Real Analysis – II

MATH 7210[1]
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[1] Based on G. B. Folland's *Real Analysis, Modern Techniques and Their Applications*, 2nd Ed.

Chapter 3. Signed measures and differentiation

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§ 3.1 Signed measures

§ 3.2 The Lebesgue-Radon-Nikodym theorem

§ 3.3 Complex measures

§ 3.4 Differentiation on Euclidean space

§ 3.5 Functions of bounded variation

In this section, we apply what we have learned in the previous section to the case

$$X = \mathbb{R}$$

Recall that if F is a nondecreasing function on $\mathbb{R}$, then

$$\mu_F((a, b]) := F(b) - F(a) \quad a \leq b,$$

defines a Borel measure.

Theorem 3.5-1 Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function and let $G(x) := F(x+)$. Then
(a) The set of discontinuous points of F is countable;
(b) Both F and G are differentiable a.e., and $F' = G'$ a.e.

Proof. (a) Notice that for all $N > 0$,

$$\sum_{|x|<N} [F(x+) - F(x)] \leq F(N) - F(-N) < \infty.$$

Hence, the set of jump points

$$\{x \in (-N, N) : F(x+) \neq F(x)\}$$

has to be at most countable.

Proof (continued). (b) G is right continuous and $G = F$ except possibly only at countable points, say, $\{x_j\}_1^\infty$.

Let μ_F and μ_G be the associate Borel measures defined as

$$\mu_G((a,b]) := G(b) - G(a), \quad a \leq b.$$

In order to the differentiability of G (and similarly for F), we need to check the existence of the limits:

$$\lim_{h \downarrow 0} \frac{G(x+h) - G(x)}{h} \quad \text{and} \quad \lim_{h \downarrow 0} \frac{G(x) - G(x-h)}{h},$$

which equal respectively

$$\lim_{h \downarrow 0} \frac{\mu_G((x, x+h])}{h} \quad \text{and} \quad \lim_{h \downarrow 0} \frac{\mu_G((x-h, x])}{h}. \tag{3}$$

Proof (continued). Notice that

- both $E_h = (x, x+h]$ and $E'_h = (x-h, x]$ shrink nicely to x as $h \downarrow 0$;
- G satisfies Properties 1 and 2 of Theorem 3.4-7.

Hence, by Theorem 3.4-7, both limits, denoted as G', in (3) exist a.e. and are equal.

It remains to show that $F' = G'$ a.e., which is equivalent to show that

$$H' = 0 \quad \text{a.e. where } H = G - F.$$

Proof (continued). It is clear that $H(x) = 0$ if $x \notin \{x_j\}_1^\infty$ and $H(x) > 0$ if $x \in \{x_j\}_1^\infty$. Hence,

$$H'(x) = \sum_{j=1}^{\infty} H(x_j)\delta_{x_j}(x) \quad \text{or equivalently} \quad \mu_H = \sum_{j=1}^{\infty} H(x_j)\delta_{x_j}.$$

Therefore, we see that $\mu_H \perp m$ because $m\left(\{x_j\}_1^\infty\right) = \mu_H\left(\left[\{x_j\}_1^\infty\right]^c\right) = 0$.

It is ready to verify that μ_H satisfies Properties 1 and 2 of Theorem 3.4-7. Finally, an application of this theorem shows that $H' = 0$ a.e. □

Definition 3.5-1 For a real-valued function $F : \mathbb{R} \to \mathbb{R}$, its *total variation* is defined as

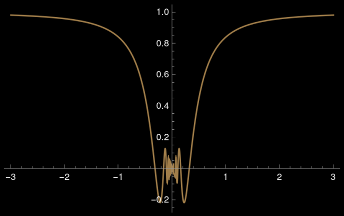
$$T_F(x) := \sup \left\{ \sum_{i=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x \right\}.$$

If $T_F(\infty) := \lim_{x\to\infty} T_F(x) < \infty$, F is said to be of *bounded variation* on $\mathbb{R}$, denoted by $F \in BV$.

Remark 3.5-1

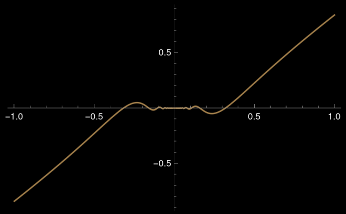
1. T_F is a nondecreasing and nonnegative function.
2. One can define similarly for $BV([a, b])$ for $-\infty < a < b < \infty$.

