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Conditional prob.

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad P(B)\neq 0.$$

$$P(A\cap B) = P(A|B)\,P(B)$$

X_n, $n=0,1,2\ldots$

$$\mathbb{P}(X_0=i_0, X_1=i_1,\cdots,X_n=i_n), \quad i_k\in\{1,\cdots,N\}$$

$\uparrow$ and

when $n=0$, initial condition.

$$\phi(i)=\mathbb{P}(X_0=i), \quad \underbrace{i=1,\cdots N}_{\text{State space (Finite)}}.$$

transition probabilities.

$$\mathbb{P}(X_n=i_n \mid X_0=i_0, X_1=i_1,\cdots,X_{n-1}=i_{n-1})$$

$$\parallel$$

$$q_n(i_n \mid i_0,\cdots,i_{n-1})$$

$$\mathbb{P}(X_n = i_n, \underline{X_{n-1} = i_{n-1}, \cdots X_0 = i_0})$$

$$\parallel$$

$$\mathbb{P}(X_n = i_n \mid \underline{X_{n-1} = i_{n-1}, \cdots X_0 = i_0}) \; \underline{\mathbb{P}(X_{n-1} = i_{n-1}, \cdots X_0 = i_0)}$$

$$\parallel$$

$$g_n(i_n \mid i_0 \cdots i_{n-1}) \; \mathbb{P}(\underline{X_{n-1} = i_{n-1}, \cdots X_0 = i_0})$$

$$\parallel$$

$$g_n(i_n \mid i_0 \cdots i_{n-1}) \; g_{n-1}(i_{n-1} \mid i_0, \cdots i_{n-2}) \cdots g_1(i_1 \mid i_0) \times \phi(i_0)$$

$\boxed{\text{Markov property}}$

$$\mathbb{P}(X_n = i_n \mid \underline{X_{n-1} = i_{n-1} \cdots X_0 = i_0}) = \underbrace{\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})}$$

$$\parallel$$

$$p_{\underset{\text{time}}{n}}(\underset{\{1, \cdots N\}}{i_{n-1}, i_n})$$

Time homogeneous Markov chain

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1} \cdots X_0 = i_0) = P(i_{n-1}, i_n)$$

transition Matrix

$$\mathbf{P} = \Big(\quad \Big)_{N \times N}$$

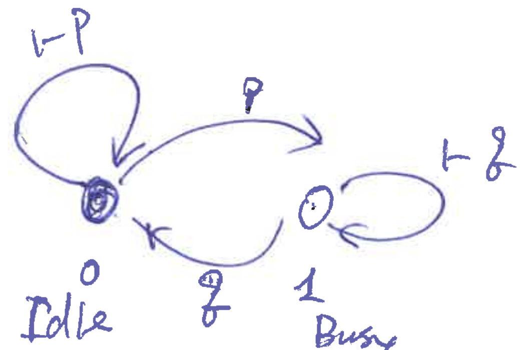
$$P = \left(P_{ij} \right)_{1 \le i,j \le N} = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} P_{11} & \cdots & P_{1N} \\ \vdots & & \vdots \\ P_{N1} & \cdots & P_{NN} \end{pmatrix} \end{array} \longrightarrow \text{row sum} = 1$$

$$\cancel{P} \quad P_{ij} = \mathbb{P}\left(X_n = j \mid X_{n-1} = i \right)$$

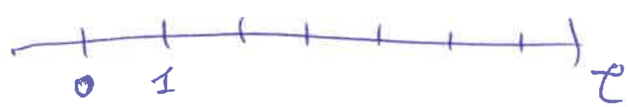
$$\begin{cases} \text{(i)} \quad 0 \le P_{ij} \le 1, \quad i,j \in \{1, \cdots, N\}. \\ \text{(ii)} \quad \sum_{j=1}^{N} P_{ij} = 1. \quad i \in \{1, \cdots, N\} \end{cases}$$

Example 1.

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-p & p \\ 1 & q & 1-q \end{array}$$

