For reflecting Bdry case, can we rigorously derive the asymptotic of

P^n

$$P = Q \Lambda Q^{-1}$$

$$P^n = Q \Lambda^n Q^{-1}$$

$\downarrow$ $n \to \infty$

??

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08/28/25
Lecture
Math 7820
Auburn

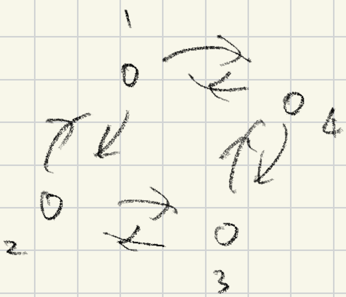
§1.3.1 Reducibility

i $\qquad$ j

i and j communicate with each other

denoted by $i \leftrightarrow j$: $\exists\, m, n$ s.t.

$$P_m(i,j) > 0,\quad P_n(j,i) > 0$$



	1	2	3	4
1	0			
2		0		0
3	0		0	
4		0		0

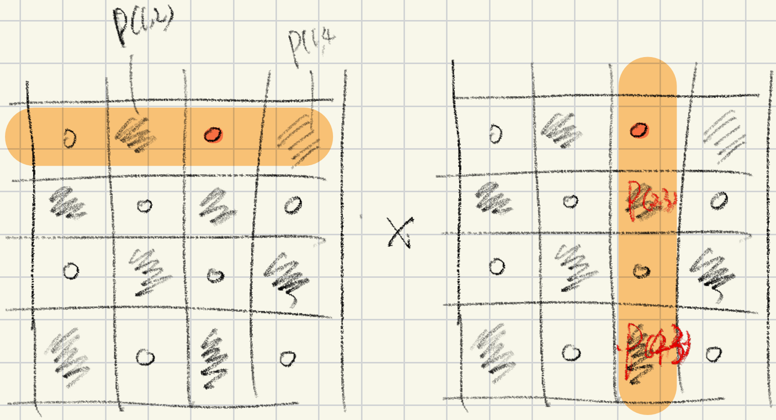
$1 \leftrightarrow 3$ in 2 steps $\qquad P(1,3) = 0$

but $P_2(1,3) \neq 0$

Notation $\quad P_n(i,j) = (i,j)$th entry of P^n

$$P_2(1,3) = P(1,4)\,P(4,3) + P(1,2)\,P(2,3)$$

$P^2 =$



$$P_2(1,3) = P(1,2)\,P(2,3) + P(1,4)\,P(4,3)$$

"$\longleftrightarrow$" is an <u>equivalent</u> relation $\Rightarrow$ Equivalent Class. Communication Class.

(i) reflexive $i \leftrightarrow i$

(ii) symmetric : $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

(iii) transive : $i \leftrightarrow k,\ k \leftrightarrow j \Rightarrow i \leftrightarrow j$.

Explanation : (i) $\exists\, n$ s.t. $P_n(i,i) > 0$

Choose $n=0$, $P^0 = I$, $P_0(i,i) = 1$

(iii) $i \leftrightarrow k$, $k \leftrightarrow j$, we need to show $i \leftrightarrow j$ 4

$\Downarrow$ $\exists\, n$ s.t. $P_n(i,k) > 0$

$\Downarrow$ $\exists\, m$, s.t. $P_m(k,j) > 0$

i.e., need to find $\tilde{n}$ s.t. $P_{\tilde{n}}(i,j) > 0$

$\tilde{n} = n+m$

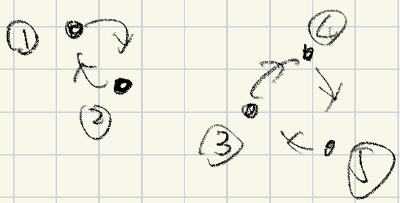
$$
\begin{aligned}
P_{n+m}(i,j) &= \mathbb{P}(X_{n+m} = j \mid X_0 = i) \\
&= \sum_{\ell} \mathbb{P}(X_{n+m} = j,\ X_n = \ell \mid X_0 = i) \\
&= \sum_{\ell} \mathbb{P}(X_{n+m} = j \mid X_n = \ell) \times \mathbb{P}(X_n = \ell \mid X_0 = i) \\
&\underset{\ell = k}{\geq} \mathbb{P}(X_{n+m} = j \mid X_n = k) \times \mathbb{P}(X_n = k \mid X_0 = i) \\
&\underset{\text{Time homogeneous}}{=} \underbrace{\mathbb{P}(X_m = j \mid X_0 = k)}_{P_m(k,j)}\ \underbrace{\mathbb{P}(X_n = k \mid X_0 = i)}_{P_n(i,k)} \\
&= P_n(i,k)\, P_m(k,j) > 0.
\end{aligned}
$$

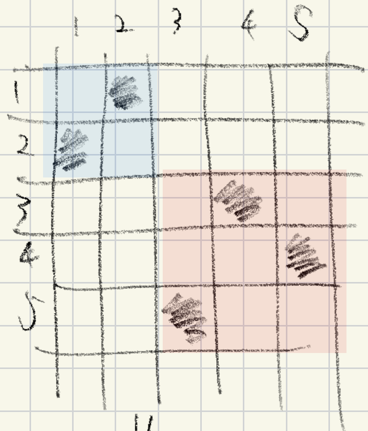
Similarly, one can show that $\exists\, n^*$ s.t.

$$P_{n^*}(j, i) > 0.$$

Therefore $i \leftrightarrow j$.

e.g.:





Two communication classes

{①, ②}, {③, ④, ⑤}

recurrent classes.

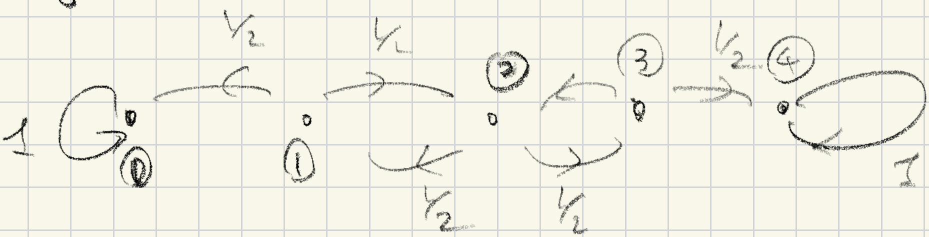
$$P = \begin{pmatrix} \boxed{P_1} & \\ & \boxed{P_2} \end{pmatrix}$$

$$P^n = \left(\begin{array}{c|c} P_1^n & 0 \\ \hline 0 & P_2^n \end{array}\right)$$

A Markov chain is IREDUCIBLE. if $\forall\, i, j$

$\exists\, n(i,j)$ s.t. $P_n(i,j) > 0$

Eg 2:



$\{0\}, \quad \{1, 2, 3\}, \quad \{4\}$

recurrent classes "otherwise, keep coming back"

transient class: "once leave never return"

$$P = \left[\begin{array}{ccc|c} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & P_r & \\ \hline & S & & Q \end{array}\right]$$

recurrent classes

transient class.

$$P^n = \begin{bmatrix} P_1^n & & & \\ & P_2^n & & \\ & & P_r^n & \\ & S_n & & Q^n \end{bmatrix}$$