Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-1. Matrix Addition, Scalar Multiplication and Transposition

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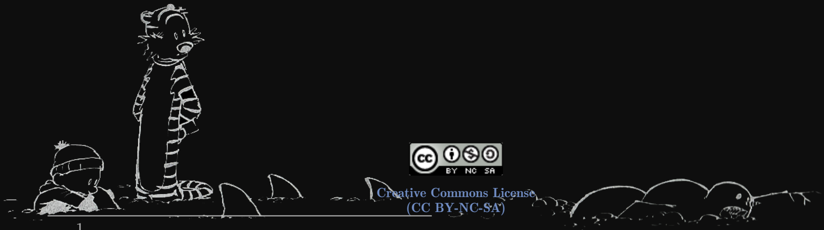

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Matrices – Definitions and Basic Properties

Matrix Addition

Scalar Multiplication

The Transpose

Matrices – Definitions and Basic Properties

Definition

Let m and n be positive integers.

- An $m \times n$ matrix is a rectangular array of numbers having m rows and n columns. Such a matrix is said to have size $m \times n$.
- A row matrix (or row) is a $1 \times n$ matrix, and a column matrix (or column) is an $m \times 1$ matrix.
- A square matrix is an $n \times n$ matrix.
- The (i, j)-entry of a matrix is the entry in row i and column j. For a matrix A, the (i, j)-entry of A is often written as a_{ij}.

General notation for an $m \times n$ matrix, A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Remark (Basic Properties)

1. Equality: two matrices are equal if and only if they have the same size and the corresponding entries are equal.
2. Zero Matrix: an $m \times n$ matrix with all entries equal to zero.
3. Addition: matrices must have the same size; add corresponding entries.
4. Scalar Multiplication: multiply each entry of the matrix by the scalar.
5. Negative of a Matrix: for an $m \times n$ matrix A, its negative is denoted $-A$ and $-A = (-1)A$.
6. Subtraction: for $m \times n$ matrices A and B, $A - B = A + (-1)B$.

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Matrix Addition

Definition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where C is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & -2 \\ 6 & 1 \end{bmatrix}$. Then,

$$\begin{aligned} A + B &= \begin{bmatrix} 1+0 & 3+-2 \\ 2+6 & 5+1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 8 & 6 \end{bmatrix} \end{aligned}$$

Theorem (Properties of Matrix Addition)

Let A, B and C be $m \times n$ matrices. Then the following properties hold.

1. $A + B = B + A$ (matrix addition is commutative).
2. $(A + B) + C = A + (B + C)$ (matrix addition is associative).
3. There exists an $m \times n$ zero matrix, 0, such that $A + 0 = A$. (existence of an additive identity).
4. There exists an $m \times n$ matrix $-A$ such that $A + (-A) = 0$. (existence of an additive inverse).

Scalar Multiplication

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let k be a scalar. Then $kA = [ka_{ij}]$.

Example

Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & -2 \\ 0 & 4 & 5 \end{bmatrix}$.

Then

$$\begin{aligned} 3A &= \begin{bmatrix} 3(2) & 3(0) & 3(-1) \\ 3(3) & 3(1) & 3(-2) \\ 3(0) & 3(4) & 3(5) \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & -3 \\ 9 & 3 & -6 \\ 0 & 12 & 15 \end{bmatrix} \end{aligned}$$

Theorem (Properties of Scalar Multiplication)

Let A, B be $m \times n$ matrices and let $k, p \in \mathbb{R}$ (scalars). Then the following properties hold.

1. $k(A + B) = kA + kB$.
   (scalar multiplication distributes over matrix addition).

2. $(k + p)A = kA + pA$.
   (addition distributes over scalar multiplication).

3. $k(pA) = (kp)A$. (scalar multiplication is associative).

4. $1A = A$. (existence of a multiplicative identity).

Example

$$2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -16 & -4 \\ 13 & 3 \end{bmatrix}$$

Problem

Let A and B be $m \times n$ matrices. Simplify the expression

$$2[9(A - B) + 7(2B - A)] - 2[3(2B + A) - 2(A + 3B) - 5(A + B)]$$

Solution

$$\begin{aligned} & 2[9(A - B) + 7(2B - A)] - 2[3(2B + A) - 2(A + 3B) - 5(A + B)] \\ = \ & 2(9A - 9B + 14B - 7A) - 2(6B + 3A - 2A - 6B - 5A - 5B) \\ = \ & 2(2A + 5B) - 2(-4A - 5B) \\ = \ & 12A + 20B \end{aligned}$$

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Matrix Transpose

Definition

If A is an $m \times n$ matrix, then its transpose, denoted A^T, is the $n \times m$ whose i^{th} row is the i^{th} column of A, $1 \leq i \leq n$; i.e., if $A = [a_{ij}]$, then

$$A^T = [a_{ij}]^T = [a_{ji}]$$

i.e., the (i, j)-entry of A^T is the (j, i)-entry of A.

Theorem (Properties of the Transpose of a Matrix)

Let A and B be $m \times n$ matrices, C be a $n \times p$ matrix, and $r \in \mathbb{R}$ a scalar. Then

1. $(A^T)^T = A$
2. $(rA)^T = rA^T$
3. $(A + B)^T = A^T + B^T$
4. $(AC)^T = C^T A^T$

To prove each these properties, you only need to compute the (i, j)-entries of the matrices on the left-hand side and the right-hand side. And you can do it!

Problem

Find the matrix A if $\left(A+3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix}\right)^T=\begin{bmatrix}2 & 1\\ 0 & 5\\ 3 & 8\end{bmatrix}$.

Solution

$$\left[\left(A+3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix}\right)^T\right]^T = \begin{bmatrix}2 & 1\\ 0 & 5\\ 3 & 8\end{bmatrix}^T$$

$$A+3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix} = \begin{bmatrix}2 & 0 & 3\\ 1 & 5 & 8\end{bmatrix}$$

$$A = \begin{bmatrix}2 & 0 & 3\\ 1 & 5 & 8\end{bmatrix} - 3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix}$$

$$A = \begin{bmatrix}-1 & 3 & 3\\ -2 & -1 & -4\end{bmatrix}$$

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the main diagonal of A.

Definition (Symmetric Matrices)

The matrix A is called symmetric if and only if $A^T = A$. Note that this immediately implies that A is a square matrix.

Examples

$$\begin{bmatrix} 2 & -3 \\ -3 & 17 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 5 \\ 0 & 2 & 11 \\ 5 & 11 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 5 & -1 \\ 2 & 1 & -3 & 0 \\ 5 & -3 & 2 & -7 \\ -1 & 0 & -7 & 4 \end{bmatrix}$$

are symmetric matrices, and each is symmetric about its main diagonal.

Definition

An $n \times n$ matrix A is said to be skew symmetric if $A^T = -A$.

Example (Skew Symmetric Matrices)

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 9 & 4 \\ -9 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix}$$

Problem

Show that if A is a square matrix, then $A - A^T$ is skew-symmetric.

Solution

We must show that $(A - A^T)^T = -(A - A^T)$. Using the properties of matrix addition, scalar multiplication, and transposition

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$