Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-5. Similarity and Diagonalization

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Similar Matrices

Diagonalization Revisited

Algebraic and Geometric Multiplicities

Characterizing Diagonalizable Matrices

Complex Eigenvalues

Eigenvalues of Real Symmetric Matrices

Similar Matrices

Definition (Similar Matrices)

Let A and B be $n \times n$ matrices. A is similar to B, written $A \sim B$, if there exists an invertible matrix P such that $B = P^{-1}AP$.

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2. if $A \sim B$, then $B \sim A$ (symmetric);
3. if $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive).

Proof.

1. Since $A = I_n A I_n$ and $I_n^{-1} = I_n$, $A = I_n^{-1} A I_n$. Therefore, $A \sim A$.
2. Suppose $A \sim B$. Then there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Multiplying both sides on the left by P, on the right by P^{-1}, and simplifying gives us $PBP^{-1} = A$. Therefore, $A = (P^{-1})^{-1}A(P^{-1})$, so $A \sim B$.

Proof. (continued)

3. Since $A \sim B$ and $B \sim C$, there exist invertible $n \times n$ matrices P and Q such that
$$B = P^{-1}AP \quad \text{and} \quad C = Q^{-1}BQ.$$
Thus
$$C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q = (Q^{-1}P^{-1})A(PQ) = (PQ)^{-1}A(PQ),$$
where PQ is invertible, and hence $A \sim C$.

■

Definition

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the **trace of A** is

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Lemma (Properties of trace)

For $n \times n$ matrices A and B, and any $k \in \mathbb{R}$,

1. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$;
2. $\mathrm{tr}(kA) = k \cdot \mathrm{tr}(A)$;
3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Proof.

The proofs of (1) and (2) are trivial. As for (3), ...

Recall that for any $n \times n$ matrix A, the characteristic polynomial of A is

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and is a polynomial of degree n.

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Theorem (Properties of Similar Matrices)

If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$;
2. $\text{rank}\,(A) = \text{rank}\,(B)$;
3. $\text{tr}(A) = \text{tr}(B)$;
4. $c_A(x) = c_B(x)$;
5. A and B have the same eigenvalues.

Proof.

Since $A \sim B$, there exists an $n \times n$ invertible matrix P so that $B = P^{-1}AP$.

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Since $A \sim B$, there exists an $n \times n$ invertible matrix P so that $B = P^{-1}AP$.

1. $\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \cdot \det(A) \cdot \det(P)$.
   Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so

   $$\det(B) = \frac{1}{\det(P)} \cdot \det(A) \cdot \det(P) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(A) = \det(A).$$

   Therefore, $\det(B) = \det(A)$.

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2. $\text{rank }(B) = \text{rank }(P^{-1}AP)$.
   Since P is invertible, $\text{rank }(P^{-1}AP) = \text{rank }(P^{-1}A)$,
   since P^{-1} is invertible, $\text{rank }(P^{-1}A) = \text{rank }(A)$.
   Therefore, $\text{rank }(B) = \text{rank }(A)$.

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   since P^{-1} is invertible, $\text{rank}\ (P^{-1}A) = \text{rank}\ (A)$.
   Therefore, $\text{rank}\ (B) = \text{rank}\ (A)$.

3. $\text{tr}(B) = \text{tr}[(P^{-1}A)P] = \text{tr}[P(P^{-1}A)] = \text{tr}[(PP^{-1})A] = \text{tr}(IA) = \text{tr}(A)$.

Proof. (continued)

4.

$$\begin{aligned} c_B(x) = \det(xI - B) &= \det(xI - P^{-1}AP) \\ &= \det(xP^{-1}P - P^{-1}AP) \\ &= \det(P^{-1}xP - P^{-1}AP) \\ &= \det[P^{-1}(xI - A)P] \\ &= \det(P^{-1}) \cdot \det(xI - A) \cdot \det(P) \\ &= \det(P^{-1}) \cdot \det(P) \cdot \det(xI - A) \end{aligned}$$

Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so

$$c_B(x) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(xI - A) = \det(xI - A) = c_A(x).$$

Proof. (continued)

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Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so

$$c_B(x) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(xI - A) = \det(xI - A) = c_A(x).$$

5. Since the eigenvalues of a matrix are the roots of the characteristic polynomial, $c_B(x) = c_A(x)$ implies that A and B have the same eigenvalues. ∎

Diagonalization Revisited

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Recall that if λ is an eigenvalue of A, then $A\vec{x} = \lambda\vec{x}$ for some nonzero vector $\vec{x}$ in $\mathbb{R}^n$. Such a vector $\vec{x}$ is called a λ-eigenvector of A or an eigenvector of A corresponding to λ.

