Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra

§2-2. Equations, Matrices, and Transformations

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Vectors

Matrix Vector Multiplication

The Dot Product

Transformations

Rotations in $\mathbb{R}^2$

Vectors

Definitions

A row matrix or column matrix is often called a vector, and such matrices are referred to as row vectors and column vectors, respectively. If $\vec{x}$ is a row vector of size $1 \times n$, and $\vec{y}$ is a column vector of size $m \times 1$, then we write

$$\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Definition (Vector form of a system of linear equations)

Consider the system of linear equations

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

Such a system can be expressed in vector form or as a vector equation by using linear combinations of column vectors:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Problem

Express the following system of linear equations in vector form:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$

Solution

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Matrix vector multiplication

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$, written $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$, and let $\vec{x}$ be an $n \times 1$ column vector,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product of matrix A and (column) vector $\vec{x}$ is the $m \times 1$ column vector given by

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \sum_{j=1}^{n} x_j\vec{a}_j$$

that is, $A\vec{x}$ is a linear combination of the columns of A.

Problem

Compute the product $A\vec{x}$ for

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solution

$$A\vec{x} = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

Problem

Compute $A\vec{y}$ for

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$

Solution

$$A\vec{y} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

Definition (Matrix form of a system of linear equations)

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Such a system can be expressed in matrix form using matrix vector multiplication,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Thus a system of linear equations can be expressed as a matrix equation

$$A\vec{x} = \vec{b},$$

where A is the coefficient matrix, $\vec{b}$ is the constant matrix, and $\vec{x}$ is the matrix of variables.

Problem

Express the following system of linear equations in matrix form.

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$

Solution

$$\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Theorem

1. Every system of m linear equations in n variables can be written in the form $A\vec{x} = \vec{b}$ where A is the coefficient matrix, $\vec{x}$ is the matrix of variables, and $\vec{b}$ is the constant matrix.

Theorem (continued)

2. The system $A\vec{x} = \vec{b}$ is consistent (i.e., has at least one solution) if and only if $\vec{b}$ is a linear combination of the columns of A.

Theorem (continued)

3. The vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the system $A\vec{x} = \vec{b}$ if and only if $x_1, x_2, \ldots, x_n$ are a solution to the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n = \vec{b}$$

where $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are the columns of A.

Problem

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Express $\vec{b}$ as a linear combination of the columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ of A, or show that this is impossible.

Solution

Solve the system $A\vec{x} = \vec{b}$ where $\vec{x}$ is a column vector with four entries. Do so by putting the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ in reduced row-echelon form.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{array}\right] \to \cdots \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 1/7 \\ 0 & 1 & 0 & 1 & -5/7 \\ 0 & 0 & 1 & -1 & 3/7 \end{array}\right]$$

Since there are infinitely many solutions (x_4 is assigned a parameter), choose any value for x_4. Choosing $x_4 = 0$ (which is the simplest thing to do) gives us

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{7}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{7}\vec{a}_1 - \frac{5}{7}\vec{a}_2 + \frac{3}{7}\vec{a}_3 + 0\vec{a}_4.$$

■

Remark

The problem may ask to to find all possible linear combinations of the columns $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$ of A.

This is equivalent to find all solutions to the corresponding system of linear equations:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} - s \\ -\frac{5}{7} - s \\ \frac{3}{7} + s \\ s \end{bmatrix}$$

Hence, all possible linear combinations are:

$$\vec{b} = \left(\frac{1}{7} - s\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{5}{7} + s\right) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{7} + s\right) \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Theorem

Let A and B be $m \times n$ matrices, and let $\vec{x}$ and $\vec{y}$ be n-vectors in $\mathbb{R}^n$. Then:

1. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$.
2. $A(a\vec{x}) = a(A\vec{x}) = (aA)\vec{x}$ for all scalars a.
3. $(A+B)\vec{x} = A\vec{x} + B\vec{x}$.

This provides a useful way to describe the solutions to a system $A\vec{x} = \vec{b}$.

Structure of solutions:

$$\text{General solution} = \text{Sol. to the Homog. Eq.} + \text{A Particular Solution}.$$

$$A\vec{x} = A\left(\vec{x}_0 + \vec{x}_1\right) = \underbrace{A\vec{x}_0}_{\vec{x}_0:\ \text{homogeneous sol.}} + \underbrace{A\vec{x}_1}_{\vec{x}_1:\ \text{particular sol.}} = \vec{0} + \vec{b} = \vec{b}.$$

- Vectors
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- Transformations
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The Dot Product

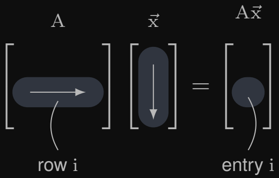
Definition

If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two ordered n-tuples, their dot product is defined to be the number

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

obtained by multiplying corresponding entries and adding the results.

This give an alternative way to carry out the matrix-vector product $A\vec{x}$.



$$A\vec{x}$$

$$\|$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\|$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + x_4 \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \qquad \text{(Def.)}$$

$$\|$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \end{bmatrix} \qquad \text{(Alternative)}$$

Problem

If $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$, compute $A\vec{x}$.