Example 3.5-1 $F(x) = x \sin(1/x)$ for $x \neq 0$ and $F(0) := \lim_{x\to 0} F(x) = 0$.



Then $F \notin BV([a, b])$ whenever $0 \in [a, b]$.

Example 3.5-2 $F(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $F(0) := \lim_{x\to 0} F(x) = 0$.



Then $F \in BV([-1,1])$. (Homework)

Lemma 3.5-2 If $F \in BV$, then $T_F + F$ and $T_F - F$ are nondecreasing.

Proof. We will prove $T_F + F$. The other one can be proved in the same way.

For $x < y$, we want to show that

$$T_F(x) + F(x) \leq T_F(y) + F(y).$$

By definition of T_F, for any $\epsilon > 0$, we can find points $x_0 < \cdots < x_n = x$ such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(x) - \epsilon.$$

Adding $|F(y) - F(x)|$ on both sides gives:

$$\begin{aligned} T_F(y) &\geq \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \\ &\geq T_F(x) + |F(y) - F(x)| - \epsilon \\ &\geq T_F(x) + (F(x) - F(y)) - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, this proves the lemma. □

Corollary 3.5-3 $F \in BV$ iff F is the difference of two *bounded nondecreasing* functions.

Proof. "$\Rightarrow$" Notice that

$$F = \underbrace{\frac{1}{2}(T_F + F)}_{:=F_1} - \underbrace{\frac{1}{2}(T_F - F)}_{:=F_2}.$$

By Lemma 3.5-2, we know that F_i are nondecreasing. It suffices to show F_i are bounded.

Indeed, $F \in BV$ implies that $T_F(\infty) < \infty$. On the other hand, for any $x > x_0$,

$$\begin{aligned} |F(x)| &\leq |F(x) - F(x_0)| + |F(x_0)| \\ &\leq T_F(x) - T_F(x_0) + |F(x_0)| \\ &\leq T_F(x) + |F(x_0)| \\ &\leq T_F(\infty) + |F(x_0)| \\ &< \infty. \end{aligned}$$

Hence, both F_i are bounded.

"$\Leftarrow$" An easy exercise ! □

Remark 3.5-2 Thanks to Corollary 3.5-3, Theorem 3.5-1 is true for all $F \in BV$, namely,

If $F \in BV$ ~~(the set of nondecreasing functions)~~, by denoting $G(x) := F(x+)$, then

(a) The set of discontinuous points of F is countable;

(b) Both F and G are differentiable a.e., and $F' = G'$ a.e.

Definition 3.5-2 For $F \in BV$,

$$F = \underbrace{\frac{1}{2}\left(T_F + F\right)}_{:=F_+} - \underbrace{\frac{1}{2}\left(T_F - F\right)}_{:=F_-}.$$

is called the *Jordan decomposition* of F. F_+ and F_- are called the *positive variation* and *negative variation* of F, respectively.

Definition 3.5-3 $F \in BV$ is called a *normalized bounded variation function*, denoted as $F \in$ *NBV*, if F is right continuous and F is nonnegative (i.e., $F(-\infty) = 0$).

Lemma 3.5-4 For any $F \in BV$, $T_F(-\infty) = 0$.

Proof. Because $T_F(x) \geq 0$, it suffices to show that for all $\epsilon > 0$, one can find $Y \in \mathbb{R}$ such that

$$T_F(y) < \epsilon \quad \text{for all } y < Y.$$

Now fix an arbitrary $\epsilon > 0$. Since $F \in BV$, we can find a sub-optimal partition $-\infty < x_0 < \cdots < x_n = x$ such that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \geq T_F(x) - \epsilon.$$

Set $Y = x_0$. For all $y < Y$,

$$\begin{aligned} T_F(x) - T_F(y) &\geq T_F(x) - T_F(x_0) \\ &\geq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \\ &\geq T_F(x) - \epsilon, \end{aligned}$$

which is $T_F(y) < \epsilon$. □

Lemma 3.5-5 If $F \in BV$ and F is right continuous, then T_F is also right continuous.

