Math 221: LINEAR ALGEBRA

Chapter 3. Determinants and Diagonalization
§3-4. Application to Linear Recurrences

Le Chen[1]

Emory University, 2021 Spring

(last updated on 01/25/2021)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Recurrences

Linear Recurrences

Example

The Fibonacci Numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

and can be defined by the linear recurrence relation

$$f_{n+2} = f_{n+1} + f_n \text{ for all } n \geq 0,$$

with the initial conditions $f_0 = 1$ and $f_1 = 1$.

Problem

Find f_{100}.

Instead of using the recurrence to compute f_{100}, we'd like to find a formula for f_n that holds for all $n \geq 0$.

Definitions

A sequence of numbers $x_0, x_1, x_2, x_3, \ldots$ is defined recursively if each number in the sequence is determined by the numbers that occur before it in the sequence.
A linear recurrence of length k has the form

$$x_{n+k} = a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots + a_k x_n, n \geq 0,$$

for some real numbers $a_1, a_2, \ldots, a_k$.

Example

The simplest linear recurrence has length one, so has the form

$$x_{n+1} = ax_n \text{ for } n \geq 0,$$

with $a \in \mathbb{R}$ and some initial value x_0.
In this case,

$$\begin{array}{rcl} x_1 & = & ax_0 \\ x_2 & = & ax_1 = a^2x_0 \\ x_3 & = & ax_2 = a^3x_0 \\ \vdots & \vdots & \vdots \\ x_n & = & ax_{n-1} = a^nx_0 \end{array}$$

Therefore, $x_n = a^nx_0$.

Example

Find a formula for x_n if

$$x_{n+2} = 2x_{n+1} + 3x_n \text{ for } n \geq 0,$$

with $x_0 = 0$ and $x_1 = 1$.

Solution. Define $V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for each $n \geq 0$. Then

$$V_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and for $n \geq 0$,

$$V_{n+1} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ 2x_{n+1} + 3x_n \end{bmatrix}$$

Example (continued)

Now express $V_{n+1} = \begin{bmatrix} x_{n+1} \\ 2x_{n+1} + 3x_n \end{bmatrix}$ as a matrix product:

$$V_{n+1} = \begin{bmatrix} x_{n+1} \\ 2x_{n+1} + 3x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = AV_n$$

This is a linear dynamical system, so we can apply the techniques from §3.3, provided that A is diagonalizable.

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x & -1 \\ -3 & x-2 \end{vmatrix} = x^2 - 2x - 3 = (x-3)(x+1)$$

Therefore A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$, and is diagonalizable.

Example (continued)

$\vec{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a basic eigenvector corresponding to $\lambda_1 = 3$, and
$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a basic eigenvector corresponding to $\lambda_2 = -1$.

Furthermore $P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$ is invertible and is the diagonalizing matrix for A, and $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

Writing $P^{-1}V_0 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, we get

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Example (continued)

Therefore,

$$\begin{aligned} V_n = \left[\begin{array}{c} x_n \\ x_{n+1} \end{array} \right] &= b_1 \lambda_1^n \vec{x}_1 + b_2 \lambda_2^n \vec{x}_2 \\ &= \frac{1}{4} 3^n \left[\begin{array}{c} 1 \\ 3 \end{array} \right] + \frac{1}{4} (-1)^n \left[\begin{array}{c} -1 \\ 1 \end{array} \right], \end{aligned}$$

and so

$$x_n = \frac{1}{4} 3^n - \frac{1}{4} (-1)^n.$$

Example

Solve the recurrence relation

$$x_{k+2} = 5x_{k+1} - 6x_k, k \geq 0$$

with $x_0 = 0$ and $x_1 = 1$.

Solution. Write

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 5x_{k+1} - 6x_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$

Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

Example (continued)

A has eigenvalues $\lambda_1 = 2$ with corresponding eigenvector $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\lambda_2 = 3$ with corresponding eigenvector $\vec{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, P^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}V_0 = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Finally,

$$V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = b_1\lambda_1^k\vec{x}_1 + b_2\lambda_2^k\vec{x}_2 = (-1)2^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Example

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = (-1)2^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and therefore

$$x_k = 3^k - 2^k.$$