Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-5. Elementary Matrices

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Elementary Matrices

Inverses of elementary matrices

Smith Normal Form

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Definition

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Remark (Three Types of Elementary Row Operations)

($\sim$ bases for genomic sequences)

- Type I: Interchange two rows.
- Type II: Multiply a row by a nonzero number.
- Type III: Add a (nonzero) multiple of one row to a different row.

Example

Switch the 2nd row and the 4th row

Multiply -2 to the 3rd row

Add -3 multiple of 1st row to the 3rd row

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

are examples of elementary matrices of types I, II and III, respectively.

Example (continued)

Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$$

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We are interested in the effect that (left) multiplication of A by E, F and G has on the matrix A.

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We are interested in the effect that (left) multiplication of A by E, F and G has on the matrix A. Computing EA, FA, and GA . . .

Example (continued)

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$$

Switch the 2nd row
and the 4th row

Example (continued)

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$$

Switch the 2nd row and the 4th row

$$FA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ -6 & -6 \\ 4 & 4 \end{bmatrix}$$

Multiply -2 to the 3rd row

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$$GA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$$

Add -3 multiple of 1st row to the 3rd row

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Remark

The elementary matrices are the programmed receipts for your cooking!

Theorem (Multiplication by an Elementary Matrix)

Let A be an $m \times n$ matrix.

If B is obtained from A by performing one single elementary row operation,

then $B = EA$

where E is the elementary matrix obtained from I_m by performing the same elementary operation on I_m as was performed on A.

$$\begin{array}{c} A \longrightarrow B \\ \text{El. Op.} \\ I \longrightarrow E \end{array} \qquad \Longrightarrow \qquad A = EB$$

Problem

Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

Find elementary matrices E and F so that $C = FEA$.

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Note. The statement of the problem implies that C can be obtained from A by a sequence of two elementary row operations, represented by elementary matrices E and F.

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where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

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where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.Thus we have the sequence $A \rightarrow EA \rightarrow F(EA) = C$, so $C = FEA$, i.e.,

$$\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}.$$

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Every elementary matrix E is invertible, and E^{-1} is also an elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E.

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The following table gives the inverse of each type of elementary row operation:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q

Note that elementary matrices of type I are self-inverse.

Inverses of Elementary Matrices

Example

Without using the matrix inversion algorithm, find the inverse of the elementary matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Hint. What row operation can be applied to G to transform it to I_4?

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$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Check by computing $G^{-1}G$.

Example (continued)

Similarly,

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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and

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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To find U so that $B = UA$, we could find $E_1, E_2, \ldots, E_k$ and multiply these together (in the correct order), but there is an easier method for finding U.

Definition

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1. there exists an invertible $m \times m$ matrix U such that $B = UA$;
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3. $U = E_k E_{k-1} \cdots E_2 E_1$, where $E_1, E_2, \ldots, E_k$ are elementary matrices corresponding, in order, to the elementary row operations used to obtain B from A.

Problem

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Solution

$$\left[\begin{array}{rrr|rr} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & -3 & -2 & -2 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array}\right] \to \left[\begin{array}{rrr|rr} 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array}\right]$$

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Therefore $R = UA$, where

$$U = \begin{bmatrix} 1/3 & 0 \\ 2/3 & -1 \end{bmatrix}.$$

■

Example (A Matrix as a product of elementary matrices)

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$$A = \begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix}.$$

Suppose we do row operations to put A in reduced row-echelon form, and write down the corresponding elementary matrices.

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[E_1]{} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[E_2]{} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_3]{}$$

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_4]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_5]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that the reduced row-echelon form of A equals I_3. Now find the matrices E_1, E_2, E_3, E_4 and E_5.

Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\begin{aligned} (E_5(E_4(E_3(E_2(E_1 A))))) &= I \\ (E_5 E_4 E_3 E_2 E_1) A &= I \end{aligned}$$

and therefore

$$A^{-1} = E_5 E_4 E_3 E_2 E_1$$

Example (continued)

Since $A^{-1} = E_5E_4E_3E_2E_1$,

$$\begin{aligned} A^{-1} &= E_5E_4E_3E_2E_1 \\ (A^{-1})^{-1} &= (E_5E_4E_3E_2E_1)^{-1} \\ A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \end{aligned}$$

Example (continued)

Since $A^{-1} = E_5E_4E_3E_2E_1$,

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This example illustrates the following result.

Example (continued)

Since $A^{-1} = E_5E_4E_3E_2E_1$,

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This example illustrates the following result.

Theorem

Let A be an $n \times n$ matrix. Then, A^{-1} exists if and only if A can be written as the product of elementary matrices.

