Math 221: LINEAR ALGEBRA

Chapter 7. Linear Transformations
§7-3. Isomorphisms and Composition

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

What is isomorphism?

Proving vector spaces are isomorphic

Characterizing isomorphisms

Composition of transformations

Inverses

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What is an isomorphism?

Example

$\mathcal{P}_1 = \{ax + b \mid a, b \in \mathbb{R}\}$, has addition and scalar multiplication defined as follows:

$$\begin{aligned} (a_1x + b_1) + (a_2x + b_2) &= (a_1 + a_2)x + (b_1 + b_2), \\ k(a_1x + b_1) &= (ka_1)x + (kb_1), \end{aligned}$$

for all $(a_1x + b_1), (a_2x + b_2) \in \mathcal{P}_1$ and $k \in \mathbb{R}$.

The role of the variable x is to distinguish a_1 from b_1, a_2 from b_2, $(a_1 + a_2)$ from $(b_1 + b_2)$, and (ka_1) from (kb_1).

Example (continued)

This can be accomplished equally well by using vectors in $\mathbb{R}^2$.

$$\mathbb{R}^2 = \left\{ \left[\begin{array}{c} a \\ b \end{array} \right] \;\middle|\; a, b \in \mathbb{R} \right\}$$

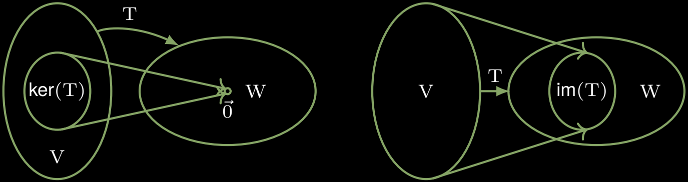
where addition and scalar multiplication are defined as follows:

$$\left[\begin{array}{c} a_1 \\ b_1 \end{array} \right] + \left[\begin{array}{c} a_2 \\ b_2 \end{array} \right] = \left[\begin{array}{c} a_1 + a_2 \\ b_1 + b_2 \end{array} \right], \; k \left[\begin{array}{c} a_1 \\ b_1 \end{array} \right] = \left[\begin{array}{c} ka_1 \\ kb_1 \end{array} \right]$$

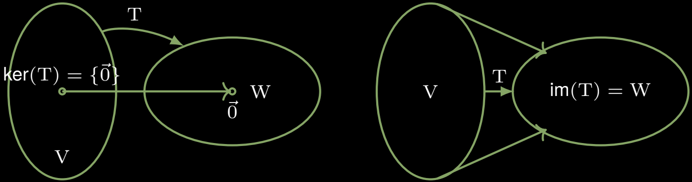
for all $\left[\begin{array}{c} a_1 \\ b_1 \end{array} \right], \left[\begin{array}{c} a_2 \\ b_2 \end{array} \right] \in \mathbb{R}^2$ and $k \in \mathbb{R}$.

Definition

Let V and W be vector spaces, and $T : V \to W$ a linear transformation. T is an isomorphism if and only if T is both one-to-one and onto (i.e., $\ker(T) = \{\mathbf{0}\}$ and $\mathrm{im}(T) = W$). If $T : V \to W$ is an isomorphism, then the vector spaces V and W are said to be isomorphic, and we write $V \cong W$.



General linear transformation T



Isomorphism T

Example

The identity operator on any vector space is an isomorphism.

Example

$T : \mathcal{P}_n \to \mathbb{R}^{n+1}$ defined by

$$T(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

for all $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in \mathcal{P}_n$ is an isomorphism. To verify this, prove that T is a linear transformation that is one-to-one and onto.

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Proving isomorphism of vector spaces

Problem

Prove that $\mathbf{M}_{22}$ and $\mathbb{R}^4$ are isomorphic.

Proof.

Let $T : \mathbf{M}_{22} \to \mathbb{R}^4$ be defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

It remains to prove that

1. T is a linear transformation;
2. T is one-to-one;
3. T is onto.

Solution (continued – 1. linear transformation)

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbf{M}_{22}$ and let $k \in \mathbb{R}$. Then

$$T(A) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \quad \text{and} \quad T(B) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

$$\Downarrow$$

$$T(A+B) = T\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \\ a_4+b_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = T(A)+T(B)$$

$$\Downarrow$$

T preserves addition.

Solution (continued – 1. linear transformation)

Also

$$T(kA) = T\begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \\ ka_4 \end{bmatrix} = k\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = kT(A)$$

$$\Downarrow$$

T preserves scalar multiplication.

