Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-1. Subspaces and Spanning

Le Chen[1]

Emory University, 2021 Spring

(last updated on 01/25/2021)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Subspaces of $\mathbb{R}^n$

The null space and the image space

The Eigenspace

Linear Combinations and Spanning Sets

Spanning sets of null(A) and im(A)

Subspaces of $\mathbb{R}^n$

The null space and the image space

The Eigenspace

Linear Combinations and Spanning Sets

Spanning sets of null(A) and im(A)

Subspaces of $\mathbb{R}^n$

Definitions

1. $\mathbb{R}$ denotes the set of real numbers, and is an example of a set of scalars.
2. $\mathbb{R}^n$ is the set of all n-tuples of real numbers, i.e.,

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}.$$

3. The vector space $\mathbb{R}^n$ consists of the set $\mathbb{R}^n$ written as column matrices, along with the (matrix) operations of addition and scalar multiplication. Unless stated otherwise, $\mathbb{R}^n$ means the vector space $\mathbb{R}^n$.

Remark

$\mathbb{R}^n$ is a concrete example of the abstract vector space will be studied in the next chapter.

A vectors is denoted by a lower case letter with an arrow written over it; for example, $\vec{u}$, $\vec{v}$, and $\vec{x}$ denote vectors.

Another example: $\vec{u} = \begin{bmatrix} -2 \\ 3 \\ 0.7 \\ 5 \\ \pi \end{bmatrix}$ is a vector in $\mathbb{R}^5$, written $\vec{u} \in \mathbb{R}^5$.

To save space on the page, the same vector $\vec{u}$ may be written instead as a row matrix by taking the transpose of the column:

$$\vec{u} = \begin{bmatrix} -2, & 3, & 0.7, & 5, & \pi \end{bmatrix}^T.$$

We are interested in nice subsets of $\mathbb{R}^n$, defined as follows.

Definition (Subspaces)

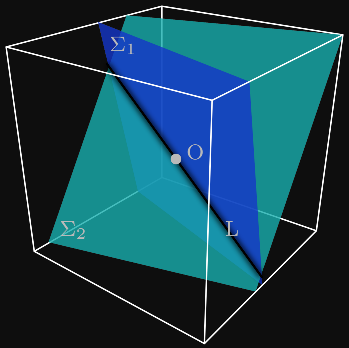
A subset U of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

- S1. The zero vector of $\mathbb{R}^n$, $\vec{0}_n$, is in U;
- S2. U is closed under addition, i.e., for all $\vec{u}, \vec{w} \in U$, $\vec{u} + \vec{w} \in U$;
- S3. U is closed under scalar multiplication, i.e., for all $\vec{u} \in U$ and $k \in \mathbb{R}$, $k\vec{u} \in U$.

Both subset $U = \left\{\vec{0}_n\right\}$ and R^n itself are subspaces of $\mathbb{R}^n$. Any other subspace of $\mathbb{R}^n$ is called a proper subspace of $\mathbb{R}^n$.

Notation

If U is a subset of $\mathbb{R}^n$, we write $U \subseteq \mathbb{R}^n$.



Σ_1
O
Σ_2
L

Example

In $\mathbb{R}^3$, the line L through the origin that is parallel to the vector $\vec{d} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$ has (vector) equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$, $t \in \mathbb{R}$, so

$$L = \left\{ t\vec{d} \mid t \in \mathbb{R} \right\}.$$

Claim. L is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3 \in L$ since $0\vec{d} = \vec{0}_3$.
- Suppose $\vec{u}, \vec{v} \in L$. Then by definition, $\vec{u} = s\vec{d}$ and $\vec{v} = t\vec{d}$, for some $s, t \in \mathbb{R}$. Thus
$$\vec{u} + \vec{v} = s\vec{d} + t\vec{d} = (s + t)\vec{d}.$$
Since $s + t \in \mathbb{R}$, $\vec{u} + \vec{v} \in L$; i.e., L is closed under addition.