Solution

The entries of $A\vec{x}$ are the dot products of the rows of A with $\vec{x}$:

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 & + & 0(-1) & + & 2 \cdot 1 & + & (-1)4 \\ 2 \cdot 2 & + & (-1)(-1) & + & 0 \cdot 1 & + & 1 \cdot 4 \\ 3 \cdot 2 & + & 1(-1) & + & 3 \cdot 1 & + & 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}. \end{aligned}$$

Of course, this agrees with the outcome of the previous example. ■

Definition (Identity Matrix)

For each $n > 2$, the identity matrix I_n is the $n \times n$ matrix with 1's on the main diagonal (upper left to lower right), and zeros elsewhere.

Example

The first few identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \ldots$$

Problem

Show that $I_n\vec{x} = \vec{x}$ for each n-vector $\vec{x}$ in $\mathbb{R}^n$, $n \geq 1$.

Solution

We verify the case $n = 4$. Given the 4-vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ the dot product rule gives

$$I_4\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 0 + 0 + 0 \\ 0 + x_2 + 0 + 0 \\ 0 + 0 + x_3 + 0 \\ 0 + 0 + 0 + x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{x}.$$

In general, $I_n\vec{x} = \vec{x}$ because entry k of $I_n\vec{x}$ is the dot product of row k of I_n with $\vec{x}$, and row k of I_n has 1 in position k and zeros elsewhere. ∎

Vectors

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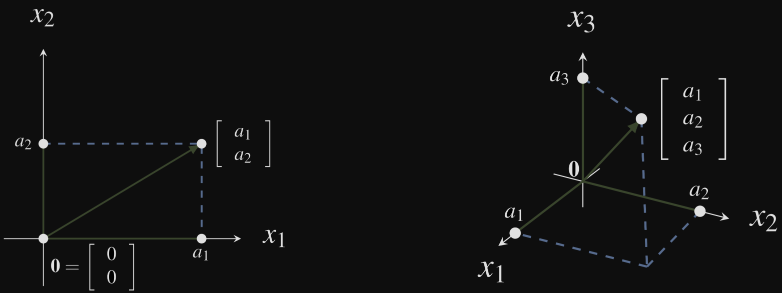
Notation and Terminology

- We have already used $\mathbb{R}$ to denote the set of real numbers.
- We use $\mathbb{R}^2$ to the denote the set of all column vectors of length two, and we use $\mathbb{R}^3$ to the denote the set of all column vectors of length three (the length of a vector is the number of entries it contains).
- In general, we write $\mathbb{R}^n$ for the set of all column vectors of length n.

$\mathbb{R}^2$ and $\mathbb{R}^3$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric interpretations as $\mathrm{position\ vectors}$ of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.



x_2
a_2
$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$
x_1
a_1
$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
x_3
a_3
$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$
$\mathbf{0}$
a_2
a_1
x_2
x_1

Definition (Transformations)

A transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, sometimes written $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, and is called a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $m = n$, then we say T is a transformation of $\mathbb{R}^n$.

What do we mean by a function?

Informally, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that, for each vector in $\mathbb{R}^n$, assigns exactly one vector of $\mathbb{R}^m$

We use the notation $T(\vec{x})$ to mean the transformation T applied to the vector $\vec{x}$.

Definition

If T acts by matrix multiplication of a matrix A (such as the previous example), we call T a matrix transformation, and write $T_A(\vec{x}) = A\vec{x}$.

Definition (Equality of Transformations)

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\vec{x}) = T(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

Example (Specifying the action of a transformation)

$T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

is a transformation that transforms the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$ into the vector

$$T\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 4+7 \\ 1-7 \\ 7-4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -6 \\ 3 \end{bmatrix}.$$

Example (Transformation by matrix multiplication)

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. By matrix multiplication, A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$. Consider the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Transforming this vector by A looks like:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}.$$

For example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

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Rotations in $\mathbb{R}^2$

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} \text{ for each } \vec{x} \in \mathbb{R}^n$$

is called the matrix transformation induced by A.

Definition

The transformation

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

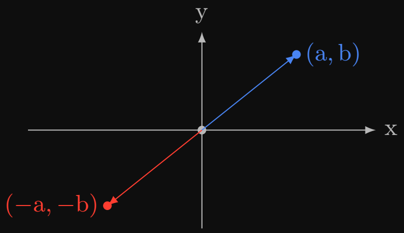
denotes counterclockwise rotation about the origin through an angle of θ.

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π.



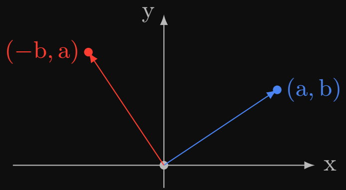
We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Example (Rotation through $\pi/2$)

We denote by

$$R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

Remark

In general, the rotation (counterclockwise) about the origin for an angle θ is

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

and

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos(\theta) - b\sin(\theta) \\ a\sin(\theta) + b\cos(\theta) \end{bmatrix}$$

$$R_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$