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An $n \times n$ matrix A is diagonalizable if $A \sim D$ for some diagonal matrix D.

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Definition (Diagonalizable – rephrased)

An $n \times n$ matrix A is diagonalizable if $A \sim D$ for some diagonal matrix D.

Remark (Diagonalizability)

Determining whether or not a square matrix A is diagonalizable is done by checking whether

the number of linearly independent eigenvectors
– geometric multiplicity

$||?$

the multiplicity of each eigenvalue
– algebraic multiplicity

Example

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\lambda = -1$ is an eigenvalue of A, and $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a (-1)-eigenvector of A since

$$A\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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Theorem

Suppose A is an $n \times n$ matrix.

1. The eigenvalues of A are the roots of $c_A(x)$.
2. The λ-eigenvectors of A are all the nonzero solutions to $(\lambda I - A)\vec{x} = \vec{0}_n$.

Problem

Determine all eigenvalues of $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 6 & 0 \\ 4 & 2 & -1 & 1 \end{bmatrix}$.

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Solution

$$\det(xI - A) = \begin{vmatrix} x+2 & 0 & 0 & 0 \\ -3 & x-6 & 0 & 0 \\ 1 & 0 & x-6 & 0 \\ -4 & -2 & 1 & x-1 \end{vmatrix} = (x+2)(x-6)(x-6)(x-1).$$

Thus, the eigenvalues of A are $-2, 6, 6$ and 1, precisely the elements on the main diagonal of A. ■

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Remark

In general, the eigenvalues of any triangular matrix are the entries on its main diagonal.

Theorem

Let A be an $n \times n$ matrix.

1. A is diagonalizable if and only if $\mathbb{R}^n$ has a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ of eigenvectors of A.
2. If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ are eigenvectors of A and form a basis of $\mathbb{R}^n$, then

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$$

is an invertible matrix such that

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where λ_i is the eigenvalue of A corresponding to $\vec{x}_i$.

This result was covered earlier, but without the use of term basis.

Theorem

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. For each i, let $\vec{x}_i$ be a λ_i-eigenvector of A. Then $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.

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Proof.

We need to show that $t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}$ only has trivial solution $t_1 = \cdots = t_k = 0$. Notice that

$$\begin{aligned}
t_1A\vec{x}_1 + t_2A\vec{x}_2 + \cdots + t_kA\vec{x}_k &= t_1\lambda_1\vec{x}_1 + t_2\lambda_2\vec{x}_2 + \cdots + t_k\lambda_k\vec{x}_k = \vec{0} \\
t_1A^2\vec{x}_1 + t_2A^2\vec{x}_2 + \cdots + t_kA^2\vec{x}_k &= t_1\lambda_1^2\vec{x}_1 + t_2\lambda_2^2\vec{x}_2 + \cdots + t_k\lambda_k^2\vec{x}_k = \vec{0} \\
\vdots \qquad & \qquad \vdots \\
t_1A^{k-1}\vec{x}_1 + \cdots\cdots + t_kA^{k-1}\vec{x}_k &= t_1\lambda_1^{k-1}\vec{x}_1 + \cdots\cdots + t_k\lambda_k^{k-1}\vec{x}_k = \vec{0}
\end{aligned}$$

Proof.

$$\begin{array}{ccccccccc}
t_1\lambda_1\vec{x}_1 & + & t_2\lambda_2\vec{x}_2 & + & \cdots & + & t_k\lambda_k\vec{x}_k & = & \vec{0} \\
t_1\lambda_1^2\vec{x}_1 & + & t_2\lambda_2^2\vec{x}_2 & + & \cdots & + & t_k\lambda_k^2\vec{x}_k & = & \vec{0} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
t_1\lambda_1^{k-1}\vec{x}_1 & + & t_2\lambda_2^{k-1}\vec{x}_2 & + & \cdots & + & t_k\lambda_k^{k-1}\vec{x}_k & = & \vec{0}
\end{array}$$

$$\Updownarrow$$

$$\begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \end{pmatrix} \begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_k \end{bmatrix} \begin{pmatrix} \lambda_1^0 & \lambda_1^1 & \cdots & \lambda_1^{k-1} \\ \lambda_2^0 & \lambda_2^1 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_k^0 & \lambda_k^1 & \cdots & \lambda_k^{k-1} \end{pmatrix} = O_{k\times k}.$$

Proof.