Example (continued)

- Suppose $\vec{u} \in L$ and $k \in \mathbb{R}$ (k is a scalar). Then $\vec{u} = t\vec{d}$, for some $t \in \mathbb{R}$, so
$$k\vec{u} = k(t\vec{d}) = (kt)\vec{d}.$$
Since $kt \in \mathbb{R}$, $k\vec{u} \in L$; i.e., L is closed under scalar multiplication.
- Therefore, L is a subspace of $\mathbb{R}^3$.

Remark

Note that there is nothing special about the vector $\vec{d}$ used in this example; the same proof works for any nonzero vector $\vec{d} \in \mathbb{R}^3$, so any line through the origin is a subspace of $\mathbb{R}^3$.

Example

In $\mathbb{R}^3$, let M denote the plane through the origin having equation $3x - 2y + z = 0$; then M has normal vector $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. If $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then

$$M = \left\{ \vec{u} \in \mathbb{R}^3 \mid \vec{n} \cdot \vec{u} = 0 \right\},$$

where $\vec{n} \cdot \vec{u}$ is the dot product of vectors $\vec{n}$ and $\vec{u}$.

Claim. M is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3 \in M$ since $\vec{n} \cdot \vec{0}_3 = 0$.
- Suppose $\vec{u}, \vec{v} \in M$. Then by definition, $\vec{n} \cdot \vec{u} = 0$ and $\vec{n} \cdot \vec{v} = 0$, so

  $$\vec{n} \cdot (\vec{u} + \vec{v}) = n \cdot \vec{u} + n \cdot \vec{v} = 0 + 0 = 0,$$

  and thus $(\vec{u} + \vec{v}) \in M$; i.e., M is closed under addition.

Example (continued)

- Suppose $\vec{u} \in M$ and $k \in \mathbb{R}$. Then $\vec{n} \cdot \vec{u} = 0$, so

$$\vec{n} \cdot (k\vec{u}) = k(\vec{n} \cdot \vec{u}) = k(0) = 0,$$

  and thus $k\vec{u} \in M$; i.e., M is closed under scalar multiplication.
- Therefore, M is a subspace of $\mathbb{R}^3$.

Remark

As in the previous example, there is nothing special about the plane chosen for this example; any plane through the origin is a subspace of $\mathbb{R}^3$.

Problem

Is $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \ a, b, c, d \in \mathbb{R} \quad \text{and} \quad 2a - b = c + 2d \right\}$ a subspace of $\mathbb{R}^4$? Justify your answer.

Solution

The zero vector of $\mathbb{R}^4$ is the vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ with $a = b = c = d = 0$.

In this case, $2a - b = 2(0) + 0 = 0$ and $c + 2d = 0 + 2(0) = 0$, so $2a - b = c + 2d$. Therefore, $\vec{0}_4 \in U$.

Solution (continued)

Suppose

$$\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \text{ are in } U.$$

Then $2a_1 - b_1 = c_1 + 2d_1$ and $2a_2 - b_2 = c_2 + 2d_2$. Now

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix},$$

and

$$\begin{aligned} 2(a_1 + a_2) - (b_1 + b_2) &= (2a_1 - b_1) + (2a_2 - b_2) \\ &= (c_1 + 2d_1) + (c_2 + 2d_2) \\ &= (c_1 + c_2) + 2(d_1 + d_2). \end{aligned}$$

Therefore, $\vec{v}_1 + \vec{v}_2 \in U$.

Solution (continued)

Finally, suppose

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U \quad \text{and} \quad k \in \mathbb{R}.$$

Then $2a - b = c + 2d$. Now

$$k\vec{v} = k \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \\ kd \end{bmatrix},$$

and

$$2ka - kb = k(2a - b) = k(c + 2d) = kc + 2kd.$$

Therefore, $k\vec{v} \in U$.

It follows from the Subspace Test that U is a subspace of $\mathbb{R}^4$.

Problem

Is $\mathrm{U} = \left\{ \begin{bmatrix} 1 \\ \mathrm{s} \\ \mathrm{t} \end{bmatrix} \;\middle|\; \mathrm{s}, \mathrm{t} \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

Solution

Note that $\vec{0}_3 \notin \mathrm{U}$, and thus U is not a subspace of $\mathbb{R}^3$.

