§1.2.

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Math 7820.
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$$P = \begin{pmatrix} 3/4, & 1/4 \\ 1/6, & 5/6 \end{pmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 0.4, & 0.6 \\ 0.4, & 0.6 \end{pmatrix}$$

$$P = \begin{pmatrix} 1-p, & p \\ q, & 1-q \end{pmatrix}$$

$$\lim_{n\to\infty} P^n = ??$$ $\Longrightarrow$ See animations on the website.

<u>Spectral decomposition</u>:

$$P\vec{v} = \lambda \vec{v}.$$

row sum $=1$ $\Rightarrow$ $\lambda_1 = 1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let's find λ_2.

$$(P - \lambda I)\vec{v} = 0$$

$$\begin{pmatrix} 1-p-\lambda & p \\ q & 1-q-\lambda \end{pmatrix} \vec{v} = 0$$

$$\det\begin{vmatrix} 1-p-\lambda & p \\ q & 1-q-\lambda \end{vmatrix} = 0$$

$$(1-p-\lambda)(1-q-\lambda) - pq = 0$$

$$\lambda^2 - (2-p-q)\lambda + (1-p)(1-q) - pq = 0$$

$$\lambda^2 - (2-p-q)\lambda + \cancel{(1-2p)q = 0} - p - q + 1 = 0$$

$$\lambda_{1,2} = \frac{2-p-q \pm \sqrt{(2-p-q)^2 - 4(1-p-q)}}{2}$$

$$= \frac{2-p-q \pm \sqrt{(2-p-q)^2 - 4(1-p-q)}}{2}$$

$$\Delta = \left(1 + (1-p-q)\right)^2 - 4(1-p-q)$$

$$= \left(1 - (1-p-q)\right)^2$$

$$= (p+q)^2$$

$$\Rightarrow \sqrt{\Delta} = p+q$$

$$\lambda_{1,2} = \frac{2-p-q \pm (p+q)}{2} = \begin{cases} \oplus & \frac{2}{2} = 1 \quad ✓ \\ \ominus & 1-p-q. \end{cases}$$

②

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Now Let's find $\vec{v_2}$:

$$\begin{pmatrix} 1-p-(1-p-q), & p \\ q & 1-q-(1-p-q) \end{pmatrix} \vec{v} = 0$$

$$\begin{pmatrix} q, & p \\ q, & p \end{pmatrix} \vec{v} = 0$$

$$\vec{v_2} = \begin{pmatrix} -p \\ q \end{pmatrix}$$

$$P = \begin{pmatrix} 1-p, & p \\ q, & 1-q \end{pmatrix}$$

$$P \underbrace{\begin{pmatrix} 1 & -p \\ 1 & q \end{pmatrix}}_{Q} = \left(P\vec{v_1}, \ P\vec{v_2} \right)$$

$$= \left(\vec{v_1} \ \ (1-p-q)\ \vec{v_2} \right)$$

$$= \left(\vec{v_1}, \vec{v_2} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & -p \\ 1 & q \end{pmatrix}}_{Q}$$

$$Q^{-1} = \frac{1}{|Q|}\begin{pmatrix} q & p \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{p+q}\begin{pmatrix} q & p \\ -1 & 1 \end{pmatrix}$$

$$PQ = Q\begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix}$$

$$P = Q\begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix}Q^{-1}$$

$$P^n = Q\begin{pmatrix} 1^n & 0 \\ 0 & (1-p-q)^n \end{pmatrix}Q^{-1}$$

$$\downarrow{\scriptstyle n\to\infty}$$

$$Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}Q^{-1}$$

$$=$$

$$\underbrace{Q\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{\parallel}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}Q^{-1}}_{\parallel}$$

$$\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{q}{p+q} & \frac{p}{p+q}\\ 0 & 0\end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix}\boxed{\frac{q}{p+q} \quad \frac{p}{p+q}}\\ \frac{q}{p+q} \quad \frac{p}{p+q}\end{pmatrix} = \lim_{n\to\infty} P^n$$

Stationary distribution.

Eg:

$$\text{If } P = \begin{pmatrix}3/4 & 1/4\\ 1/6 & 5/6\end{pmatrix} \qquad \begin{matrix}p = 1/4\\ q = 1/6\end{matrix}$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix}\frac{1/6}{1/6+1/4} & \frac{1/4}{1/6+1/4}\\ \vdots & \vdots\end{pmatrix} = \begin{pmatrix}\frac{2}{2+3} & \frac{3}{2+3}\\ \frac{2}{5} & \frac{3}{5}\end{pmatrix}.$$

If P is $n \times n$ Stochastic Matrix, 6

$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Schur decomposition.

$$D = Q^T P Q$$

$$\uparrow$$

$$\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{M} \\ 0 & \end{bmatrix}$$

Additional Assumptions

① $\lambda_1 = 1$ is simple.

② $|\lambda_i| < 1$, $i = 2, \dots d$.

