and so proves the independence of B. □

Proof of Theorem 7.4.1. By Lemma 7.4.1, it suffices to prove that $\dim_{\mathbb{C}}(\mathbf{D}_n^*) = n$. This holds for $n = 1$ because the proof of Theorem 3.5.1 goes through to show that $\mathbf{D}_1^* = \mathbb{C}e^{a_0 x}$. Hence we proceed by induction on n. With an eye on equation (7.3), consider the polynomial

$$p(t) = t^n - a_{n-1}t^{n-1} - a_{n-2}t^{n-2} - \cdots - a_2 t^2 - a_1 t - a_0$$

(called the *characteristic polynomial* of equation (7.3)). Now define a map $D : \mathbf{D}_\infty \to \mathbf{D}_\infty$ by $D(f) = f'$ for all f in $\mathbf{D}_\infty$. Then D is a linear operator, whence $p(D) : \mathbf{D}_\infty \to \mathbf{D}_\infty$ is also a linear operator. Moreover, since $D^k(f) = f^{(k)}$ for each $k \geq 0$, equation (7.3) takes the form $p(D)(f) = 0$. In other words,

$$\mathbf{D}_n^* = \ker[p(D)]$$

By the fundamental theorem of algebra,[5] let w be a complex root of $p(t)$, so that $p(t) = q(t)(t - w)$ for some complex polynomial $q(t)$ of degree $n - 1$. It follows that $p(D) = q(D)(D - w1_{\mathbf{D}_\infty})$. Moreover $D - w1_{\mathbf{D}_\infty}$ is onto by Lemma 7.4.2, $\dim_{\mathbb{C}}[\ker(D - w1_{\mathbf{D}_\infty})] = 1$ by the case $n = 1$ above, and $\dim_{\mathbb{C}}(\ker[q(D)]) = n - 1$ by induction. Hence Lemma 7.4.3 shows that $\ker[P(D)]$ is also finite dimensional and

$$\dim_{\mathbb{C}}(\ker[p(D)]) = \dim_{\mathbb{C}}(\ker[q(D)]) + \dim_{\mathbb{C}}(\ker[D - w1_{\mathbf{D}_\infty}]) = (n-1) + 1 = n.$$

Since $\mathbf{D}_n^* = \ker[p(D)]$, this completes the induction, and so proves Theorem 7.4.1. □

7.5 More on Linear Recurrences[6]

In Section 3.4 we used diagonalization to study linear recurrences, and gave several examples. We now apply the theory of vector spaces and linear transformations to study the problem in more generality.

Consider the linear recurrence

$$x_{n+2} = 6x_n - x_{n+1} \quad \text{for } n \geq 0$$

If the initial values x_0 and x_1 are prescribed, this gives a sequence of numbers. For example, if $x_0 = 1$ and $x_1 = 1$ the sequence continues

$$x_2 = 5,\ x_3 = 1,\ x_4 = 29,\ x_5 = -23,\ x_6 = 197,\ \dots$$

[5]This is the reason for allowing our solutions to (7.3) to be *complex* valued.

[6]This section requires only Sections 7.1-7.3.