(You could also show that U is not closed under addition, or not closed under scalar multiplication.)

Problem

Is $\mathrm{U} = \left\{ \begin{bmatrix} r \\ 0 \\ s \end{bmatrix} \middle| \; r, s \in \mathbb{R} \quad \text{and} \quad r^2 + s^2 = 0 \right\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

Solution

Since $r \in \mathbb{R}$, $r^2 \geq 0$ with equality if and only if $r = 0$. Similarly, $s \in \mathbb{R}$ implies $s^2 \geq 0$, and $s^2 = 0$ if and only if $s = 0$. This means $r^2 + s^2 = 0$ if and only if $r^2 = s^2 = 0$; thus $r^2 + s^2 = 0$ if and only if $r = s = 0$. Therefore U contains only $\vec{0}_3$, the zero vector, i.e., $\mathrm{U} = \{\vec{0}_3\}$. As we already observed, $\{\vec{0}_n\}$ is a subspace of $\mathbb{R}^n$, and therefore U is a subspace of $\mathbb{R}^3$.

The null space and the image space

Definitions (Null Space and Image Space)

Let A be an $m \times n$ matrix. The null space of A is defined as

$$\mathrm{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}_m\},$$

and the image space of A is defined as

$$\mathrm{im}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$

Remark

1. Since A is $m \times n$ and $\vec{x} \in \mathbb{R}^n$, $A\vec{x} \in \mathbb{R}^m$, so $\mathrm{im}(A) \subseteq \mathbb{R}^m$ while $\mathrm{null}(A) \subseteq \mathbb{R}^n$.
2. Image space is also called column space of A, denoted as $\mathrm{col}(A)$:

$$\mathrm{col}(A) = \mathrm{span}\left(\vec{a}_1, \cdots, \vec{a}_n\right) = \mathrm{im}(A).$$

Problem

Prove that if A is an $m \times n$ matrix, then $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$.

Proof.

S1. Since $A\vec{0}_n = \vec{0}_m$, $\vec{0}_n \in \mathrm{null}(A)$.

S2. Let $\vec{x}, \vec{y} \in \mathrm{null}(A)$. Then $A\vec{x} = \vec{0}_m$ and $A\vec{y} = \vec{0}_m$, so

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$

and thus $\vec{x} + \vec{y} \in \mathrm{null}(A)$.

S3. Let $\vec{x} \in \mathrm{null}(A)$ and $k \in \mathbb{R}$. Then $A\vec{x} = \vec{0}_m$, so

$$A(k\vec{x}) = k(A\vec{x}) = k\vec{0}_m = \vec{0}_m,$$

and thus $k\vec{x} \in \mathrm{null}(A)$.

Therefore, $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$. $\blacksquare$

Problem

Prove that if A is an $m \times n$ matrix, then $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^m$.

Proof.

S1. Since $\vec{0}_n \in \mathbb{R}^n$ and $A\vec{0}_n = \vec{0}_m$, $\vec{0}_m \in \mathrm{im}(A)$.

S2. Let $\vec{x}, \vec{y} \in \mathrm{im}(A)$. Then $\vec{x} = A\vec{u}$ and $\vec{y} = A\vec{v}$ for some $\vec{u}, \vec{v} \in \mathbb{R}^n$, so

$$\vec{x} + \vec{y} = A\vec{u} + A\vec{v} = A(\vec{u} + \vec{v}).$$

Since $\vec{u} + \vec{v} \in \mathbb{R}^n$, it follows that $\vec{x} + \vec{y} \in \mathrm{im}(A)$.

S3. Let $\vec{x} \in \mathrm{im}(A)$ and $k \in \mathbb{R}$. Then $\vec{x} = A\vec{u}$ for some $\vec{u} \in \mathbb{R}^n$, and thus

$$k\vec{x} = k(A\vec{u}) = A(k\vec{u}).$$

Since $k\vec{u} \in \mathbb{R}^n$, it follows that $k\vec{x} \in \mathrm{im}(A)$.