Proof. Fix an arbitrary $x \in \mathbb{R}$ and $\epsilon > 0$. Let $\alpha := T_F(x+) - T_F(x)$. It suffices to prove that for some constant $C > 0$, $\alpha \leq C\epsilon$.

By the right-continuity of F and $T_F(x+)$, there exist $\delta > 0$ such that for all $h \in (0, \delta)$,

$$|F(x+h) - F(x)| < \epsilon \quad \text{and} \quad |T_F(x+h+) - T_F(x+)| < \epsilon.$$

Since $|T_F(x+h+) - T_F(x+)| < \epsilon$ is true for all $h \in (0, \delta)$, it also holds that

$$|T_F(x+h) - T_F(x+)| < \epsilon, \qquad \text{for all } h \in (0, \delta).$$

Proof (continued). Fix any $h \in (0, \delta)$. We can find a sufficient good partition $x = x_0^1 < \cdots < x_{n_1}^1 = x + h$ such that

$$\sum_{i=1}^{n_1} \left| F(x_i^1) - F(x_{i-1}^1) \right| \geq \frac{3}{4} \left[T_F \left(x_{n_1}^1 \right) - T_F(x) \right] \geq \frac{3\alpha}{4}.$$

Similarly, we can find another sufficient good partition $x = x_0^2 < \cdots < x_{n_2}^2 = x_1^1$ such that

$$\sum_{i=1}^{n_2} \left| F(x_i^2) - F(x_{i-1}^2) \right| \geq \frac{3}{4} \left[T_F \left(x_{n_2}^2 \right) - T_F(x) \right] \geq \frac{3\alpha}{4}.$$

We can continue this refinement but these two steps will be sufficient:

Proof (continued). Hence,

$$
\begin{aligned}
\alpha + \epsilon &\geq [T_F(x+) - T_F(x)] + [T_F(x+h) - T_F(x)] \\
&= T_F(x+h) - T_F(x) \\
&\geq \left[T_F(x_1^1) - T_F(x)\right] + \left[T_F(x+h) - T_F(x_1^1)\right] \\
&\geq \sum_{i=1}^{n_2} \left|F(x_i^2) - F(x_{i-1}^2)\right| + \sum_{i=2}^{n_1} \left|F(x_i^1) - F(x_{i-1}^1)\right| \\
&= \sum_{i=1}^{n_2} \left|F(x_i^2) - F(x_{i-1}^2)\right| + \sum_{i=1}^{n_1} \left|F(x_i^1) - F(x_{i-1}^1)\right| - \left|F(x_1^1) - F(x_0)\right| \\
&\geq \sum_{i=1}^{n_2} \left|F(x_i^2) - F(x_{i-1}^2)\right| + \sum_{i=1}^{n_1} \left|F(x_i^1) - F(x_{i-1}^1)\right| - \epsilon \\
&\geq \frac{3\alpha}{4} + \frac{3\alpha}{4} - \epsilon \\
&\geq \frac{3\alpha}{2} - \epsilon
\end{aligned}
$$

which is equivalent to $\alpha \leq 4\epsilon$. □

There is a one-to-one correspondence between
the finite signed measure[2] μ on $\mathbb{R}$ and $F \in NBV$.

Theorem 3.5-6 (a) If μ is a finite signed measure on $\mathbb{R}$, then $F \in NBV$ where $F(x) := \mu((-\infty, x])$.

(b) Conversely, if $F \in NBV$, there is a unique signed finite Borel measure μ_F on $\mathbb{R}$ determined through $\mu_F((-\infty, x]) = F(x)$.

(c) Moreover, if $F \in NBV$, then μ_{T_F} is a finite measure and

$$|\mu| = \mu_{T_F}.$$

[2] $|\mu(\mathbb{R})| = \mu_+(\mathbb{R}) + \mu_-(\mathbb{R}) < \infty$

Proof of Theorem 3.5-6. (a) Assume that μ is a finite signed measure on $\mathbb{R}$. Let $\mu = \mu_+ - \mu_-$ be its Jordan decomposition.

