Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-4. Matrix Inverses

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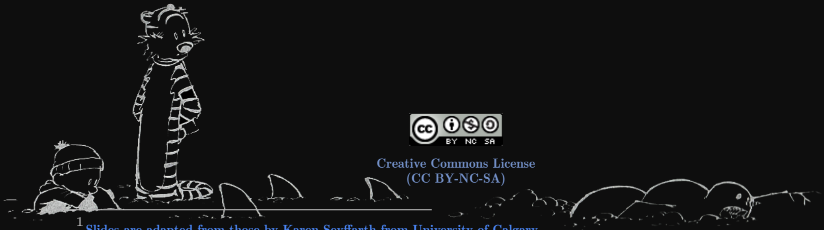

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

The Identity and Inverse Matrices

Finding the Inverse of a Matrix

Properties of the Inverse

Inverse of Transformations

The Identity and Inverse Matrices

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The Identity and Inverse Matrices

Definition

For each $n \geq 2$, the $n \times n$ identity matrix, denoted I_n, is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition

Let $n \geq 2$. For each j, $1 \leq j \leq n$, we denote by $\vec{e}_j$ the j^{th} column of I_n.

Example

When $n = 3$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Theorem

Let A be an $m \times n$ matrix. Then $AI_n = A$ and $I_mA = A$.

Proof.

The (i, j)-entry of AI_n is the product of the i^{th} row of $A = [a_{ij}]$, namely $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{bmatrix}$ with the j^{th} column of I_n, namely $\vec{e}_j$. Since $\vec{e}_j$ has a one in row j and zeros elsewhere,

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{bmatrix} \vec{e}_j = a_{ij}$$

Since this is true for all $i \leq m$ and all $j \leq n$, $AI_n = A$.

The proof of $I_mA = A$ is analogous—work it out! ■

Instead of AI_n and I_mA we often write AI and IA, respectively, since the size of the identity matrix is clear from the context: the sizes of A and I must be compatible for matrix multiplication.

Thus

$$AI = A \quad \text{and} \quad IA = A$$

which is why I is called an identity matrix – it is an identity for matrix multiplication.

Definition (Matrix Inverses)

Let A be an $n \times n$ matrix. Then B is an inverse of A if and only if $AB = I_n$ and $BA = I_n$.

Remark

Note that since A and I_n are both $n \times n$, B must also be an $n \times n$ matrix.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, B is an inverse of A.

Problem

Does every square matrix have an inverse?

Solution

No! Take e.g. the zero matrix $\mathbf{O_n}$ (all entries of $\mathbf{O_n}$ are equal to 0)

$$A\mathbf{O_n} = \mathbf{O_n}A = \mathbf{O_n}$$

for all $n \times n$ matrices A: The (i, j)-entry of $\mathbf{O_n}A$ is equal to $\sum_{k=1}^{n} 0a_{kj} = 0$. ■

Problem

Does every nonzero square matrix have an inverse?

Problem

Does the following matrix A have an inverse?

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution

No! To see this, suppose

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is an inverse of A. Then

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$$

which is never equal to I_2. (Why?) ■

Theorem (Uniqueness of an Inverse)

If A is a square matrix and B and C are inverses of A, then $B = C$.

Proof.

Since B and C are inverses of A, $AB = I = BA$ and $AC = I = CA$. Then

$$C = CI = C(AB) = CAB$$

and

$$B = IB = (CA)B = CAB$$

so $B = C$. ∎

Example (revisited)

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, we saw that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The preceding theorem tells us that B is the inverse of A, rather than just an inverse of A.

Remark (notation)

Let A be a square matrix, i.e., an $n \times n$ matrix.

- The inverse of A, if it exists, is denoted A^{-1}, and

$$AA^{-1} = I = A^{-1}A$$

- If A has an inverse, then we say that A is invertible.