Therefore, $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^m$. ■

The Eigenspace

Definition (Eigenspace)

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ is the set

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

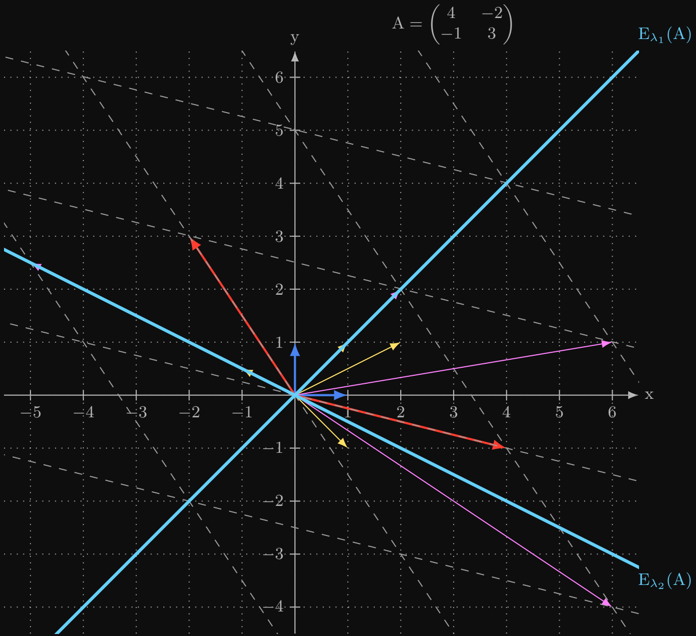
Example

$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$ has two eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 5$ with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

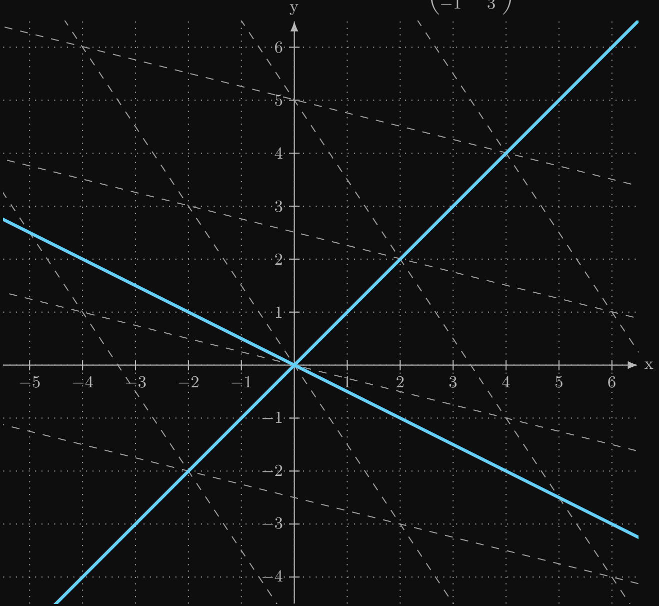
Hence,

$$E_{\lambda_1}(A) = E_2(A) = \{t\vec{v}_1 | t \in \mathbb{R}\}$$
$$E_{\lambda_2}(A) = E_5(A) = \{t\vec{v}_2 | t \in \mathbb{R}\}$$



$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
$E_{\lambda_1}(A)$
$E_{\lambda_2}(A)$
x
y
−5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4

$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$



$$E_2(A) = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$$

$$E_5(A) = \left\{ t \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$$

Note that

$$\begin{aligned}
E_\lambda(A) &= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} \\
&= \left\{\vec{x} \in \mathbb{R}^n \mid \lambda\vec{x} - A\vec{x} = \vec{0}_n\right\} \\
&= \left\{\vec{x} \in \mathbb{R}^n \mid (\lambda I - A)\vec{x} = \vec{0}_n\right\}
\end{aligned}$$

showing that

$$E_\lambda(A) = \mathrm{null}(\lambda I - A).$$

It follows that

- if λ is not an eigenvalue of A, then $E_\lambda(A) = \{\vec{0}_n\}$;
- the nonzero vectors of $E_\lambda(A)$ are the eigenvectors of A corresponding to λ;
- the eigenspace of A corresponding to λ is a subspace of $\mathbb{R}^n$.