Example (revisited – Matrix inversion algorithm)

$$[\ A \mid I\ ] = \left[\begin{array}{rrr|c} 1 & 2 & -4 & \\ -3 & -6 & 13 & I \\ 0 & -1 & 2 & \end{array}\right]$$

$$E_1 [\ A \mid I\ ] = \left[\begin{array}{rrr|c} 1 & 2 & -4 & \\ 0 & 0 & 1 & E_1 \\ 0 & -1 & 2 & \end{array}\right] = \left[\begin{array}{rrr|rrr} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$E_2E_1 [\ A \mid I\ ] = \left[\begin{array}{rrr|c} 1 & 2 & -4 & \\ 0 & -1 & 2 & E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{rrr|rrr} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

Example (continued)

$$E_3E_2E_1[\,A \mid I\,] = \left[\begin{array}{ccc|c} 1 & 2 & -4 & \\ 0 & 1 & -2 & E_3E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

$$E_4E_3E_2E_1[\,A \mid I\,] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & -2 & E_4E_3E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

$$E_5E_4E_3E_2E_1[\,A \mid I\,] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & E_5E_4E_3E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 6 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

$$A^{-1} = E_5E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

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Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \underset{E_2}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix}$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \underset{E_2}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \underset{E_3}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_1]{} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_2]{} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \xrightarrow[E_3]{} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[E_4]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \underset{E_2}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \underset{E_3}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \underset{E_4}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

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$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \underset{E_2}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \underset{E_3}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \underset{E_4}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix},$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_1]{} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_2]{} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \xrightarrow[E_3]{} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[E_4]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix},$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \underset{E_2}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \underset{E_3}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \underset{E_4}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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with

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}, E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Since $E_4 E_3 E_2 E_1 A = I$, $A^{-1} = E_4 E_3 E_2 E_1$, and hence

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

Solution (continued)

Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1/11 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1}$$

Solution (continued)

Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1/11 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1}$$

i.e.,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$\blacksquare$

One result that we have assumed in all our work involving reduced row-echelon matrices is the following.

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Theorem (Uniqueness of the Reduced Echelon Form)

If A is an $m \times n$ matrix and R and S are reduced row-echelon forms of A, then $R = S$.

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Theorem (Uniqueness of the Reduced Echelon Form)

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Remark

This theorem ensures that the reduced row-echelon form of a matrix is unique,

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Theorem (Uniqueness of the Reduced Echelon Form)

If A is an $m \times n$ matrix and R and S are reduced row-echelon forms of A, then $R = S$.

Remark

This theorem ensures that the reduced row-echelon form of a matrix is unique, and its proof follows from the results about elementary matrices.

Elementary Matrices

Inverses of elementary matrices

Smith Normal Form

Smith Normal Form

Smith Normal Form

Definition

If A is an $m \times n$ matrix of rank r, then the matrix $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$ is called the **Smith normal form** of A.

Theorem

If A is an $m \times n$ matrix of rank r, then there exist invertible matrices U and V of size $m \times m$ and $n \times n$, respectively, such that

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$$

Proof.

1. Apply the elementary row operations:

$$[\mathrm{A}|\mathrm{I_m}] \xrightarrow{\text{e.r.o.}} [\mathrm{rref}\,(\mathrm{A})\,|\mathrm{U}]$$

2. Apply the elementary column operations:

$$\begin{pmatrix} \mathrm{rref}(\mathrm{A}) \\ \mathrm{I_n} \end{pmatrix} \xrightarrow{\text{e.c.o.}} \begin{pmatrix} \begin{pmatrix} \mathrm{I_r} & 0 \\ 0 & 0 \end{pmatrix}_{\mathrm{m\times n}} \\ \mathrm{V} \end{pmatrix}_{\mathrm{2m\times n}}$$

■

Remark

The elementary column operations above are equivalent to the elementary row operations on the transpose:

$$\left[\mathrm{rref}(\mathrm{A})^{\mathrm{T}}\middle|\mathrm{I_n}\right] \xrightarrow{\text{e.r.o.}} \left[\begin{pmatrix} \mathrm{I_r} & 0 \\ 0 & 0 \end{pmatrix}_{\mathrm{n\times m}}\middle|\mathrm{V^T}\right]_{\mathrm{n\times 2m}}$$

Problem

Find the decomposition of $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ into the Smith normal form: $A = \widetilde{U} N \widetilde{V}$, where N is the Smith normal form of A and $\widetilde{U}, \widetilde{V}$ are some invertible matrices.

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Solution

We have seen that

$$[A|I_2] = \left[\begin{array}{ccc|cc} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array}\right] = [\mathrm{rref}(A)|U]$$

Now,

$$\left(\mathrm{rref}(A)^T \,\middle|\, I_3\right) = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] = \left[N^T \middle| V^T\right]$$

Solution (Continued)

Hence, we find $\mathrm{N} = \mathrm{UAV}$, namely,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, since U and V are invertible, we see that

$$\mathrm{A} = \mathrm{U}^{-1}\mathrm{N}\mathrm{V}^{-1},$$

namely,

$$\begin{aligned} \mathrm{A} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} &= \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \widetilde{\mathrm{U}}\mathrm{N}\widetilde{\mathrm{V}}. \end{aligned}$$

$\blacksquare$