Finding the inverse of a 2×2 matrix

Example

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then there is a formula for A^{-1}:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This can easily be verified by computing the products AA^{-1} and $A^{-1}A$.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Showing that $A^{-1}A = I_2$ is left as an exercise.

Remark

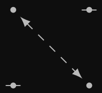
Here are some terminology related to this example:

1. Determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - cd$$

2. Adjugate:

$$\operatorname{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$



Problem

Suppose that A is any $n \times n$ matrix.

- How do we know whether or not A^{-1} exists?
- If A^{-1} exists, how do we find it?

Solution

The matrix inversion algorithm!

Although the formula for the inverse of a 2×2 matrix is quicker and easier to use than the matrix inversion algorithm, the general formula for the inverse an $n \times n$ matrix, $n \geq 3$ (which we will see later), is more complicated and difficult to use than the matrix inversion algorithm. To find inverses of square matrices that are not 2×2, the matrix inversion algorithm is the most efficient method to use.

The Matrix Inversion Algorithm

Let A be an $n \times n$ matrix. To find A^{-1}, if it exists,

Step 1 take the $n \times 2n$ matrix

$$\left[\begin{array}{c|c} A & I_n \end{array}\right]$$

obtained by augmenting A with the $n \times n$ identity matrix, I_n.

Step 2 Perform elementary row operations to transform $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ into a reduced row-echelon matrix.

Theorem (Matrix Inverses)

Let A be an $n \times n$ matrix. Then the following conditions are equivalent.

1. A is invertible.
2. the reduced row-echelon form on A is I.
3. $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ can be transformed into $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$ using the Matrix Inversion Algorithm.

Problem

Find, if possible, the inverse of $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$.

Solution

Using the matrix inversion algorithm

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

From this, we see that A has no inverse. ■

Problem

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists.

Solution

Using the matrix inversion algorithm

$$\left[\,A \mid I\,\right] = \left[\begin{array}{rrr|rrr} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 25 & -3 & 5 & 4 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{10}{25} & 0 & -\frac{5}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] = \left[\,I \mid A^{-1}\,\right]$$

Solution (continued)

Therefore, A^{-1} exists, and

$$A^{-1} = \begin{bmatrix} \frac{10}{25} & 0 & -\frac{5}{25} \\ \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 10 & 0 & -5 \\ 1 & -10 & 7 \\ -3 & 5 & 4 \end{bmatrix}.$$

$\blacksquare$

Remark

It is always a good habit to check your answer by computing AA^{-1} and $A^{-1}A$.

One can use matrix inverse to solve $A\vec{x} = \vec{b}$ when there are n linear equations in n variables, i.e., A is a square matrix.

Example

The system of linear equations

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

can be written in matrix form as $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

You can check that $A^{-1} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$.

Example (continued)

Since A^{-1} exists and has the property that $A^{-1}A = I$, we obtain the following.

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}(A\vec{x}) &= A^{-1}\vec{b} \\ (A^{-1}A)\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

i.e., $A\vec{x} = \vec{b}$ has the unique solution given by $\vec{x} = A^{-1}\vec{b}$. Therefore,

$$\vec{x} = A^{-1}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

is the unique solution to the system. ∎

Remark

The last example illustrates another method for solving a system of linear equations when the coefficient matrix is square and invertible. Unless that coefficient matrix is 2×2, this is generally NOT an efficient method for solving a system of linear equations.

Example

Let A, B and C be matrices, and suppose that A is invertible.

1. If $AB = AC$, then

$$\begin{aligned} A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ IB &= IC \\ B &= C \end{aligned}$$

2. If $BA = CA$, then

$$\begin{aligned} (BA)A^{-1} &= (CA)A^{-1} \\ B(AA^{-1}) &= C(AA^{-1}) \\ BI &= CI \\ B &= C \end{aligned}$$

Problem

Can you find square matrices A, B and C for which $AB = AC$ but $B \neq C$?