Subspaces of $\mathbb{R}^n$

The null space and the image space

The Eigenspace

Linear Combinations and Spanning Sets

Spanning sets of null(A) and im(A)

Linear Combinations and Spanning Sets

Definition (Linear Combinations and Spanning)

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$ and $t_1, t_2, \ldots, t_k \in \mathbb{R}$. Then the vector

$$\vec{x} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$$

is called a linear combination of the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$; the (scalars) $t_1, t_2, \ldots, t_k \in \mathbb{R}$ are the coefficients. The set of all linear combinations of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ is called the span of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$, and is written

$$\mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} = \{t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k \mid t_1, t_2, \ldots, t_k \in \mathbb{R}\}.$$

Additional Terminology. If $U = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$, then

- U is spanned by the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$.
- the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ span U.
- the set of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a spanning set for U.

Example

Let $\vec{x} \in \mathbb{R}^3$ be a nonzero vector. Then $\text{span}\{\vec{x}\} = \{k\vec{x} \mid k \in \mathbb{R}\}$ is a line through the origin having direction vector $\vec{x}$.

Example

Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be nonzero vectors that are not parallel. Then

$$\text{span}\{\vec{x}, \vec{y}\} = \{k\vec{x} + t\vec{y} \mid k, t \in \mathbb{R}\}$$

is a plane through the origin containing $\vec{x}$ and $\vec{y}$.

How would you find the equation of this plane?

Problem

Let $\vec{x} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ 20 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}$. Is $\vec{x} \in \operatorname{span}\{\vec{y}, \vec{z}\}$?

Solution

An equivalent question is: can $\vec{x}$ be expressed as a linear combination of $\vec{y}$ and $\vec{z}$?

Suppose there exist $a, b \in \mathbb{R}$ so that $\vec{x} = a\vec{y} + b\vec{z}$. Then

$$\begin{bmatrix} 8 \\ 3 \\ -13 \\ 20 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solve this system of four linear equations in the two variables a and b.

Solution (continued)

$$\left[\begin{array}{rr|r} 2 & -1 & 8 \\ 1 & 0 & 3 \\ -3 & 2 & -13 \\ 5 & -3 & 20 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right]$$

Since the system has no solutions, $\vec{x} \notin \mathrm{span}\{\vec{y}, \vec{z}\}$. ■

Problem

Let $\vec{w} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ 21 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}$. Is $\vec{w} \in \mathrm{span}\{\vec{y}, \vec{z}\}$?

This is almost identical to a previous problem, except that $\vec{w}$ (above) has one entry that is different from the vector $\vec{x}$ of that problem.

Solution

In this case, the system of linear equations is consistent, and gives us $\vec{w} = 3\vec{y} - 2\vec{z}$, so $\vec{w} \in \mathrm{span}\{\vec{y}, \vec{z}\}$.

Theorem

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$ and let $U = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. Then

1. U is a subspace of $\mathbb{R}^n$ containing each $\vec{x}_i$, $1 \leq i \leq k$;
2. if W is a subspace of $\mathbb{R}^n$ and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in W$, then $U \subseteq W$.

Remark

Property 2 is saying that U is the "smallest" subspace of $\mathbb{R}^n$ that contains $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$.

Proof. (Part 1 of Theorem)

Since $U = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ and $0\vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_k = \vec{0}_n$, $\vec{0}_n \in U$.