Since λ_i are distinct, the Vandermonde matrix is invertible, hence,

$$\begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \end{pmatrix} \begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_k \end{bmatrix} = O_{k\times k}.$$

$$\Updownarrow$$

$$t_i\vec{x}_i = 0 \quad \text{for all } i = 1, \cdots, k$$

$$\Downarrow$$

$$t_i = 0 \quad \text{for all } i = 1, \cdots, k$$

Only trivial solution is found. Hence, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is independent. ■

Proof. (Another proof left for you to study)

The proof is by induction on k, the number of distinct eigenvalues.

Basis. If $k = 1$, then $\{\vec{x}_1\}$ is an independent set because $\vec{x}_1 \neq \vec{0}_n$.
Suppose that for some $k \geq 1$, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is independent, where $\vec{x}_i$ is an eigenvector of A corresponding to λ_i, $1 \leq i \leq k$, and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. (This is the Inductive Hypothesis.) Now suppose $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are distinct eigenvalues of A that have corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{k+1}$, respectively. Consider

$$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_{k+1}\vec{x}_{k+1} = \vec{0}_n, \text{ for } t_1, t_2, \ldots, t_{k+1} \in \mathbb{R}. \tag{1}$$

Multiplying equation (1) by A (on the left) gives us

Proof. (continued)

$$t_1 A\vec{x}_1 + t_2 A\vec{x}_2 + \cdots + t_{k+1} A\vec{x}_{k+1} = \vec{0}_n,$$
$$\Downarrow$$
$$t_1 \lambda_1 \vec{x}_1 + t_2 \lambda_2 \vec{x}_2 + \cdots + t_{k+1} \lambda_{k+1} \vec{x}_{k+1} = \vec{0}_n. \tag{2}$$

Also, multiplying (1) by λ_{k+1} gives us

$$t_1 \lambda_{k+1} \vec{x}_1 + t_2 \lambda_{k+1} \vec{x}_2 + \cdots + t_{k+1} \lambda_{k+1} \vec{x}_{k+1} = \vec{0}_n, \tag{3}$$

and subtracting (3) from (2) results in

$$t_1(\lambda_1 - \lambda_{k+1})\vec{x}_1 + t_2(\lambda_2 - \lambda_{k+1})\vec{x}_2 + \cdots + t_k(\lambda_k - \lambda_{k+1})\vec{x}_k = \vec{0}_n.$$

Proof. (continued)

By the inductive hypothesis, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is independent, so

$$t_i(\lambda_i - \lambda_{k+1}) = 0 \text{ for } i = 1, 2, \ldots k.$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, $(\lambda_i - \lambda_{k+1}) \neq 0$ for $i = 1, 2, \ldots, k$, and thus $t_i = 0$ for $i = 1, 2, \ldots, k$. Substituting these values into (1) yields

$$t_{k+1}\vec{x}_{k+1} = \vec{0}_n,$$

implying that $t_{k+1} = 0$, since $\vec{x}_{k+1} \neq \vec{0}_n$.

Therefore, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{k+1}\}$ is an independent set, and the result follows by induction. ■

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Theorem (Covered earlier, but now with a proof)

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

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If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Proof.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denote the n (distinct) eigenvalues of A, and let $\vec{x}_i$ be an eigenvector of A corresponding to λ_i, $1 \leq i \leq n$. By the previous Theorem, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is an independent set. A subset of n linearly independent vectors of $\mathbb{R}^n$ also spans $\mathbb{R}^n$, and thus $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a basis of $\mathbb{R}^n$. Thus A is diagonalizable. ■

Problem

Is the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 8 & 6 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

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Solution

Because A has characteristic polynomial

$$c_A(x) = (x+3)(x-2)(x-4),$$

A has distinct eigenvalues $-3, 2$ and 4.

Since A has three distinct eigenvalues, A is diagonalizable. ■

Problem (Covered earlier, but with different wording)

Is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ diagonalizable? Explain.

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Is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ diagonalizable? Explain.