The Identity and Inverse Matrices

Finding the Inverse of a Matrix

Properties of the Inverse

Inverse of Transformations

Properties of the Inverse

Example

Suppose A is an invertible matrix. What is the $(A^T)^{-1}$?
We need to find:

$$A^T \boxed{???} = \boxed{???} A^T = I.$$

Notice that

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Hence, $\boxed{???} = (A^{-1})^T$, i.e., $(A^T)^{-1} = (A^{-1})^T$. ∎

Properties of the Inverse

Example

Suppose A and B are invertible $n \times n$ matrices. What is $(AB)^{-1}$?

We need to find:

$$(AB)\,\boxed{???} = \boxed{???}\,(AB) = I.$$

Notice that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Hence, $\boxed{???} = B^{-1}A^{-1}$, i.e., $(AB)^{-1} = B^{-1}A^{-1}$. ■

The previous two examples prove the first two parts of the following theorem.

Theorem (Properties of Inverses)

1. If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$.
2. If A and B are invertible matrices, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

3. If $A_1, A_2, \ldots, A_k$ are invertible, then $A_1 A_2 \cdots A_k$ is invertible and

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}.$$

Theorem (More Properties of Inverses)

1. I is invertible, and $I^{-1} = I$.
2. If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.
3. If A is invertible, so is A^k, and $(A^k)^{-1} = (A^{-1})^k$.
   (A^k means A multiplied by itself k times)
4. If A is invertible and $p \in \mathbb{R}$ is nonzero, then pA is invertible, and $(pA)^{-1} = \frac{1}{p}A^{-1}$.

Example

Given $(3\mathrm{I} - \mathrm{A}^{\mathrm{T}})^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, we wish to find the matrix A. Taking inverses of both sides of the equation:

$$\begin{aligned} 3\mathrm{I} - \mathrm{A}^{\mathrm{T}} &= \left(2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\right)^{-1} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \\ &= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Example (continued)

$$
\begin{aligned}
3I - A^T &= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \\
-A^T &= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} - 3I \\
-A^T &= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\
-A^T &= \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -1 & -\frac{5}{2} \end{bmatrix} \\
A &= \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & \frac{5}{2} \end{bmatrix}
\end{aligned}
$$

Problem

True or false? Justify your answer.

$$\text{If } A^3 = 4I, \text{ then } A \text{ is invertible.}$$

Solution

To show A is invertible, We need to find:

$$A \boxed{???} = \boxed{???} A = I.$$

Because $A^3 = 4I$, we see that

$$\frac{1}{4}A^3 = I$$

so

$$(\frac{1}{4}A^2)A = I \quad \text{and} \quad A(\frac{1}{4}A^2) = I.$$

Therefore, A is invertible, and $\boxed{???} = \frac{1}{4}A^2$, i.e., $A^{-1} = \frac{1}{4}A^2$. ■

Theorem (Inverse Theorem)

Let A be an $n \times n$ matrix, and let $\vec{x}$, $\vec{b}$ be $n \times 1$ vectors. The following conditions are equivalent.

1. A is invertible.
2. The rank of A is n.
3. The reduced row echelon form of A is I_n.
4. $A\vec{x} = \vec{0}$ has only the trivial solution, $\vec{x} = \vec{0}$.
5. A can be transformed to I_n by elementary row operations.
6. The system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for any choice of $\vec{b}$.
7. The system $A\vec{x} = \vec{b}$ has at least one solution $\vec{x}$ for any choice of $\vec{b}$.
8. There exists an $n \times n$ matrix C with the property that $CA = I_n$.
9. There exists an $n \times n$ matrix C with the property that $AC = I_n$.

$$\begin{array}{ccccccccccc} (1) & \Leftrightarrow & (2) & \Leftrightarrow & (3) & \Leftrightarrow & (4) & \Leftrightarrow & (5) & \Leftrightarrow & (6) \\ \Updownarrow & & & & & & \Uparrow & & & & \Downarrow \\ (9) & & & & & & (8) & & \Leftarrow & & (7) \end{array}$$

Proof.