Suppose $\vec{x}, \vec{y} \in U$. Then for some $s_i, t_i \in \mathbb{R}$, $1 \leq i \leq k$,

$$\begin{aligned} \vec{x} &= s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k \\ \vec{y} &= t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k \end{aligned}$$

Thus

$$\begin{aligned} \vec{x} + \vec{y} &= (s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k) + (t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k) \\ &= (s_1 + t_1)\vec{x}_1 + (s_2 + t_2)\vec{x}_2 + \cdots + (s_k + t_k)\vec{x}_k. \end{aligned}$$

Since $s_i + t_i \in \mathbb{R}$ for all $1 \leq i \leq k$, $\vec{x} + \vec{y} \in U$, i.e., U is closed under addition.

Proof. (Part 1 of Theorem – continued)

Suppose $\vec{x} \in U$ and $a \in \mathbb{R}$. Then for some $s_i \in \mathbb{R}$, $1 \leq i \leq k$,

$$\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$$

Thus

$$\begin{aligned} a\vec{x} &= a(s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k) \\ &= (as_1)\vec{x}_1 + (as_2)\vec{x}_2 + \cdots + (as_k)\vec{x}_k. \end{aligned}$$

Since $as_i \in \mathbb{R}$ for all $1 \leq i \leq k$, $a\vec{x} \in U$. Hence, U is closed under scalar multiplication.

Therefore, U is a subspace of $\mathbb{R}^n$. Furthermore, since

$$\vec{x}_i = \sum_{j=1}^{i-1} 0\vec{x}_j + 1\vec{x}_i + \sum_{j=i+1}^{k} 0\vec{x}_j,$$

it follows that $\vec{x}_i \in U$ for all i, $1 \leq i \leq k$.

Proof. (Part 2 of Theorem)

Let $W \subset \mathbb{R}^n$ be a subspace that contains $\vec{x}_1, \cdots, \vec{x}_n$. We need to prove that $U \subseteq W$.

Suppose $\vec{x} \in U$. Then $\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$ for some $s_i \in \mathbb{R}$, $1 \leq i \leq k$. Since W contain each $\vec{x}_i$ and W is closed under scalar multiplication, it follows that $s_i\vec{x}_i \in W$ for each i, $1 \leq i \leq k$. Furthermore, since W is closed under addition, $\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k \in W$. Therefore, $U \subseteq W$.

Problem (revisited)

Is $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \quad \text{and} \quad 2a - b = c + 2d \right\}$ a subspace of $\mathbb{R}^4$? Justify your answer.

Solution (Another)

Let $\vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U$. Since $2a - b = c + 2d$, $c = 2a - b - 2d$, and thus

$$U = \left\{ \begin{bmatrix} a \\ b \\ 2a - b - 2d \\ d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

By a previous Theorem, U is a subspace of $\mathbb{R}^4$.

Problem

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $U_1 = \text{span}\{\vec{x}, \vec{y}\}$, and $U_2 = \text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\}$. Prove that $U_1 = U_2$.

Solution

To show that $U_1 = U_2$, prove that $U_1 \subseteq U_2$, and $U_2 \subseteq U_1$. We begin by noting that, by the first part of the previous Theorem, U_1 and U_2 are subspaces of $\mathbb{R}^n$.

Since $2\vec{x} - \vec{y}, 2\vec{y} + \vec{x} \in U_1$, it follows from the second part of the previous Theorem that $\text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\} \subseteq U_1$, i.e., $U_2 \subseteq U_1$.

Also, since

$$\begin{aligned} \vec{x} &= \frac{2}{5}(2\vec{x} - \vec{y}) + \frac{1}{5}(2\vec{y} + \vec{x}), \\ \vec{y} &= -\frac{1}{5}(2\vec{x} - \vec{y}) + \frac{2}{5}(2\vec{y} + \vec{x}), \end{aligned}$$

$\vec{x}, \vec{y} \in U_2$. Therefore, by the second part of the previous Theorem, $\text{span}\{\vec{x}, \vec{y}\} \subseteq U_2$, i.e., $U_1 \subseteq U_2$. The result now follows.

Problem

Show that $\mathbb{R}^n = \operatorname{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, where $\vec{e}_j$ denote the j^{th} column of I_n.