Solution

First, $c_A(x) = (x-2)(x+1)^2$, so the eigenvalues of A are $\lambda_1 = 2, \lambda_2 = -1$, and $\lambda_3 = -1$. Since the eigenvalues are not distinct, it isn't immediately obvious that A is diagonalizable. The general solution to $(-I - A)\vec{x} = \vec{0}_3$:

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is $x_1 = -s - t, x_2 = s$, and $x_3 = t$ for $s, t \in \mathbb{R}$, leading to basic solutions

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

that are linearly independent. Therefore, there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A, so A is diagonalizable. ■

Algebraic and Geometric Multiplicities

Algebraic and Geometric Multiplicities

Lemma (Technical but useful)

Let A be an $n \times n$ matrix, with independent eigenvectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. Extend $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ to a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k, \ldots, \vec{x}_n\}$ of $\mathbb{R}^n$, and let $P = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right]$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the (not necessarily distinct) eigenvalues corresponding to $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$, then

$$P^{-1}AP = \left[\begin{array}{cc} \mathrm{diag}(\lambda_1, \ldots, \lambda_k) & B \\ 0_{(n-k)\times k} & A_1 \end{array} \right],$$

where B is an $k \times (n-k)$ matrix and A_1 is an $(n-k) \times (n-k)$ matrix.

Proof.

$$\left[\ A\vec{x}_1 \mid \cdots \mid A\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n\ \right] = \left[\ \lambda_1\vec{x}_1 \mid \cdots \mid \lambda_k\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n\ \right]$$

$$||$$

$$A\left[\ \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n\ \right] \qquad ||$$

Proof.

$$\left[\, A\vec{x}_1 \mid \cdots \mid A\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n \,\right] = \left[\, \lambda_1\vec{x}_1 \mid \cdots \mid \lambda_k\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n \,\right]$$

$$\| \qquad\qquad \|$$

$$A\left[\, \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n \,\right]$$

$$\left[\, \vec{x}_1 \mid \cdots \mid \vec{x}_k \mid \vec{x}_{k+1} \mid \cdots \mid \vec{x}_n \,\right] \left[\begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & 0 & & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array}\right]$$

Proof.

$$\left[\, A\vec{x}_1 \mid \cdots \mid A\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n \,\right] = \left[\, \lambda_1\vec{x}_1 \mid \cdots \mid \lambda_k\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n \,\right]$$

$$\left[\, A\vec{x}_1 \mid \cdots \mid A\vec{x}_n \,\right] = A\left[\, \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n \,\right]$$

$$\left[\, \lambda_1\vec{x}_1 \mid \cdots \mid \lambda_k\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n \,\right] = \left[\, \vec{x}_1 \mid \cdots \mid \vec{x}_k \mid \vec{x}_{k+1} \mid \cdots \mid \vec{x}_n \,\right] \left[\begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & 0 & & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array}\right]$$

$$\uparrow \qquad \cdots \qquad \uparrow$$

$$P^{-1}A\vec{x}_{k+1} \qquad \cdots \qquad P^{-1}A\vec{x}_n$$

Proof.

$$\left[\, A\vec{x}_1 \;\middle|\; \cdots \;\middle|\; A\vec{x}_k \;\middle|\; A\vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; A\vec{x}_n \,\right] = \left[\, \lambda_1\vec{x}_1 \;\middle|\; \cdots \;\middle|\; \lambda_k\vec{x}_k \;\middle|\; A\vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; A\vec{x}_n \,\right]$$

$$\|$$

$$A\left[\, \vec{x}_1 \;\middle|\; \vec{x}_2 \;\middle|\; \cdots \;\middle|\; \vec{x}_n \,\right] \qquad \|$$

$$\left[\, \vec{x}_1 \;\middle|\; \cdots \;\middle|\; \vec{x}_k \;\middle|\; \vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; \vec{x}_n \,\right] \left[\begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & 0 & & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array}\right]$$

$$\uparrow \qquad \cdots \qquad \uparrow$$

$$P^{-1}A\vec{x}_{k+1} \qquad \cdots \qquad P^{-1}A\vec{x}_n$$

$$\Updownarrow$$

$$AP = P\begin{bmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_k) & B \\ 0_{(n-k)\times k} & A_1 \end{bmatrix}$$

Proof.