Since T preserves addition and scalar multiplication, T is a linear transformation.

Solution (continued – 2. One-to-one)

By definition,

$$\begin{aligned}\ker(T) &= \{A \in \mathbf{M}_{22} \mid T(A) = \mathbf{0}\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.\end{aligned}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker T$, then $a = b = c = d = 0$, and thus $\ker(T) = \{\mathbf{0}_{22}\}$.

$$\Downarrow$$

T is one-to-one.

Solution (continued – 3. Onto)

Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4,$$

and define matrix $A \in \mathbf{M}_{22}$ as follows:

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Then $T(A) = X$, and therefore T is onto.

Finally, since T is a linear transformation that is one-to-one and onto, T is an isomorphism. Therefore, $\mathbf{M}_{22}$ and $\mathbb{R}^4$ are isomorphic vector spaces. $\blacksquare$

Example (Other isomorphic vector spaces)

1. For all integers $n \geq 0$, $\mathcal{P}_n \cong \mathbb{R}^{n+1}$.
2. For all integers m and n, $m, n \geq 1$, $\mathbf{M}_{mn} \cong \mathbb{R}^{m\times n}$.
3. For all integers m and n, $m, n \geq 1$, $\mathbf{M}_{mn} \cong \mathcal{P}_{mn-1}$.

You should be able to define appropriate linear transformations and prove each of these statements.

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Characterizing isomorphisms

Theorem

Let V and W be finite dimensional vector spaces and $T : V \to W$ a linear transformation. The following are equivalent.

1. T is an isomorphism.
2. If $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is any basis of V, then $\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)\}$ is a basis of W.
3. There exists a basis $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of V such that $\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)\}$ is a basis of W.

Proof.

(1) $\Rightarrow$ (2): This is because

- One-to-one linear transformations preserve independent sets.
- Onto linear transformations preserve spanning sets.

(2) $\Rightarrow$ (3) is trivial.

Proof. (Continued)

(3) $\Rightarrow$ (1). We need to prove that T is both onto and one-to-one.

If $T(\vec{v}) = \vec{0}$, write $\vec{v} = v_1\vec{b}_1 + \cdots + v_n\vec{b}_n$ where each v_i is in $\mathbb{R}$. Then

$$\vec{0} = T(\vec{v}) = v_1 T(\vec{b}_1) + \cdots + v_n T(\vec{b}_n)$$

so $v_1 = \cdots = v_n = 0$ by (3). Hence $\vec{v} = \vec{0}$, so $\ker T = \{\vec{0}\}$ and T is one-to-one.

To show that T is onto, let $\vec{w}$ be any vecor in W. By (3) there exist $w_1, \ldots, w_n$ in $\mathbb{R}$ such that

$$\vec{w} = w_1 T(\vec{b}_1) + \cdots + w_n T(\vec{b}_n) = T(w_1\vec{b}_1 + \cdots + w_n\vec{b}_n)$$

Thus T is onto. ∎

Suppose V and W are finite dimensional vector spaces with $\dim(V) = \dim(W)$, and let

$$\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\} \quad \text{and} \quad \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\}$$

be bases of V and W respectively. Then $T : V \to W$ defined by

$$T(\vec{b}_i) = \vec{f}_i \text{ for } 1 \leq k \leq n$$

is a linear transformation that maps a basis of V to a basis of W. By the previous Theorem, T is an isomorphism.

Conversely, if V and W are isomorphic and $T : V \to W$ is an isomorphism, then (by the previous Theorem) for any basis $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of V, $\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)\}$ is a basis of W, implying that $\dim(V) = \dim(W)$.

This proves the next theorem.

Theorem

Finite dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Corollary

If V is a vector space with $\dim(V) = n$, then V is isomorphic to $\mathbb{R}^n$.

Problem

Let V denote the set of 2×2 real symmetric matrices. Then V is a vector space with dimension three. Find an isomorphism $T : \mathcal{P}_2 \to V$ with the property that $T(1) = I_2$ (the 2×2 identity matrix).

Solution

$$V = \left\{ \left[\begin{array}{cc} a & b \\ b & c \end{array} \right] \;\middle|\; a, b, c \in \mathbb{R} \right\} = \operatorname{span} \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Let

$$B = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Then B is independent, and $\operatorname{span}(B) = V$, so B is a basis of V. Also, $\dim(V) = 3 = \dim(\mathcal{P}_2)$. However, we want a basis of V that contains I_2.

