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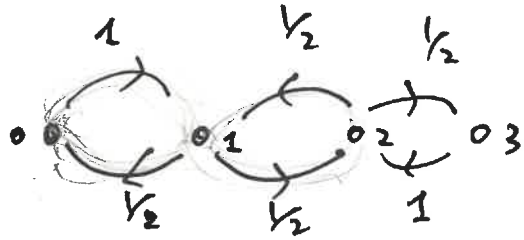
1.3.2. Periodicity.

Let P be the transition matrix of an irreducible Markov chain. Let i be a state of the chain. Define the period of a state i to be the greatest common divisor of

$$J_i := \{n \geq 0 : P_n(i,i) > 0\}.$$

Start from i, after n steps you have positive prob. to come back to i.

Eg:



$d(0) \overset{??}{=}$

$J_0 = \{2, 4, 6, \cdots\} \Rightarrow d(0) = 2.$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & 0 & 0 & 1 & \end{array}, \quad P^2 = \cdots \text{ see (Chatgpt. computation).}$$

$d(1) = ??$

$J_1 = \{2, 4, 6, \dots\} \Rightarrow d(1) = 2.$

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Proposition: Irreducible M.C. All states have the same period.

proof (Illustration).

$d(i) = d(j)$ $\quad i \neq j$. $\qquad$ Irreducible means $\exists\, n, m$ s.t.

$$\begin{cases} P_n(i,j) > 0 \\ P_m(j,i) > 0. \end{cases}$$

$$P_{n+d(j)k+m}(i,i) > 0. \Rightarrow$$

$n + d(j) \times k + m \in J_i$ $\quad$ ---- $(*)$

for $k = 0, 1, 2, \ldots$

If $k=0$, this means $n+m \in J_i$. i.e.,

$\exists\, \tilde{k}$ s.t. $\quad n+m = \tilde{k}\, d(i)$.

If $k=1$, then $(*)$ means, $\quad n+m+d(j) \in J_i$

$\Rightarrow \exists\, \hat{k}$ s.t. $\quad n+m+d(j) = \hat{k}\, d(i)$,

Recall $n+m = \tilde{k}\, d(i)$. Therefore, $d(j) = (\hat{k} - \tilde{k})\, d(i)$.

$d(j)$ is multiple of $d(i)$.

Similarly, we also have: $d(i)$ is multiple of $d(j)$.

Therefore, $d(i) = d(j)$. $\square$

§1.3.3. Irreducible & aperiodic Chains.

Df: An irreducible Markov chain P is <u>aperiodic</u> iff $d = 1$.

<u>Prop.</u> P is an irreducible & aperiodic. Then $\exists\, n$ s.t. P^n has all entries strictly positive.

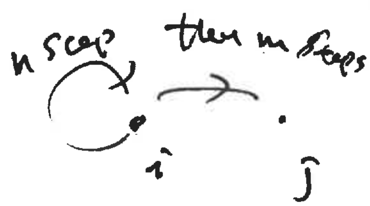
<u>Proof</u>: Because P is <u>irreducible</u>, $\forall i, j$, $\exists\, m(i,j)$ s.t.

$$P_{m(i,j)}(i, j) > 0.$$

Since P is <u>aperiodic</u>, $\exists\, M(i)$ s.t. $\forall n \geq M(i)$

$$P_n(i, i) > 0.$$

$$J_i = \{ \cdots\cdots, \underset{\substack{\uparrow \\ \exists M}}{M}, M+1, M+2, M+3, \cdots \}.$$



Then
$$P_{m+n}(i, j) \geq P_n(i, i)\, P_m(i, j) > 0.$$

Since we only have finite many pairs of (i,j), ~~we~~ we can find M s.t. $\forall n \geq M$

$P_n(i,j) > 0.$ $\blacksquare$.