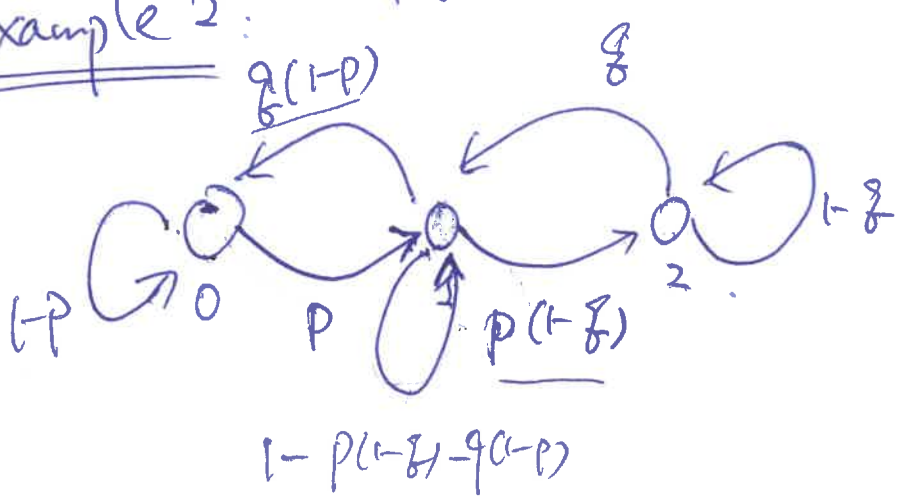


$N = 2.$



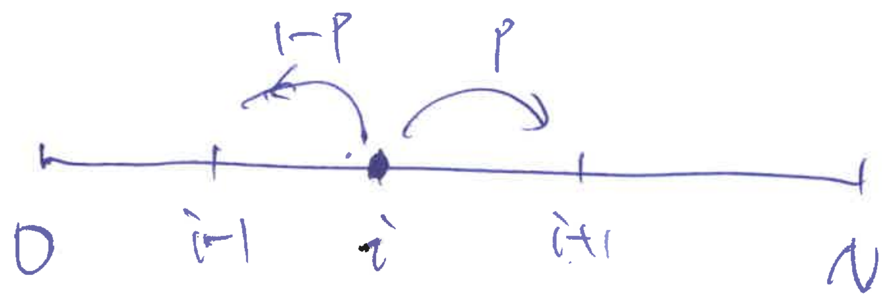
$\begin{cases} p: \text{ new custom.} \\ q: \text{ complete the service.} \end{cases}$

Example 2:



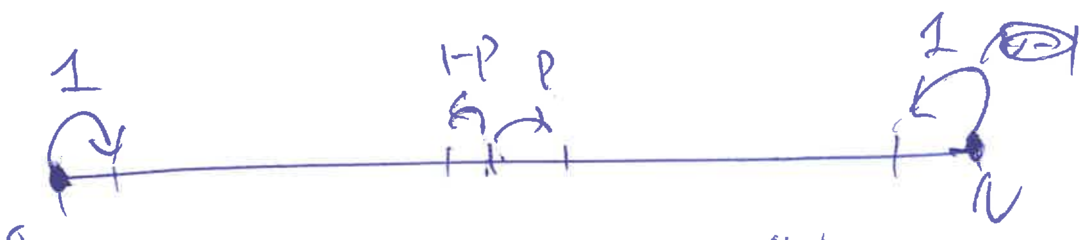
	0	1	2
0	$1-p$	p	0
1	$q(1-p)$	$1-p(1-q)-q(1-p)$	$p(1-q)$
2	0	q	$1-q$

Example 3. R.W. with Reflecting Boundary.



$$1 \leq i \leq N-1$$

$$p(0,1)=1, \quad p(N,N-1)=1$$



$$\begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & N-1 & N \\ \hline 0 & 0 & 1 & 0 & \cdots & & 0 \\ 1 & 1-p & 0 & p & 0 & \cdots & \\ & & 1-p & 0 & p & & \\ & & & 1-p & 0 & p & \\ N-1 & & & & 1-p & 0 & p \\ N & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Reflecting Boundary

Example 4: R.W. with absorbing boundaries.

$$p(0,0) = 1. \qquad p(N,N) = 1.$$

$$P = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ \hline 1-p & 0 & p & & \\ & \ddots & \ddots & \ddots & \\ & & 1-p & 0 & p \\ \hline 0 & \cdots & & 0 & 1 \end{pmatrix}$$

P Stochastic Matrix.

P^2

$$P(X_2 = j \mid X_0 = i) = \sum_{k=1}^{N} P(X_2 = j, X_1 = k \mid X_0 = i)$$

$$= \sum_{k=1}^{N} P(X_2 = j \mid X_1 = k, \cancel{X_0 = i}) \quad \leftarrow \text{Markov property}$$

$$\times P(X_1 = k \mid X_0 = i)$$

$$= \sum_{k=1}^{N} P_{kj} P_{ik} = (P^2)_{ij}$$

$$P(X_n = j \mid X_0 = i) = (P^n)_{ij}.$$

n-Step transition probability.

$$\bar{\phi}_0 = (\phi_0(1), \cdots \phi_0(N))$$ initial prob.

What is the distribution ~~after~~ at a time n?

$$P(X_1 = i) = \sum_{k=1}^{N} P(X_1 = i, X_0 = k)$$

$$= \sum_{k=1}^{N} \underbrace{P(X_1 = i \mid X_0 = k)}_{P_{ki}} \underbrace{P(X_0 = k)}_{\phi_0(k)}$$

$$= \sum_{k=1}^{N} \phi_0(k) P_{ki}$$

$$\underline{(\phi_0(1) \ldots \phi_0(N))} \begin{pmatrix} P_{ij} \end{pmatrix} = (\;,\; , \bigcirc \; \ldots)$$

ith. $P(X_1 = i)$

$\underline{\vec{\phi}_0 P^n}$ distribution at time n.

Example 6

$$\vec{\phi}_0 P^6 = (0.424, 0.576)$$

$\uparrow$

$(1,0)$

$\uparrow$

idle

$$P\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \overset{\substack{1\\ \parallel}}{\lambda}\begin{pmatrix}1\\\vdots\\1\end{pmatrix}$$

P has $\lambda=1$ and $\begin{pmatrix}1\\\vdots\\1\end{pmatrix}$ as the right eigenvalue and right eigenvector.

$\vec{v}P = \lambda\vec{v}$. left version.

$$\lim_{n\to\infty} P^n$$