Solution (continued)

Let

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Since B' consists of $\dim(V)$ symmetric independent matrices, B' is a basis of V. Note that $I_2 \in B'$. Define

$$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T(x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then for all $ax^2 + bx + c \in \mathcal{P}_2$,

$$T(ax^2 + bx + c) = \begin{bmatrix} c & b \\ b & a + c \end{bmatrix},$$

and $T(1) = I_2$.

By the previous Theorem, $T : \mathcal{P}_2 \to V$ is an isomorphism. $\blacksquare$

Theorem

Let V and W be vector spaces, and $T : V \to W$ a linear transformation. If $\dim(V) = \dim(W) = n$, then T is an isomorphism if and only if T is either one-to-one or onto.

Proof.

($\Rightarrow$) By definition, an isomorphism is both one-to-one and onto.

($\Leftarrow$) Suppose that T is one-to-one. Then $\ker(T) = \{\vec{0}\}$, so $\dim(\ker(T)) = 0$. By the Dimension Theorem,

$$\begin{aligned} \dim(V) &= \dim(\mathrm{im}(T)) + \dim(\ker(T)) \\ n &= \dim(\mathrm{im}(T)) + 0 \end{aligned}$$

so $\dim(\mathrm{im}(T)) = n = \dim(W)$. Furthermore $\mathrm{im}(T) \subseteq W$, so it follows that $\mathrm{im}(T) = W$. Therefore, T is onto, and hence is an isomorphism.

Proof. (continued)

($\Leftarrow$) Suppose that T is onto. Then $\mathrm{im}(T) = W$, so $\dim(\mathrm{im}(T)) = \dim(W) = n$. By the Dimension Theorem,

$$\begin{aligned} \dim(V) &= \dim(\mathrm{im}(T)) + \dim(\ker(T)) \\ n &= n + \dim(\ker(T)) \end{aligned}$$

so $\dim(\ker(T)) = 0$. The only vector space with dimension zero is the zero vector space, and thus $\ker(T) = \{\vec{0}\}$. Therefore, T is one-to-one, and hence is an isomorphism. ■

Composition of transformations

Definition

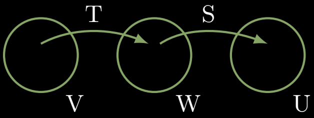
Let V, W and U be vector spaces, and let

$$T : V \to W \quad \text{and} \quad S : W \to U$$

be linear transformations. The composite of T and S is

$$ST : V \to U$$

where $(ST)(\vec{v}) = S(T(\vec{v}))$ for all $\vec{v} \in V$. The process of obtaining ST from S and T is called composition.



Example

Let $S : \mathbf{M}_{22} \to \mathbf{M}_{22}$ and $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be linear transformations such that

$$S(A) = -A^T \quad \text{and} \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \text{ for all } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Then

$$(ST)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = S\begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} -b & -d \\ -a & -c \end{bmatrix},$$

and

$$(TS)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = T\begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} = \begin{bmatrix} -c & -a \\ -d & -b \end{bmatrix}.$$

If a, b, c and d are distinct, then $(ST)(A) \neq (TS)(A)$.

This illustrates that, in general, $ST \neq TS$.

Theorem

Let V, W, U and Z be vector spaces and

$$V \xrightarrow{T} W \xrightarrow{S} U \xrightarrow{R} Z$$

be linear transformations. Then

1. ST is a linear transformation.
2. $T1_V = T$ and $1_W T = T$.
3. $(RS)T = R(ST)$.

Problem (The composition of onto transformations is onto)

Let V, W and U be vector spaces, and let

$$\mathrm{V} \xrightarrow{\mathrm{T}} \mathrm{W} \xrightarrow{\mathrm{S}} \mathrm{U}$$

be linear transformations. Prove that if T and S are onto, then ST is onto.

Proof.

Let $\mathbf{z} \in \mathrm{U}$. Since S is onto, there exists a vector $\mathbf{y} \in \mathrm{W}$ such that $\mathrm{S}(\mathbf{y}) = \mathbf{z}$. Furthermore, since T is onto, there exists a vector $\mathbf{x} \in \mathrm{V}$ such that $\mathrm{T}(\mathbf{x}) = \mathbf{y}$. Thus

$$\mathbf{z} = \mathrm{S}(\mathbf{y}) = \mathrm{S}(\mathrm{T}(\mathbf{x})) = (\mathrm{ST})(\mathbf{x}),$$

showing that for each $\mathbf{z} \in \mathrm{U}$ there exists and $\mathbf{x} \in \mathrm{V}$ such that $(\mathrm{ST})(\mathbf{x}) = \mathbf{z}$. Therefore, ST is onto. $\blacksquare$

Problem (The composition of one-to-one transformations is one-to-one)

Let V, W and U be vector spaces, and let

$$V \xrightarrow{T} W \xrightarrow{S} U$$

be linear transformations. Prove that if T and S are one-to-one, then ST is one-to-one.