③ Q^T — first row has all entries nonnegative.

$$P = Q D Q^T$$

$$P^n = Q D^n Q^T$$

$$= Q \begin{pmatrix} 1^n & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{M^n} \\ 0 & \end{pmatrix} Q^T$$

$$M^n \xrightarrow{n \to \infty} [0]_{d-1 \times d-1}$$

$$\downarrow n \to +\infty$$

$$Q \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} Q^T$$

$$Q^T = \begin{bmatrix} \overline{\pi}_1 \\ \overline{\pi}_2 \\ \vdots \\ \overline{\pi}_n \end{bmatrix}$$

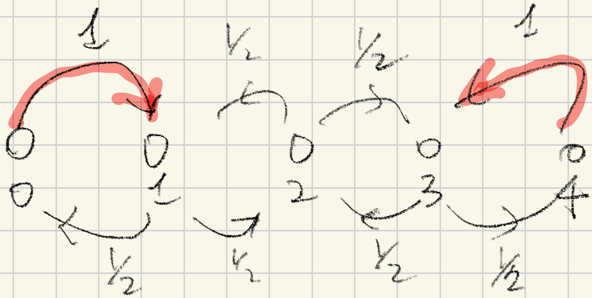
row vector

$$= \begin{pmatrix} \Big| & 0 \end{pmatrix} \begin{pmatrix} \overline{\pi}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\pi}_1 \overline{\pi}_1 \\ \overline{\pi}_1 \overline{\pi}_1 \cdots \\ \overline{\pi}_1 \overline{\pi}_1 \end{pmatrix}$$

§1.3. Classification of states.

Example 1 Reflecting Boundary:



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P^n \xrightarrow[n\to+\infty]{} ?? \begin{cases} n \text{ even} \\ n \text{ odd} \end{cases}$$

pair

Large n.

even states

$\downarrow$

even states

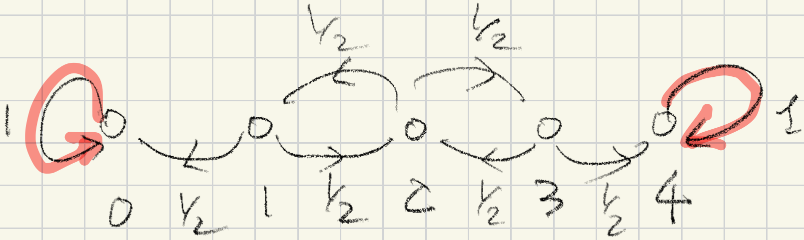
$$P^{2n} \approx \begin{pmatrix} \frac{1}{4}, & 0, & \frac{1}{2}, & 0, & \frac{1}{4} \\ 0, & \frac{1}{2}, & 0, & \frac{1}{2}, & 0 \\ \frac{1}{4}, & 0, & \frac{1}{2}, & 0, & \frac{1}{4} \\ 0 & \frac{1}{2} & 0, & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2}, & 0 & \frac{1}{4} \end{pmatrix}$$

$$P^{2n+1} \approx \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/4 & 0 & \frac{1}{2} & 0 & 1/4 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/4 & 0 & 1/2 & 0 & 1/4 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

even $\longleftrightarrow$ odd

states states

Exaple 2: Aborbing bdry.



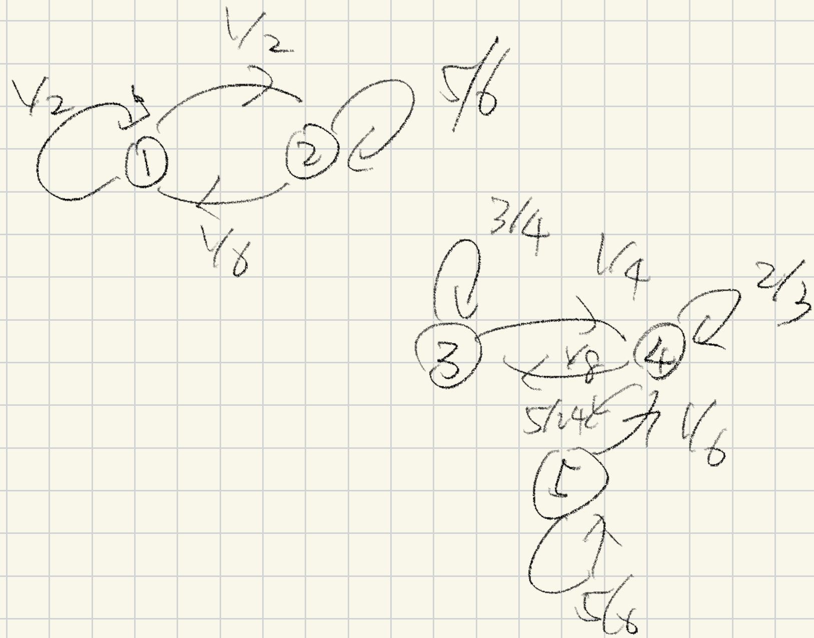
$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

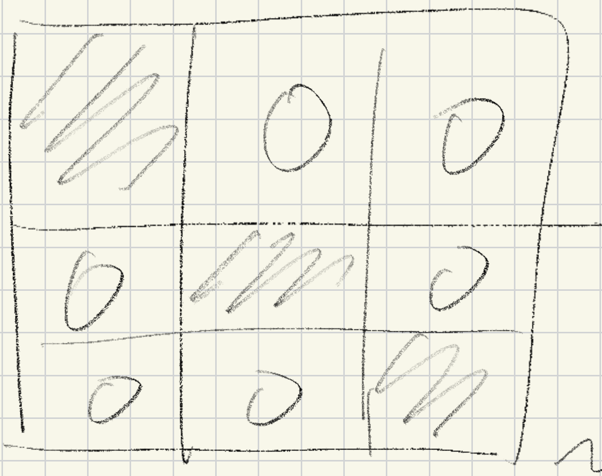
$$P^n \approx \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 2/4 & 0 & 0 & 0 & 2/4 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

States 1, 2, 3

are **Transient**.

Example 3: Reducible Markov Chain.





three sub-chains