Define

$$\begin{aligned} F_+(x) &:= \mu_+((-\infty, x]), \\ F_-(x) &:= \mu_-((-\infty, x]), \\ F(x) &:= F_+(x) - F_-(x). \end{aligned}$$

Then we see that

1. $F_\pm(x)$ and $F(x)$ are right continuous;
2. $F_\pm(x)$ and $F(x)$ are bounded: $|F(x)| \le F_+(\infty) + F_-(\infty) = |\mu|(\mathbb{R}) < \infty$;
3. $F_\pm(-\infty) = F(-\infty) = 0$;
4. $F(x)$ is a difference of two bounded nondecreasing functions $\Rightarrow F \in BV$; see Corollary 3.5-3.
5. Hence, $F \in NBV$.

This proves (a).

Proof (continued). (b) Now we assume that $F \in NBV$. Let

$$F = \underbrace{\frac{1}{2}(T_F + F)}_{:=F_+} - \underbrace{\frac{1}{2}(T_F - F)}_{:=F_-}$$

be its Jordan decomposition.

By Lemmas 3.5-4 and 3.5-5, we see that both $F_\pm$ are right-continuous and bounded.

Define

$$\mu_\pm((-\infty, x]) := F_\pm(x) - F(-\infty) = F_\pm(x).$$

Then we see that $\mu_\pm$ are finite Borel measures: $\mu_\pm(\mathbb{R}) = F_\pm(\infty) \le |\mu|(\mathbb{R})$.

As a consequence,

$$\mu((-\infty, x]) := \mu_+((-\infty, x]) - \mu_-((-\infty, x])$$

is a finite signed measure.

This proves (b).

Proof (continued). (c) Let $F \in NBV$ and the corresponding finite signed measure be μ_F.

In order to show that $|\mu_F| = \mu_{T_F}$, it suffices to show that

$$|\mu_F|((-\infty, x]) = \mu_{T_F}((-\infty, x]), \quad \text{for all } x \in \mathbb{R}. \qquad (\star)$$

By Lemmas 3.5-4 and 3.5-5, we see that both T_F is right-continuous and bounded. Hence, $T_F \in NBV$. By Part (b), μ_{T_F} is a finite (positive) measure such that

$$\mu_{T_F}((-\infty, x]) = T_F(x).$$

Apply part (a) to $|\mu_F|$ to see that $G(x) := |\mu_F|((-\infty, x]) \in NBV$, and moreover G is nondecreasing.

Hence,

$$(\star) \quad \Leftrightarrow \quad G(x) = T_F(x).$$

To prove $G = T_F$, following steps in Exercise 28. This proves (c) and hence the whole theorem. □

Corollary 3.5-7 Suppose that $F \in NBV$ and let μ_F be its corresponding finite signed measure. Let the corresponding Jordan decompositions of F and μ_F be

$$F = \underbrace{\frac{1}{2}(T_F + F)}_{:=F_+} - \underbrace{\frac{1}{2}(T_F - F)}_{:=F_-} \quad \text{and} \quad \mu_F = \mu_+ - \mu_-, \quad \text{respectively.}$$

Then

$$\mu_+ = \mu_{F_+} \quad \text{and} \quad \mu_- = \mu_{F_-}.$$

Proof. See Exercise 29. $\square$

More properties:

1. $F \in NBV \Rightarrow F' \in L^1(m)$.
2. If $F \in NBV$, then $\mu_F \perp m \iff F' = 0$ a.e.
3. If $F \in NBV$, then

$$\begin{aligned} \mu_F \ll m \quad &\iff \quad F \text{ is absolutely continuous} \\ &\iff \quad F(x) = \int_{-\infty}^{x} F'(t)dt \end{aligned}$$

4. If μ is a signed measure on $\mathbb{R}$, then

$$\mu = \underbrace{\mu_d + \mu_{sc}}_{=\mu_s} + \mu_{ac}.$$

Theorem 3.5-8 If $F, G \in NBV$ and at least one of them is continuous, then for all $-\infty < a < b < \infty$,

$$\int_{(a,b]} FdG + \int_{(a,b]} GdF = F(b)G(b) - F(a)G(a).$$

Remark 3.5-3 The integrals in the above theorem is called the *Lebesgue-Stieltjes integral*.

Proof. Without loss of generality, we may assume that (1) both F and G are nondecreasing; (2) G is continuous.

Fix arbitrary $a, b \in \mathbb{R}$ such that $a < b$. Let

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : a < x \leq y \leq b\right\}$$

By Tonelli's theorem, compute $\mu_F \times \mu_G\,(\Omega)$ in two ways...

□