The proof of this is left as an exercise.

What is isomorphism?

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Theorem

Let V and W be finite dimensional vector spaces, and $T : V \to W$ a linear transformation. Then the following statements are equivalent.

1. T is an isomorphism.
2. There exists a linear transformation $S : W \to V$ so that
$$ST = 1_V \quad \text{and} \quad TS = 1_W.$$
In this case, the isomorphism S is uniquely determined by T:
$$\text{if } \vec{w} \in W \quad \text{and} \quad \vec{w} = T(\vec{v}), \text{ then } S(\vec{w}) = \vec{v}.$$

Given an isomorphism $T : V \to W$, the unique isomorphism satisfying the second condition of the theorem is the inverse of T, and is written T^{-1}.

Remark (Fundamental Identities (relating T and T^{-1}))

If V and W are vector spaces and $T : V \to W$ is an isomorphism, then $T^{-1} : W \to V$ is a linear transformation such that

$$(T^{-1}T)(\vec{v}) = \vec{v} \quad \text{and} \quad (TT^{-1})(\vec{w}) = \vec{w}$$

for each $\vec{v} \in V$, $\vec{w} \in W$. Equivalently,

$$T^{-1}T = 1_V \quad \text{and} \quad TT^{-1} = 1_W.$$

Problem

The function $T : \mathcal{P}_2 \to \mathbb{R}^3$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a - c \\ 2b \\ a + c \end{bmatrix} \text{ for all } a + bx + cx^2 \in \mathcal{P}_2$$

is a linear transformation (this is left for you to verify). Does T have an inverse? If so, find T^{-1}.

Solution

Since $\dim(\mathcal{P}_2) = 3 = \dim(\mathbb{R}^3)$, it suffices to prove that T is either one-to-one or onto.

Suppose $a + bx + cx^2 \in \ker(T)$. Then

$$\begin{cases} a - c = 0 \\ 2b = 0 \\ a + c = 0 \end{cases} \implies \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Therefore, $\ker(T) = \{\mathbf{0}\}$, and hence T is one-to-one. By our earlier observation, it follows that T is onto, and thus is an isomorphism.

Solution (continued)

To find T^{-1}, we need to specify $T^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ for any $\begin{bmatrix} p \\ q \\ r \end{bmatrix} \in \mathbb{R}^3$.

Let $a + bx + cx^2 \in \mathcal{P}_2$, and suppose

$$T(a + bx + cx^2) = \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

By the definition of T, $p = a - c$, $q = 2b$ and $r = a + c$. We now solve for a, b and c in terms of p, q and r.

$$\left[\begin{array}{rrr|c} 1 & 0 & -1 & p \\ 0 & 2 & 0 & q \\ 1 & 0 & 1 & r \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & (r+p)/2 \\ 0 & 1 & 0 & q/2 \\ 0 & 0 & 1 & (r-p)/2 \end{array}\right].$$

Solution (continued)

We now have $a = \frac{r+p}{2}$, $b = \frac{q}{2}$ and $c = \frac{r-p}{2}$, and thus

$$T(a + bx + cx^2) = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = T\left(\frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2\right)$$

Therefore,

$$\begin{aligned} T^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix} &= T^{-1}\left(T\left(\frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2\right)\right) \\ &= (T^{-1}T)\left(\frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2\right) \\ &= \frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2. \end{aligned}$$

$\blacksquare$

Definition

Let V be a vector space with $\dim(V) = n$, let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a fixed basis of V, and let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denote the standard basis of $\mathbb{R}^n$. We define a transformation $C_B : V \to \mathbb{R}^n$ by

$$C_B(a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n) = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Then C_B is a linear transformation such that $C_B(\vec{b}_i) = \vec{e}_i$, $1 \le i \le n$, and thus C_B is an isomorphism, called the coordinate isomorphism corresponding to B.

Example

Let V be a vector space and let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a fixed basis of V. Then $C_B : V \to \mathbb{R}^n$ is invertible, and it is clear that $C_B^{-1} : \mathbb{R}^n \to V$ is defined by

$$C_B^{-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n \text{ for each } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n.$$