Solution

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then $\vec{x} = x_1\vec{e_1} + x_2\vec{e_2} + \cdots + x_n\vec{e_n}$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Therefore, $\vec{x} \in \operatorname{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, and thus $\mathbb{R}^n \subseteq \operatorname{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

Conversely, since $\vec{e}_i \in \mathbb{R}^n$ for each i, $1 \leq i \leq n$ (and $\mathbb{R}^n$ is a vector space), it follows that $\operatorname{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$. The equality now follows.

Problem

Let $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Does $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ span $\mathbb{R}^4$? (Equivalently, is $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} = \mathbb{R}^4$?)

Solution

To prove $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} = \mathbb{R}^4$, we need to prove two directions:

$$\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} \subseteq \mathbb{R}^4 \quad \text{and} \quad \mathbb{R}^4 \subseteq \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}.$$

For the first relation, since $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4 \in \mathbb{R}^4$ (and $\mathbb{R}^4$ is a vector space), $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} \subseteq \mathbb{R}^4$.

Solution (continued)

For the second relation, notice that

$$\begin{aligned}\vec{e}_1 &= \vec{x}_1 - \vec{x}_2\\ \vec{e}_2 &= \vec{x}_2 - \vec{x}_3\\ \vec{e}_3 &= \vec{x}_3 - \vec{x}_4\\ \vec{e}_4 &= \vec{x}_4,\end{aligned}$$

showing that $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \in \mathrm{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$. Therefore, since $\mathrm{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ is a vector space,

$$\mathbb{R}^4 = \mathrm{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \mathrm{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\},$$

and the equality follows.

Problem

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

If you check, you'll find that $\vec{e}_2$ can not be written as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and $\vec{u}_4$.

Spanning sets of null(A) and im(A)

Lemma

Let A be an $m \times n$ matrix, and let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ denote a set of basic solutions to $A\vec{x} = \vec{0}_m$. Then

$$\mathrm{null}(A) = \mathrm{span}\{\vec{x}_1, \cdots, \vec{x}_k\}.$$

Lemma

Let A be an $m \times n$ matrix with columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$. Then

$$\mathrm{im}(A) = \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

Proof. (of $\mathrm{null}(A) = \mathrm{span}\{\vec{x}_1, \cdots, \vec{x}_k\}$)

"$\supseteq$:" Because $\vec{x}_i \in \mathrm{null}(A)$ for each i, $1 \leq i \leq k$, it follows that

$$\mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathrm{null}(A).$$

"$\subseteq$:" Every solution to $A\vec{x} = \vec{0}_m$ can be expressed as a linear combination of basic solutions, implying that

$$\mathrm{null}(A) \subseteq \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}.$$

Therefore, $\mathrm{null}(A) = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. ■

Proof. (of $\mathrm{im}(A) = \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$)

"$\subseteq$:" Suppose $\vec{y} \in \mathrm{im}(A)$. Then (by definition) there is a vector $\vec{x} \in \mathbb{R}^n$ so that $\vec{y} = A\vec{x}$. Write $\vec{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$. Then

$$\vec{y} = A\vec{x} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \ldots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n.$$

Therefore, $\vec{y} \in \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$, implying that

$$\mathrm{im}(A) \subseteq \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

Proof. (continued)

Notice that for each j, $1 \leq j \leq n$,

$$\begin{aligned} A\vec{e}_j &= \left[\begin{array}{cccc} \vec{c}_1 & \vec{c}_2 & \ldots & \vec{c}_n \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{c} \\ \\ \\ \\ \leftarrow \text{ jth row} \\ \\ \\ \\ \end{array} \\ &= 0\vec{c}_1 + 0\vec{c}_2 + \cdots + 0\vec{c}_{j-1} + 1\vec{c}_j + 0\vec{c}_{j+1} + \cdots + 0\vec{c}_n \\ &= \vec{c}_j. \end{aligned}$$

Thus $\vec{c}_j \in \mathrm{im}(A)$ for each j, $1 \leq j \leq n$. It follows that

$$\mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\} \subseteq \mathrm{im}(A),$$

and therefore

$$\mathrm{im}(A) = \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

■