$$\left[\, A\vec{x}_1 \,\middle|\, \cdots \,\middle|\, A\vec{x}_k \,\middle|\, A\vec{x}_{k+1} \,\middle|\, \cdots \,\middle|\, A\vec{x}_n \,\right] = \left[\, \lambda_1\vec{x}_1 \,\middle|\, \cdots \,\middle|\, \lambda_k\vec{x}_k \,\middle|\, A\vec{x}_{k+1} \,\middle|\, \cdots \,\middle|\, A\vec{x}_n \,\right]$$

$$\|$$

$$A\left[\, \vec{x}_1 \,\middle|\, \vec{x}_2 \,\middle|\, \cdots \,\middle|\, \vec{x}_n \,\right] \qquad \|$$

$$\left[\, \vec{x}_1 \,\middle|\, \cdots \,\middle|\, \vec{x}_k \,\middle|\, \vec{x}_{k+1} \,\middle|\, \cdots \,\middle|\, \vec{x}_n \,\right] \left[\begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & 0 & & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array}\right]$$

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$$\Updownarrow$$

$$P^{-1}AP = \begin{bmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_k) & B \\ 0_{(n-k)\times k} & A_1 \end{bmatrix}$$

∎

Proof. (Another proof)

Recall that $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis of R_n. Since $I_n = P^{-1}P$,

$$\begin{aligned} \left[\begin{array}{cccc} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{array}\right] = P^{-1}P &= P^{-1}\left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right] \\ &= \left[\begin{array}{cccc} P^{-1}\vec{x}_1 & P^{-1}\vec{x}_2 & \cdots & P^{-1}\vec{x}_n \end{array}\right] \end{aligned}$$

Thus for each j, $1 \leq j \leq n$, $P^{-1}\vec{x}_j = \vec{e}_j$. Also,

$$\begin{aligned} P^{-1}AP &= P^{-1}A\left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right] \\ &= \left[\begin{array}{cccc} P^{-1}A\vec{x}_1 & P^{-1}A\vec{x}_2 & \cdots & P^{-1}A\vec{x}_n \end{array}\right], \end{aligned}$$

so the j^th column of $P^{-1}AP$, $1 \leq j \leq k$, is equal to

$$P^{-1}(A\vec{x}_j) = P^{-1}(\lambda_j\vec{x}_j) = \lambda_j(P^{-1}\vec{x}_j) = \lambda_j\vec{e}_j.$$

This gives us the first k columns of $P^{-1}AP$, and the result follows. ■

Definition

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ is the set

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

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$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

Remark

1. The eigenspace $E_\lambda(A)$ is indeed a subspace of $\mathbb{R}^n$ because

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} = \{\vec{x} \in \mathbb{R}^n \mid (\lambda I - A)\vec{x} = \vec{0}_n\} = \text{null}(\lambda I - A).$$

2. If λ is not an eigenvalue of A, then $E_\lambda(A) = \{0\}$.

Definition

1. If A is an $n \times n$ matrix and λ is an eigenvalue of A, then the (algebraic) multiplicity of λ is the largest value of m for which

$$c_A(x) = (x - \lambda)^m g(x)$$

   for some polynomial $g(x)$, i.e., the multiplicity of λ is the number of times that λ occurs as a root of $c_A(x)$.

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Lemma

If A is an $n \times n$ matrix, and λ is an eigenvalue of A of multiplicity m, then

$$\dim(E_\lambda(A)) \leq m,$$

that is,

$$\text{Geometric multiplicity} \leq \text{Algebraic multiplicity}.$$

Proof.

Let $d = \dim(E_\lambda(A))$, and let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d\}$ be a basis of $E_\lambda(A)$. As a consequence, we know that there exists an invertible $n \times n$ matrix P so that

$$P^{-1}AP = \begin{bmatrix} \operatorname{diag}(\lambda, \ldots, \lambda) & B \\ 0_{(n-d)\times d} & A_1 \end{bmatrix} = \begin{bmatrix} \lambda I_d & B \\ 0_{(n-d)\times d} & A_1 \end{bmatrix}$$

where B is $d \times (n-d)$ and A_1 is $(n-d) \times (n-d)$.