(1), (2), (4), (5) and (6) are all equivalent.

(6) $\Rightarrow$ (7) is clear. As for (7) $\Rightarrow$ (8), let $\vec{c}_j$ be one of the solution of $A\vec{x} = \vec{e}_j$. The

$$A[\vec{c}_1, \cdots, \vec{c}_n] = [\vec{e}_1, \cdots, \vec{e}_n] = I$$

Hence, (8) holds with $C = [\vec{c}_1, \cdots, \vec{c}_n]$.

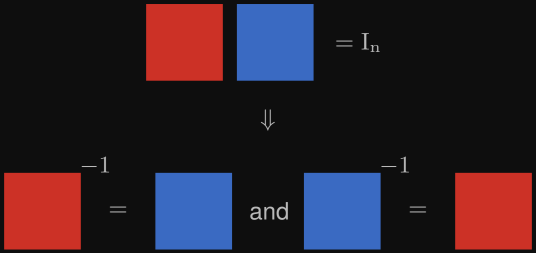
(1) $\Rightarrow$ (8) and (9): Using $C = A^{-1}$.

(8) $\Rightarrow$ (4): Whenever $\vec{x}$ is a sol. i.e., $A\vec{x} = \vec{0}$, then $\vec{x} = I\vec{x} = CA\vec{x} = C\vec{0} = \vec{0}$. Hence, $\vec{0}$ is the only solution. (4) holds true.

(9) $\Rightarrow$ (1): By reversing the roles of A and C and apply (8) to see that C is invertible. Thus A is the inverse of C, and hence A is itself invertible. $\blacksquare$

Corollary

If A and B are $n \times n$ matrices such that $AB = I$, then $BA = I$. Furthermore, A and B are invertible, with $B = A^{-1}$ and $A = B^{-1}$.



Remark

Important Fact In Corollary, it is essential that the matrices be square.

Theorem

If A and B are matrices such that $AB = I$ and $BA = I$, then A and B are square matrices (of the same size).

Example

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \neq I_3.$$

Remark

This example illustrates why "an inverse" of a non-square matrix doesn't make sense. If A is $m \times n$ and B is $n \times m$, where $m \neq n$, then even if $AB = I$, it will never be the case that $BA = I$.

The Identity and Inverse Matrices

Finding the Inverse of a Matrix

Properties of the Inverse

Inverse of Transformations

Inverse of Transformations

Definition

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ are transformations such that for each $\vec{x} \in \mathbb{R}^n$,

$$(S \circ T)(\vec{x}) = \vec{x} \quad \text{and} \quad (T \circ S)(\vec{x}) = \vec{x}.$$

Then T and S are invertible transformations; S is called an inverse of T, and T is called an inverse of S. (Geometrically, S reverses the action of T, and T reverses the action of S.)

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix transformation induced by matrix A. Then we have:

1. A is invertible if and only if T has an inverse.
2. In the case where T has an inverse, the inverse is unique and is denoted T^{-1}.
3. Furthermore, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is induced by the matrix A^{-1}.

Fundamental Identities relating T and T^{-1}

1. $T^{-1} \circ T = 1_{\mathbb{R}^n}$
2. $T \circ T^{-1} = 1_{\mathbb{R}^n}$

Example

Let $\mathrm{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a transformation given by

$$\mathrm{T}\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} x+y \\ y \end{array}\right]$$

Then T is a linear transformation induced by $\mathrm{A} = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$.

Notice that the matrix A is invertible. Therefore the transformation T has an inverse, T^{-1}, induced by

$$\mathrm{A}^{-1} = \left[\begin{array}{cr} 1 & -1 \\ 0 & 1 \end{array}\right]$$

Example (continued)

Consider the action of T and T^{-1}:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix};$$

$$T^{-1}\begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore,

$$T^{-1}\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

■