Define $A' = P^{-1}AP$. Then $A \sim A'$, so A and A' have the same characteristic polynomial. Thus

$$\begin{aligned} c_A(x) = c_{A'}(x) = \det(xI - A') &= \det \begin{bmatrix} (x-\lambda)I_d & -B \\ 0_{(n-d)\times d} & xI_{n-d} - A_1 \end{bmatrix} \\ &= \det[(x-\lambda)I_d] \det(xI_{n-d} - A_1) \\ &= (x-\lambda)^d c_{A_1}(x) \\ &= (x-\lambda)^d g(x). \end{aligned}$$

Since λ has multiplicity m, $d \leq m$, and therefore $\dim(E_\lambda(A)) \leq m$ as required. ∎

Similar Matrices

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Algebraic and Geometric Multiplicities

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Complex Eigenvalues

Eigenvalues of Real Symmetric Matrices

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$$\text{Diagonalizable}$$
$$\Updownarrow$$
$$\text{Geometric multiplicity} = \text{Algebraic multiplicity}, \quad \text{for all } \lambda.$$

Problem (Covered earlier, here with new terminology)

If possible, diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$. Otherwise, explain why A is not diagonalizable.

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Solution

$c_A(x) = (x-3)(x+1)^2$, so A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = -1$. Find the dimension of $E_{-1}(A)$ by solving the linear system $(-I - A)\vec{x} = \vec{0}_3$.

$$\left[\begin{array}{rrr|r} 4 & -1 & -6 & 0 \\ -2 & -2 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this, we see that $\dim(E_{-1}(A)) = 1$. Since -1 is an eigenvalue of multiplicity two, A is not diagonalizable. ■

Problem (Covered earlier, here with new terminology)

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Show that A is diagonalizable, and that B is not diagonalizable.

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Show that A is diagonalizable, and that B is not diagonalizable.

Solution

Both A and B are triangular matrices, so we immediately see that A and B have the same eigenvalues: $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Thus for each matrix, 1 is an eigenvalue of multiplicity two.

Solving the system $(I - A)\vec{x} = \vec{0}_3$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that there are two parameters in the general solution, so $\dim(E_1(A)) = 2$. Therefore, A is diagonalizable.

Solution (continued)

Solving the system $(I - B)\vec{x} = \vec{0}_3$:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that the general solution has only one parameter, so $\dim(E_1(B)) = 1$. However, the algebraic multiplicity of $\lambda = 1$ is 2. Therefore, B is not diagonalizable. ■

Complex Eigenvalues

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If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

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Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & -1 \\ 1 & x-1 \end{vmatrix} = x^2 - 2x + 2.$$

The roots of $c_A(x)$ are distinct complex numbers: $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, so A is diagonalizable. Corresponding eigenvectors are

$$\vec{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix},$$

respectively.

Solution (continued)

A diagonalizing matrix for A is

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix},$$

and

$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

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Remark

Notice that A is a real matrix, but has complex eigenvalues (and eigenvectors).

Eigenvalues of Real Symmetric Matrices

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Theorem

The eigenvalues of any real symmetric matrix are real.

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Proof.

Let A be an $n \times n$ real symmetric matrix, and let λ be an eigenvalue of A. To prove that λ is real, it is enough to prove that $\overline{\lambda} = \lambda$, i.e., λ is equal to its (complex) conjugate.

We use $\overline{A}$ to denote the matrix obtained from A by replacing each entry by its conjugate. Since A is real, $\overline{A} = A$.

Suppose

$$\vec{x} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

is a λ-eigenvector of A. Then $A\vec{x} = \lambda\vec{x}$.

Proof. (continued)

Let $c = \vec{x}^T \overline{\vec{x}} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{bmatrix}$.

Then $c = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$; since $\vec{x} \neq \vec{0}$, c is a positive real number. Now

$$
\begin{aligned}
\lambda c &= \lambda(\vec{x}^T \overline{\vec{x}}) = (\lambda \vec{x}^T)\overline{\vec{x}} = (\lambda \vec{x})^T \overline{\vec{x}} \\
&= (A\vec{x})^T \overline{\vec{x}} = \vec{x}^T A^T \overline{\vec{x}} \\
&= \vec{x}^T A \overline{\vec{x}} \qquad \text{(since A is symmetric)} \\
&= \vec{x}^T \, \overline{A} \, \overline{\vec{x}} \qquad \text{(since A is real)} \\
&= \vec{x}^T (\overline{A\vec{x}}) = \vec{x}^T (\overline{\lambda \vec{x}}) = \vec{x}^T \, \overline{\lambda} \, \overline{\vec{x}} \\
&= \overline{\lambda}(\vec{x}^T \overline{\vec{x}}) \\
&= \overline{\lambda} c.
\end{aligned}
$$

Thus, $\lambda c = \overline{\lambda} c$. Since $c \neq 0$, it follows that $\lambda = \overline{\lambda}$, and therefore λ is real. ■